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A REMARK ON MINIMAL FOLIATIONS

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1. Introduction. A foliation $\mathscr{F}$ of a closed manifold M is said to be geometrically taut if there is a Riemannian metric g on M for which the leaves become minimal submanifolds (see [4]). We call the triple $(M, \mathscr{F}, g)$ a minimal foliation. Recently, Sullivan [4] gave a necessary and sufficient condition for a foliation to be geometrically taut. In particular, a codimension-one foliation is geometrically taut if every compact leaf is cut out by a closed transversal. Thus we have many examples of minimal foliations. In this paper, we shall study the converse with restricted Riemannian metrics, that is, if $(M, \mathscr{F}, g)$ is an oriented minimal codimension-one foliation on an oriented closed Riemannian manifold with non-negative Ricci tensor, then the unit vector field on M perpendicular to $\mathscr{F}$ is parallel. Consequently, $\mathscr{F}$ can be defined by a closed 1-form.

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2. Preliminaries and statement of result. We shall consider only codimension-one foliations and work in C^∞-category.

Let $\mathscr{F}$ be an oriented codimension-one foliation on a closed oriented Riemannian manifold (M, g). Then we can choose a unit vector field N on M perpendicular to $\mathscr{F}$ so that the orientation of M coincides with the one given by $\mathscr{F}$ and N. Define the second fundamental form $\bar{A}$ of $(M, \mathscr{F}, g)$ by

$$\bar{A}_p\colon T_p\mathscr{F} \to T_p\mathscr{F}, \quad \bar{A}_p(V) = D_V N \quad \text{for} \quad V \in T_p\mathscr{F},$$

where D means the Riemannian connection of (M, g) (see [2]). Note that $g(\bar{A}(V), W) = g(V, \bar{A}(W))$ for $V, W \in T_p\mathscr{F}$. Hereafter, we always assume these situations.

DEFINITION. A triple $(M, \mathscr{F}, g)$ is said to be a *minimal foliation* if $\mathrm{Trace}(\bar{A}_p) = 0$ for all $p \in M$. A triple $(M, \mathscr{F}, g)$ is said to be a *totally geodesic foliation* if $\bar{A}_p = 0$ for all $p \in M$.

Let X be a vector field on (M, g). Define a (1, 1)-tensor A_X by $A_X(V) = D_V X$ for $V \in T_pM$. Also define a smooth function δX by $\delta X = -\mathrm{div}\, X$, where div means the divergence. Then we have $\delta X = -\mathrm{Trace}(A_X)$ (see

[2]). The following theorem is well-known (cf. [2]).

THEOREM (Green's theorem). *Let X be a vector field on a closed oriented Riemannian manifold M. Then*

$$\int_M \mathrm{Ric}(X, X)dM + \int_M \mathrm{Trace}(A_X^2)dM - \int_M (\delta X)^2 dM = 0 ,$$

where $\mathrm{Ric}(X, X)$ *means the Ricci curvature of* X.

Now we state our theorem.

THEOREM. *Let* $(M, \mathscr{F}, g, N)$ *be an oriented minimal codimension-one foliation on a closed Riemannian manifold with nonnegative Ricci curvature. Then N is a parallel vector field. Hence, in particular,* $\mathscr{F}$ *can be defined by a closed 1-form.*

3. **Proof of Theorem.** Let $\dim(M) = n + 1$ and let $\{E_1, \cdots, E_n, N\}$ be a local oriented orthonormal basis of TM. Then $\{E_1, \cdots, E_n\}$ is a local oriented orthonormal basis of $T\mathscr{F}$. Throughout this section we shall use this basis.

LEMMA 1. *If* $\mathscr{F}$ *is a minimal foliation, then* $\delta N = 0$.

PROOF. Define an n-form $\chi_{\mathscr{F}}$ on M^{n+1} by

$$\chi_{\mathscr{F}}(V_1, \cdots, V_n) = \det(\langle E_i, V_j \rangle) ,$$

where $\langle X, Y\rangle$ means $g(X, Y)$. By Rummler's calculation [3], $\mathscr{F}$ is minimal if and only if $d\chi_{\mathscr{F}} = 0$. Define a 1-form ω by $\omega(X) = \langle X, N\rangle$. Then $\chi_{\mathscr{F}} \wedge \omega = dM$. Thus we have $\delta N = \delta\omega = \delta * \chi_{\mathscr{F}} = \pm * d\chi_{\mathscr{F}} = 0$, where $*$ is Hodge's star operator and $\delta\omega$ is the co-differential of ω.

LEMMA 2. $\mathrm{Trace}(\bar{A}^2) = \mathrm{Trace}(A_N^2)$.

PROOF. We have only to show that $\langle A_N^2(N), N\rangle = 0$. This follows from the fact that $2\langle D_V N, N\rangle = V\langle N, N\rangle = V(1) = 0$.

LEMMA 3. *Define 1-forms ω and θ by* $\omega(X) = \langle N, X\rangle$ *and* $\theta(X) = \langle D_N N, X\rangle$. *Then* $d\omega = \omega \wedge \theta$.

PROOF. We have to show $d\omega(E_i, E_j) = (\omega \wedge \theta)(E_i, E_j)$ and $d\omega(E_i, N) = (\omega \wedge \theta)(E_i, N)$, but these are clear by the definitions of d and the exterior product.

LEMMA 4. $\langle D_X D_N N, Y\rangle = \langle D_Y D_N N, X\rangle$ *for* $X, Y \in \Gamma(T\mathscr{F})$.

PROOF. By Lemma 3, we have $d\theta = \omega \wedge \eta$ for some 1-form η. By the definition of d and the fact that $[X, Y] = D_X Y - D_Y X$, we see that $(\omega \wedge \eta)(X, Y) = 0$ and $d\theta(X, Y) = \langle D_X D_N N, Y\rangle - \langle D_Y D_N N, X\rangle$ for $X, Y \in$

$\Gamma(T\mathscr{F})$.

PROOF OF THEOREM. First we show that $\mathscr{F}$ is totally geodesic, that is, $\bar{A}=0$. By Green's theorem and Lemmas 1 and 2, we have

$$\int_M \mathrm{Ric}(N, N)dM + \int_M \mathrm{Trace}(\bar{A}^2)dM = 0 . \tag{3.1}$$

Since $\bar{A}$ is symmetric with respect to g, it follows that $\mathrm{Trace}(\bar{A}^2) = \sum_{i=1}^n \langle \bar{A}^2(E_i), E_i\rangle = \sum_{i=1}^n \langle \bar{A}(E_i), \bar{A}(E_i)\rangle \geqq 0$. Thus, combining the above with the hypothesis that $\mathrm{Ric}(N, N) \geqq 0$, we have $\mathrm{Trace}(\bar{A}^2) = 0$ by (3.1). This implies $\bar{A}=0$. Note that we also have $\mathrm{Ric}(N, N)=0$.

Next we show that $\delta(D_N N)=0$. By the definition of the Ricci curvature we have $\mathrm{Ric}(N, N) = \sum_{i=1}^n K(N, E_i)$, where $K(N, E_i)$ is the sectional curvature of the 2-plane spanned by N and E_i. We have

$$\begin{aligned} K(N, E_i) &= \langle R(E_i, N)N, E_i\rangle = \langle D_{E_i}D_N N - D_N D_{E_i}N - D_{[E_i,N]}N, E_i\rangle \\ &= \langle D_{E_i}D_N N, E_i\rangle + \langle D_{D_N E_i}N, E_i\rangle \\ &= \langle D_{E_i}D_N N, E_i\rangle + \langle D_N E_i, N\rangle\langle D_N N, E_i\rangle \\ &= \langle D_{E_i}D_N N, E_i\rangle - \langle D_N N, E_i\rangle^2 . \end{aligned}$$

Thus we have $\mathrm{Ric}\,(N, N) = \sum_{i=1}^n \langle D_{E_i}D_N N, E_i\rangle - \sum_{i=1}^n \langle D_N N, E_i\rangle^2 = \sum_{i=1}^n \langle D_{E_i}D_N N, E_i\rangle - \langle D_N N, D_N N\rangle = \mathrm{Trace}(A_{D_N N})$. We have already pointed out that $\mathrm{Ric}(N, N)=0$ and $\delta(D_N N) = -\mathrm{Trace}(A_{D_N N})$. Hence $\delta(D_N N)=0$.

Finally we show that N is parallel. As $\mathscr{F}$ is totally geodesic, it is sufficient to show that $D_N N = 0$. By Green's theorem and $\delta(D_N N)=0$ we have

$$\int_M \mathrm{Ric}\,(D_N N, D_N N)\,dM + \int_M \mathrm{Trace}(A^2_{D_N N})dM = 0 . \tag{3.2}$$

$$\begin{aligned} \mathrm{Trace}(A^2_{D_N N}) &= \sum_{i=1}^n \langle A^2_{D_N N}(E_i), E_i\rangle + \langle A^2_{D_N N}(N), N\rangle \\ &= \sum_{i=1}^n \langle D_{D_{E_i}D_N N}D_N N, E_i\rangle + \langle D_{D_N D_N N}D_N N, N\rangle \\ &= \sum_{i,j=1}^n \langle D_{E_i}D_N N, E_j\rangle\langle D_{E_j}D_N N, E_i\rangle + \langle D_N D_N N, N\rangle^2 \\ &\quad + 2\sum_{i=1}^n \langle D_{E_i}D_N N, N\rangle\langle D_N D_N N, E_i\rangle \\ &= \sum_{i,j=1}^n \langle D_{E_i}D_N N, E_j\rangle\langle D_{E_j}D_N N, E_i\rangle + \langle D_N D_N N, N\rangle^2 , \end{aligned}$$

because $\langle D_{E_i}D_N N, N\rangle = E_i\langle D_N N, N\rangle - \langle D_N N, D_{E_i}N\rangle = 0$. Thus by Lemma 4 we have

$$\mathrm{Trace}(A^2_{D_N N}) = \sum_{i,j=1}^{n} \langle D_{E_i} D_N N, E_j \rangle^2 + \langle D_N D_N N, N \rangle^2 \geqq 0 .$$

As $\mathrm{Ric}(D_N N, D_N N) \geqq 0$, we have $\mathrm{Trace}(A^2_{D_N N}) = 0$ by (3.2). Hence $0 = \langle D_N D_N N, N \rangle = -\langle D_N N, D_N N \rangle$, that is, $D_N N = 0$.

By Lemma 3, it is clear that $\mathscr{F}$ can be defined by a closed 1-form. This completes the proof.

4. **Concluding remarks.** Cheeger-Gromoll [1] proved the following theorem.

THEOREM [1]. *Let M be a compact manifold of nonnegative Ricci curvature. Then the universal covering $\tilde{M}$ of M splits isometrically as $\bar{M} \times \boldsymbol{R}^k$, where $\bar{M}$ is compact and $\boldsymbol{R}^k$ has its standard flat metric.*

Using this, we have the following.

COROLLARY. *Let $(\tilde{M} = \bar{M} \times \boldsymbol{R}^k, \tilde{\mathscr{F}}, \tilde{g}, \tilde{N})$ be the canonical lifting of $(M, \mathscr{F}, g, N)$ to the universal covering $\tilde{M}$ of M. Then $\tilde{N}$ is perpendicular to $\bar{M} \times \{x\}$, $x \in \boldsymbol{R}^k$. Consequently, $\tilde{\mathscr{F}} = \bar{M} \times (\boldsymbol{R}^k, \mathscr{F}')$, where $(\boldsymbol{R}^k, \mathscr{F}')$ is a totally geodesic foliation by flat planes.*

PROOF. Let $p\colon \tilde{M} = \bar{M} \times \boldsymbol{R}^k \to \bar{M}$ (resp. $q\colon \tilde{M} \to \boldsymbol{R}^k$) be the canonical projection onto the first factor (resp. the second factor) of $\tilde{M}$. Then $T\tilde{M} = p^*(T\bar{M}) \oplus q^*(T\boldsymbol{R}^k)$. Thus the vector field $\tilde{N}$ has the unique expression $\tilde{N} = X + Y$, where $X \in \Gamma(p^*(T\bar{M}))$ and $Y \in \Gamma(q^*(T\boldsymbol{R}^k))$. As $\tilde{N}$ is a parallel vector field, $\tilde{D}_V \tilde{N} = \tilde{D}_V X + \tilde{D}_V Y = 0$. We also have $\langle \tilde{D}_V X, \tilde{D}_V Y \rangle = -\langle Y, \tilde{D}_V \tilde{D}_V X \rangle = 0$ for $V \in \Gamma(p^*(T\bar{M}))$. Thus $X|_{\bar{M} \times \{x\}}$ is a parallel vector field on $\bar{M} \times \{x\}$, $x \in \boldsymbol{R}^k$. If $\tilde{N}$ is not perpendicular to $\bar{M} \times \{x\}$ at $(m, x) \in \bar{M} \times \{x\}$, then $\bar{M} \times \{x\}$ has a nonvanishing parallel vector field. As $\bar{M}$ is simply connected, de Rham's decomposition theorem (see [2]) implies that $\bar{M}$ splits isometrically as $M' \times \boldsymbol{R}$, which contradicts the compactness of $\bar{M}$.

The same argument as in the proof of Theorem also gives the proof of the following (see also Tanno [5]).

PROPOSITION. *Let $(M, \mathscr{F}, g, N)$ be a totally geodesic foliation on a closed Riemannian manifold with nonpositive sectional curvature. Then N is a parallel vector field.*

PROOF. We have $D_V D_N N = \langle D_N N, V \rangle \cdot D_N N$ for $V \in T\mathscr{F}$ by the assumptions. Using this formula, we can show directly that $\delta(D_N N) = 0$ and $\mathrm{Ric}(D_N N, D_N N) = 0$. Thus (3.2) gives the desired conclusion.

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