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A NOTE ON IMAGES OF REDUCTION OPERATORS

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Consider a nonnegative locally Hölder continuous 2-form P on a *hyperbolic* Riemann surface R. We denote by $P(R)$ the space of solutions of the equation $d*du = uP$ on R. By $PB(R)$, $PD(R)$ and $PBD(R)$ we denote the subspaces of bounded, Dirichlet-finite and bounded Dirichlet-finite solutions. The reduction operator T is a linear order preserving mapping of a subspace of $P(R)$ into $H(R)$ defined by

$$(1) \qquad Tu = u + \frac{1}{2\pi}\int_R g_R(\cdot,\zeta)u(\zeta)P(\zeta)\,,$$

where $g_R(\cdot,\zeta)$ is harmonic Green's function for R. In case $u \in PY(R)$, $Y = B$, D or BD, it is known that Tu exists and $Tu \in HY(R)$ (cf. [3]). We denote by T_Y the restriction $T|PY(R)$. Since T_Y is an injection (cf. [3]) it can be used to reduce questions concerning $PY(R)$ to questions concerning a subspace of $HY(R)$, $Y = B$, D or BD.

Denote by X_Y^P the image of $PY(R)$ under T_Y, $T = BD$ or D. The problem of characterizing X_D^P is central to the study of $PD(R)$. Singer [6], [7] gave the first substantial results in this direction. In [2] we extended his technique to give a complete characterization of X_D^P. Although this result has significant practical applications, it is nonetheless cumbersome to apply. The motivation of the present note is to give a more efficient characterization of X_D^P. However, we will not make use of any result of [2] here.

To each function $h \in HD^+(R)$ we associate a sequence $\{h_k\} \subset HBD^+(R)$, called *the standard HBD-approximation to h*, as follows. Set $\psi_k = (h \cap k) \cup k^{-1}$ and $h_k = \Pi\psi_k - k^{-1}$, $k = 1, 2, \cdots$, where $\Pi\psi_k$ is the harmonic projection of ψ_k and $\cap$ (resp. $\cup$) denotes the pointwise minimum (resp. maximum). Later we shall elaborate on the useful properties of $\{h_k\}$. Consider the family

$$\mathscr{D} = \{u \in PD(R) \,|\, 0 \leqq u \leqq 1\}\,.$$

Define a function $\delta = \sup_{u\in\mathscr{D}} u$. Our main result can be stated as

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follows:

THEOREM. *Let* $h \in HD^+(R)$. *Then* $h \in X_D^P$ *if and only if* $\{h_k\} \subset X_{BD}^P$ *and* $D_R(\delta h) < +\infty$.

1. In order to simplify our arguments we use the Royden ideal boundary theory adapted to the equation $d*du = uP$. We begin by reviewing some facts here but refer to [5] and [1] for more details. Let $\tilde{M}(R)$ be the space of continuous Tonelli functions on R with finite Dirichlet integrals over R and let $M(R)$ be the space of bounded functions in $\tilde{M}(R)$, i.e., $M(R)$ is the Royden algebra associated to R. Denote by R^* the Royden compactification of R and by $\varDelta$ the harmonic boundary. The set $\varDelta_P$ of Green's energy nondensity points is the set of points $q^* \in \varDelta$ such that q^* has a neighborhood U^* in R^* with $\langle 1, 1\rangle_{U^* \cap R}^P < +\infty$. Here,

$$\langle \varphi, \varphi \rangle_\Omega^P = \frac{1}{2\pi} \int_{\Omega \times \Omega} g_\Omega(z, \zeta)\varphi(z)P(z)\varphi(\zeta)P(\zeta) ,$$

for an open set $\Omega \subset R$ and a suitable function φ on Ω. The following alternative description of $\varDelta_P$ is useful:

$$\varDelta_P = \{q^* \in \varDelta \mid u(q^*) \neq 0, \text{ for some } u \in PD(R)\} .$$

Moreover, $\varDelta_P$ serves for a maximum principle for $PD(R)$: For an open set $\Omega \subset R$ and a function $u \in PD(\Omega)$, $|u| \leqq M$ holds whenever $\limsup_{q \to q^*} |u(q)| \leqq M$ for each $q^* \in \partial\Omega \cup (\bar{\Omega} \cap \varDelta_P)$.

The modified Royden decomposition theorem may be formulated as follows: Let W be an open subset of R with a C^1 relative boundary and let $f \in \tilde{M}(R)$. Then there is a unique function $h \in HD(W) \cap \tilde{M}(R)$ such that $(f - h) | \varDelta \cup (\overline{R \setminus W}) = 0$. Moreover, the Dirichlet principle holds: $D_R(h - f, h) = 0$. The notation $h = \Pi_{\overline{R \setminus W}} f$ is used. Concerning the existence of solutions of $d*du = uP$ we have the following: Let $f \in \tilde{M}(R)$ and assume either that f is a nonnegative subsolution of $d*du = uP$ on R or that f is bounded and $\mathrm{Supp}(f | \varDelta) \subset \varDelta_P$. Then there is a unique function $u \in PD(R)$ with $(u - f) | \varDelta = 0$. Here, we use the symbol $\Pi^P f$ to denote u.

For $u \in PD^+(R)$, the function $T_D u - u$ is a potential on R and belongs to $M(R)$. Thus it vanishes on $\varDelta$. On the other hand, we also have $(\Pi u - u) | \varDelta = 0$ and we conclude by the maximum principle that $T_D u = \Pi u$. By the Dirichlet principle $D_R(u) = D_R(T_D u) + D_R(u - T_D u)$. Since

$$D_R\Big(\frac{1}{2\pi} \int_R g_R(\cdot, \zeta)u(\zeta)P(\zeta)\Big) = \langle u, u \rangle_R^P$$

(cf. [3]), we have the formula

$$D_R(u) = D_R(T_D u) + \langle u, u\rangle_R^P \,. \tag{2}$$

2. Let W be an open subset of R with ∂W being C^1. We denote by $HD(W; \partial W)$ the functions in $HD(W) \cap C(R)$ which vanish on $R \setminus W$. It is easily seen that $HD(W; \partial W)$ is generated by its nonnegative functions. The extremization μ_D: $HD(W; \partial W) \to HD(R)$ is defined to be the linear mapping such that $\mu_D u - u$ is a potential for each $u \in HD^+(W; \partial W)$. Since C^1-coordinate lines are removable sets for Tonelli functions we see that $HD(W; \partial W) \subset \tilde{M}(R)$. Consequently, $\Pi(\mu_D u - u) = 0$ for each $u \in HD^+(W; \partial W)$. We see that $\mu_D u = \Pi u$ for each $u \in HD(W; \partial W)$. For a function to be in the image of μ_D we have the following test (cf. [4]).

LEMMA. *Let $\mathcal{O}$ be an open subset of Δ and W an open subset of R with C^1 relative boundary such that $\mathcal{O} \subset \bar{W}$. Let w be a bounded nonnegative Tonelli function on R which is continuous on $R \cup \mathcal{O}$ and $w|\mathcal{O} = 1$, $w|R \setminus W = 0$. If $h \in HD^+(R)$ such that $h|\Delta \setminus \mathcal{O} = 0$ and $D_W(wh) < +\infty$, then h is in the image of μ_D.*

Since $wh \in \tilde{M}(R)$ and $wh|\overline{R \setminus W} = 0$, the function $v = \Pi_{\overline{R\setminus W}}(wh)$ has the properties $v|\Delta = wh|\Delta$ and $v \in HD(W; \partial W)$. Clearly, $wh|\mathcal{O} = h|\mathcal{O}$. For any $q^* \in \Delta \setminus \mathcal{O}$ take a net $\{q_\lambda\} \subset R$ with $q^* = \lim q_\lambda$. Then $0 \leqq \lim wh(q_\lambda) \leqq \limsup w(q_\lambda) \lim h(q_\lambda) = 0$ because w is bounded and $h(q^*) = 0$. Therefore, $h|\Delta = wh|\Delta = v|\Delta$. We conclude that $h = \Pi v = \mu_D v$.

3. For an $h \in HD^+(R)$, let $\{h_k\}$ be the standard HBD-approximation to h. Set $F_k = \{p^* \in \Delta \,|\, h(p^*) \geqq k^{-1}\}$, a compact subset of Δ, $k = 1, 2, \cdots$. The properties of $\{h_k\}$ that we shall use are contained in the

LEMMA. (i) $\operatorname{Supp}(h_k|\Delta) \subset F_k$;
(ii) $\lim (h_k|\Delta) = h|\Delta$;
(iii) $\{h_k\} \subset X_{BD}$ *if and only if* $h|\Delta \setminus \Delta_P = 0$;
(iv) $D_R(h_k) \leqq D_R(h)$;
(v) $h = CD\text{-}\lim h_k$.

Note that $h_k|\Delta = (((h|\Delta) \cap k) \cup k^{-1}) - k^{-1}$. This implies (i) and (ii). For the proof of (iii) assume that $\{h_k\} \subset X_{BD}$. Fix k and choose $u_k \in PBD(R)$ such that $T_{BD} u_k = h_k$. Since $u_k|\Delta \setminus \Delta_P = 0$ and $\Pi u_k = h_k$, we have $h_k|\Delta \setminus \Delta_P = 0$. By (ii) we conclude that $h|\Delta \setminus \Delta_P = 0$. Conversely, assume that $h|\Delta \setminus \Delta_P = 0$. For any fixed k, we have $F_k \subset \Delta_P$ and hence by (i), $\operatorname{Supp}(h_k|\Delta) \subset \Delta_P$. Therefore we may consider $u_k = \Pi^P h_k$. By the maximum principle we conclude that $T_{BD} u_k = h_k$, and the proof of (iii) is complete. Clearly $D_R(\psi_k) \leqq D_R(h)$ and thus (iv) follows from the Dirichlet principle.

By comparing boundary values we see that $h_k \leqq h_{k+1} \leqq h$. Thus $\hat{h} = C\text{-}\lim h_k$ exists on R and $\hat{h} \leqq h$. By (iv) and Fatou's lemma we conclude that $\hat{h} \in HD(R)$. In view of $\hat{h}|\varDelta \geqq h_k|\varDelta$ and (ii) we see that $\hat{h} \geqq h$ on R. We conclude that $h = C\text{-}\lim h_k$. Since $h - h_k = \Pi(h - \psi_k + k^{-1})$, the Dirichlet principle implies that $D_R(h - h_k) \leqq D_R(h - \psi_k + k^{-1}) = D_{A_k}(h)$, where $A_k = \{p \in R \,|\, h(p) < k^{-1}$ or $h(p) > k\}$. This shows that also $h = D\text{-}\lim h_k$.

4. If $u \in PD^+(R)$, then in a natural way we may define a sequence $\{u_k\}$ called the *standard PBD-approximation to u*. In fact, set $h = T_R u$. Then $h|\varDelta\setminus\varDelta_P = u|\varDelta\setminus\varDelta_P = 0$ and hence Lemma 3 (iii) implies that $\{h_k\}$, the standard HBD-appoximation to h, is contained in X_{BD}^P. Set $u_k = T_{BD}^{-1} h_k$.

LEMMA. (i) $\operatorname{Supp}(u_k|\varDelta) \subset F_k \subset \varDelta_P$;
(ii) $\lim (u_k|\varDelta) = u|\varDelta$;
(iii) $D_R(u_k) \leqq D_R(u)$;
(iv) $u = CD\text{-}\lim u_k$.

The facts $h|\varDelta = u|\varDelta$, $h_k|\varDelta = u_k|\varDelta$ together with Lemma 3 (i) and 3 (ii) imply that (i) and (ii) hold. By comparing the boundary values we see that $u_k \leqq u_{k+1} \leqq u$. From (2) we see that $D_R(u) = D_R(h) + \langle u, u\rangle_R^P$ and $D_R(u_k) = D_R(h_k) + \langle u_k, u_k\rangle_R^P$. Thus (iii) follows from Lemma 3 (iii). By an argument analogous to that used in proving Lemma 3 (v) we see that $u = C\text{-}\lim u_k$. Again by (2) $D_R(u - u_k) = D_R(h - h_k) + \langle u - u_k, u - u_k\rangle_R^P$. By Lemma 3 (v) and the monotone convergence theorem we conclude that $u = D\text{-}\lim u_k$, which completes the proof.

It is worthwhile to point out here that although for $h \in HD^+(R)$ the assumption $h \in X_D^P$ implies $\{h_k\} \subset X_{BD}^P$, the converse is not true even if h is bounded. Indeed in [1] we constructed 2-forms P and Q on a Riemann surface T^∞ such that $\varDelta_P = \varDelta_Q$ yet there is a function $v \in QBD(T^\infty)$ such that $v|\varDelta \neq u|\varDelta$ for every $u \in PBD(T^\infty)$. Thus if we set $h = T_{BD}v$, then $h|\varDelta\setminus\varDelta_P = v|\varDelta\setminus\varDelta_Q = 0$, i.e., $\{h_k\} \subset X_{BD}^P$ but $h \notin X_{BD}^P$.

5. Consider the family $\mathscr{D}$ and the function δ defined in the beginning of this paper. For each $p^* \in \varDelta_P$ there is a function $f_{p^*} \in M(R)$ with $0 \leqq f_{p^*} \leqq 1$, $f_{p^*}(p^*) = 1$ and $\operatorname{Supp}(f_{p^*}|\varDelta) \subset \varDelta_P$. Thus we may consider $u_{p^*} = \Pi^P f_{p^*}$. Note that $u_{p^*} \in \mathscr{D}$ and hence $u_{p^*} \leqq \delta$. We conclude that $1 = \liminf_{p\to p^*} u_{p^*}(p) \leqq \liminf_{p\to p^*} \delta(p) \leqq \limsup_{p\to p^*} \delta(p) \leqq 1$. We extend the function δ to $\varDelta_P$ by setting $\delta|\varDelta_P = 1$. Then we have shown that δ is continuous on $R \cup \varDelta_P$.

It is easily seen that $\mathscr{D}$ is a Perron family with respect to $d*du = uP$. Clearly $0 \in \mathscr{D}$. If $u_1, u_2 \in \mathscr{D}$, then $u_1 \cup u_2$ is a nonnegative subsolution in $M(R)$. Thus $\Pi^P(u_1 \cup u_2)$ exists, is the least solution majorant of u_1 and u_2 and belongs to $\mathscr{D}$. Since $\mathscr{D}$ is a Perron family we have that $\delta \in PB^+(R)$ and that there is an increasing sequence $\{\tilde{\delta}_k\} \subset \mathscr{D}$ such that $\delta = B\text{-}\lim \tilde{\delta}_k$.

LEMMA. *Let $h \in HD^+(R)$. Under the assumption that $\{h_k\} \subset X_{BD}^P$ there exists a sequence $\{\delta_k\} \subset \mathscr{D}$ such that*

(i) $\delta_k | F_k = 1$;

(ii) $\operatorname{Supp}(\delta_k | \varDelta) \subset \varDelta_P$;

(iii) $\delta = B\text{-}\lim \delta_k$.

We shall call the sequence $\{\delta_k\}$ the *PBD-approximation to δ determined by h.* Although $\{\delta_k\} \subset PBD(R)$, δ need not be in $PBD(R)$. We begin the proof by replacing $\{\tilde{\delta}_k\}$ by a sequence $\{\hat{\delta}_k\} \subset \mathscr{D}$ with the property that $\operatorname{Supp}(\hat{\delta}_k | \varDelta) \subset \varDelta_P$ as well as $\delta = B\text{-}\lim \hat{\delta}_k$. To accomplish this we consider the standard PBD-approximation $\{\hat{\delta}_{kn}\}_{n=1}^{\infty}$ to $\tilde{\delta}_k$ and note that the diagonal sequence $\hat{\delta}_k = \hat{\delta}_{kk}$ has the required properties. Now consider the functions

$$g_k = (k^2 + k)[(h \cap k^{-1}) \cup (k+1)^{-1} - (k+1)^{-1}]\,, \qquad k = 1, 2, \cdots .$$

Clearly, $g_k \in M(R)$, $0 \leqq g_k \leqq 1$, $g_k | F_k = 1$ and since $\{h_k\} \subset X_{BD}^P$ we also have $\operatorname{Supp}(g_k | \varDelta) \subset F_{k+1} \subset \varDelta_P$. Since $\operatorname{Supp}((\hat{\delta}_k \cup g_k) | \varDelta) \subset \varDelta_P$, we may define $\delta_k = \Pi^P(\hat{\delta}_k \cup g_k)$. It is easily seen that $\delta_k \in \mathscr{D}$ and satisfies (i) and (ii). By the maximum principle $\hat{\delta}_k \leqq \delta_k$ and since $\delta = B\text{-}\lim \hat{\delta}_k$ we conclude that (iii) holds.

6. In [2] we characterized X_D^P as follows. If $h \in HD^+(R)$, then $h \in X_D^P$ if and only if $\{h_k\} \subset X_{BD}^P$ and $D_R(\delta_k h_k) = \mathscr{O}(1)$, where $\{h_k\}$ is the standard HBD-approximation to h and $\{\delta_k\}$ is the PBD-approximation to δ determined by h. The condition $D_R(\delta_k h_k) = \mathscr{O}(1)$ is difficult to verify in practice. By Fatou's lemma it implies that $D_R(\delta h) < +\infty$ and this gives the hope that $D_R(\delta_k h_k) = \mathscr{O}(1)$ and $D_R(\delta h) < +\infty$ are equivalent. On the other hand, Singer [6] showed that with a slightly different δ the two conditions are not equivalent. In spite of this doubt our main theorem shows that indeed the two conditions are equivalent. For the sake of completeness we present here the proof of the necessity of the condition of our main theorem.

Let $h \in HD^+(R)$ and assume that $h \in X_D^P$. Let $u \in PD(R)$ such that $T_D u = h$. Choose the standard HBD-approximation $\{h_k\}$ to h, the standard PBD-approximation $\{u_k\}$ to u and the PBD-approximation $\{\delta_k\}$ to δ

determined by h. The function $u_k(1-\delta_k)\in M^+(R)$ and hence by Lemmas 4 (i) and 5 (i) we have $u_k(1-\delta_k)|\Delta=0$. In view of the duality between Δ and $M_\Delta(R)$ (cf. [5]) we may choose a sequence $\{f_n\}\subset M_0^+(R)$ with $u_k(1-\delta_k)=BD\text{-}\lim f_n$. By this and Green's formula we obtain

$$(3)\quad \begin{aligned} D_R(u_k(1-\delta)) &= \lim_n D_R(f_n, u_k(1-\delta_k)) = \lim_n\left(-\int_R f_n d*d(u_k(1-\delta_k))\right)\\ &= \lim_n\left(-\int_R f_n u_k(1-\delta_k)P+\int_R f_n u_k\delta_k P+2\int_R f_n du_k\wedge *d\delta_k\right)\\ &\leqq -\liminf_n\int_R f_n u_k(1-\delta_k)P+\limsup_n\int_R f_n u_k\delta_k P\\ &\qquad +2\limsup_n\int_R f_n du_k\wedge *d\delta_k\ . \end{aligned}$$

In view of $u_k(1-\delta_k)\geqq 0$ and $f_n\geqq 0$, the first term on the right hand side of (3) is nonpositive. We estimate the second term:

$$(4)\quad \begin{aligned} \limsup_n\int_R f_n u_k\delta_k P &\leqq \limsup_n\int_R f_n u_k P = -\lim_n D_R(f_n, u_k)\\ &= -D_R(u_k(1-\delta_k), u_k)\ . \end{aligned}$$

By the Schwarz inequality $\int_R |du_k\wedge *d\delta_k|\leqq D_R^{1/2}(u_k)D_R^{1/2}(\delta_k)<+\infty$ and since $\{f_n\}$ is uniformly bounded, we conclude by the Lebesgue dominated convergence theorem that

$$(5)\quad \lim_n\int_R f_n du_k\wedge *d\delta_k=\int_R u_k(1-\delta_k)du_k\wedge *d\delta_k\ .$$

Substituting (4) and (5) into (3) and applying the Schwarz inequality repeatedly, we get

$$\begin{aligned} D_R(u_k(1-\delta_k)) &\leqq -D_R(u_k(1-\delta_k), u_k)+2\int_R u_k(1-\delta_k)du_k\wedge *d\delta_k\\ &= -D_R(u_k(1-\delta_k), u_k)-2\int_R(1-\delta_k)du_k\wedge *d(u_k(1-\delta_k))\\ &\qquad +2\int_R(1-\delta_k)^2 du_k\wedge *du_k\\ &\leqq 3D_R^{1/2}(u_k(1-\delta_k))D_R^{1/2}(u_k)+2D_R(u_k)\ . \end{aligned}$$

This implies that $D_R^{1/2}(u_k(1-\delta_k))\leqq 4D_R^{1/2}(u_k)$ and by the triangle inequality we obtain

$$(6)\quad D_R^{1/2}(\delta_k u_k)\leqq 5D_R^{1/2}(u_k)\ .$$

7. Set $\varphi_k=h_k-u_k$. In this section we give an estimate on $D_R(\delta_k\varphi_k)$ which together with (6) will give the desired bound on $D_R^{1/2}(\delta_k h_k)$. Note that $\varphi_k|\Delta = 0$ and $\varphi_k \geqq 0$. Thus $\delta_k\varphi_k \in M^+(R)$ and $\delta_k\varphi_k|\Delta = 0$. Consequently we may choose a sequence $\{f_n\} \subset M_0^+(R)$ with $\delta_k\varphi_k = BD\text{-}\lim f_n$. We estimate $D_R(\delta_k\varphi_k)$ as follows:

$$\begin{aligned}
D_R(\delta_k\varphi_k) &= \lim_n D_R(f_n, \delta_k\varphi_k) = \lim_n \left(-\int_R f_n d * d(\delta_k\varphi_k)\right) \\
&\leqq -\liminf_n \int_R f_n\delta_k\varphi_k P + \limsup_n \int_R f_n\delta_k u_k P \\
&\quad - 2\liminf_n \int_R f_n d\delta_k \wedge * d\varphi_k \leqq -D_R(\delta_k\varphi_k, u_k) - 2\int_R \delta_k\varphi_k d\delta_k \wedge * d\varphi_k \\
&= -D_R(\delta_k\varphi_k, u_k) - 2\int_R \delta_k d(\delta_k\varphi_k) \wedge * d\varphi_k + 2\int_R \delta_k^2 d\varphi_k \wedge * d\varphi_k \\
&\leqq D_R^{1/2}(\delta_k\varphi_k)D_R^{1/2}(u_k) + 2D_R^{1/2}(\delta_k\varphi_k)D_R^{1/2}(\varphi_k) + 2D_R(\varphi_k) .
\end{aligned}$$

In view of the Dirichlet principle, $D_R(\varphi_k) \leqq D_R(u_k)$ which implies that $D_R(\delta_k\varphi_k) \leqq 3D_R^{1/2}(\delta_k\varphi_k)D_R^{1/2}(u_k) + 2D_R(u_k)$. Hence,

$$D_R^{1/2}(\delta_k\varphi_k) \leqq 4D_R^{1/2}(u_k) .$$

From this and (6) we see that $D_R^{1/2}(\delta_k h_k) \leqq 9D_R^{1/2}(u_k)$ and by Lemma 4 (iii) we arrive at $D_R(\delta_k h_k) = \mathcal{O}(1)$. Finally by Fatou's lemma we conclude that $D_R(\delta h) < +\infty$. This establishes the necessity of our condition.

8. We shall establish the sufficiency in Sections 8 - 13. We begin with two simple inequalities. Assume Ω is an open subset of R and $\varphi, \psi \in M(\Omega)$. Then

$$\begin{aligned}
(7) \qquad D_\Omega(\varphi\psi) &= \int_\Omega \psi^2 d\varphi \wedge * d\varphi + 2\int_\Omega \varphi\psi d\varphi \wedge * d\psi + \int_\Omega \varphi^2 d\psi \wedge * d\psi \\
&\leqq 2\int_\Omega \psi^2 d\varphi \wedge * d\varphi + 2b^2 D_\Omega(\psi) ,
\end{aligned}$$

where $\sup_\Omega |\varphi| = b$. Also,

$$\begin{aligned}
(8) \qquad \int_\Omega \psi^2 d\varphi \wedge * d\varphi &\leqq D_\Omega(\varphi\psi) - 2\int_\Omega \varphi\psi d\varphi \wedge * d\psi \\
&= D_\Omega(\varphi\psi) - 2\int_\Omega \varphi d(\varphi\psi) \wedge * d\psi + 2\int_\Omega \varphi^2 d\psi \wedge * d\psi \\
&\leqq D_\Omega(\varphi\psi) + 2bD_\Omega^{1/2}(\varphi\psi)D_\Omega^{1/2}(\psi) + 2b^2 D_\Omega(\psi) .
\end{aligned}$$

We shall use (7) and (8) in case φ, ψ are merely continuous Tonelli functions on Ω. To see the validity of (7) and (8) in this case, note that $\varphi, \psi \in M(\Omega')$, where Ω' is a relatively compact open set in Ω. Apply (7) and (8) with Ω replaced by Ω'. Then let $\Omega' \to \Omega$ on the right hand sides and then on the left hand sides. Of course, the right hand sides or both sides may be $+\infty$. The application of (7) and (8) that we intend to make is in the case where φ is a bounded continuous Tonelli function on Ω and ψ is in $\tilde{M}(\Omega)$. In this case we see from (7) that $\int_\Omega \psi^2 d\varphi \wedge *d\varphi < +\infty$ implies that $D_\Omega(\varphi\psi) < +\infty$ and from (8) that $D_\Omega(\varphi\psi) < +\infty$ implies that $\int_\Omega \psi^2 d\varphi \wedge *d\varphi < +\infty$.

9. Let $h \in HD^+(R)$ and assume that $\{h_k\} \subset X^P_{BD}$ and $D_R(\delta h) < +\infty$. By Sard's theorem we may choose an $\alpha \in (0, 1)$ such that $W = \{p \in R \mid \delta(p) > \alpha\}$ has a C^1 relative boundary. Let δ^* be the lower semicontinuous extension of δ to R^*. Then $W^* = \{p^* \in R^* \mid \delta^*(p^*) > \alpha\}$ is open in R^* and since δ is continuous on $R \cup \Delta_P$ with $\delta \mid \Delta_P = 1$ we have $\Delta_P \subset W^*$. Since $W^* \cap R = W$, the denseness of $W^* \cap R$ in W^* gives $\Delta_P \subset W^* \subset \bar{W}$.

Set $w = (1-\alpha)^{-1}(\delta - \alpha) \cup 0$ and note that the hypotheses of Lemma 2 with Δ_P playing the role of $\mathcal{O}$ are met. Thus there is a function $v \in HD(W; \partial W)$ such that $\mu_D v = h$. The proof will be complete when we demonstrate a function $u \in PD(R)$ with $u \mid \Delta = v \mid \Delta$.

Note that by (8) we have

$$(9) \qquad \int_R h^2 d\delta \wedge *d\delta < +\infty$$

and in view of $0 \leqq v \leqq h$ this implies that

$$(10) \qquad \int_W v^2 d\delta \wedge *d\delta < +\infty \,.$$

By (7) we conclude that

$$(11) \qquad D_W(\delta v) < +\infty \,.$$

10. Set $r = T_{B,W}\delta$; i.e.,

$$(12) \qquad r = \delta + \frac{1}{2\pi}\int_W g_W(\cdot, \zeta)\delta(\zeta)P(\zeta) \,.$$

Let $\{W_n\}$ be a regular exhaustion of W; specifically, $W_n \subset \bar{W}_n \subset W_{n+1} \subset W$, $\bar{W}_n$ is compact, $W = \bigcup_{n=1}^\infty W_n$ and ∂W_n consists of analytic curves. Define a sequence $\{r_n\}$ of functions on W by $r_n \mid W \setminus W_n = \delta$ and $r_n \mid W_n = T_{B,W_n}\delta$, i.e.,

$$r_n \mid W_n = \delta \mid W_n + \frac{1}{2\pi}\int_{W_n} g_{W_n}(\cdot, \zeta)\delta(\zeta)P(\zeta) \,.$$

The following can easily be verified: r_n is a continuous Tonelli function on W; $r_n \mid W_n$ is harmonic; $\delta \leqq r_n \leqq r_{n+1} \leqq r$ and $r = B\text{-}\lim r_n$ on W. We further claim that

$$D_{W_n}(r_n v) = \mathcal{O}(1)\ . \tag{13}$$

Using Green's formula, we get

$$\begin{aligned} D_{W_n}(r_n v) &= \int_{\partial W_n} r_n v * d(r_n v) - 2\int_{W_n} r_n v dr_n \wedge * dv \\ &= \int_{\partial W_n} \delta v * d(r_n v) - 2\int_{W_n} r_n d(r_n v) \wedge * dv + 2\int_{W_n} r_n^2 dv \wedge * dv \end{aligned} \tag{14}$$

and by another application we obtain

$$\begin{aligned} \int_{\partial W_n} \delta v * d(r_n v) &= \int_{W_n} d(\delta v) \wedge * d(r_n v) + 2\int_{W_n} \delta v dr_n \wedge * dv \\ &= \int_{W_n} d(\delta v) \wedge * d(r_n v) + 2\int_{W_n} \delta d(r_n v) \wedge * dv \\ &\quad - 2\int_{W_n} \delta r_n dv \wedge * dv\ . \end{aligned} \tag{15}$$

We substitute (15) into (14) and apply the Schwarz inequality to obtain

$$D_{W_n}(r_n v) \leqq D_{W_n}^{1/2}(r_n v)(D_W^{1/2}(\delta v) + 4D_W^{1/2}(v)) + 2D_W(v)\ .$$

In view of (11) we conclude that (13) holds.

11. In this section we establish

$$\int_{W_n} v^2(r_n - \delta)\delta P = \mathcal{O}(1)\ . \tag{16}$$

We begin by applying Green's formula:

$$\begin{aligned} \int_{W_n} v^2(r_n - \delta)\delta P &= -D_{W_n}(v^2(r_n - \delta), \delta) \\ &= -2\int_{W_n} v(r_n - \delta) dv \wedge * d\delta - \int_{W_n} v^2 d(r_n - \delta) \wedge * d\delta\ . \end{aligned} \tag{17}$$

By the Schwarz inequality we obtain

$$\begin{aligned} \left|\int_{W_n} v(r_n - \delta) dv \wedge * d\delta\right| &\leqq \int_{W_n} v\,|dv \wedge * d\delta| \\ &\leqq D_W^{1/2}(v)\Big(\int_W v^2 d\delta \wedge * d\delta\Big)^{1/2}, \end{aligned} \tag{18}$$

as well as,

$$\begin{aligned} \left|\int_{W_n} v^2 d(r_n - \delta) \wedge * d\delta\right| &\leqq \int_{W_n} v^2\,|dr_n \wedge * d\delta| + \int_{W_n} v^2 d\delta \wedge * d\delta \\ &\leqq \Big(\int_{W_n} v^2 dr_n \wedge * dr_n\Big)^{1/2}\Big(\int_W v^2 d\delta \wedge * d\delta\Big)^{1/2} + \int_W v^2 d\delta \wedge * d\delta\ . \end{aligned} \tag{19}$$

We apply (8):

$$\int_{W_n} v^2 dr_n \wedge *dr_n \leqq D_{W_n}(r_n v) + 2D_{W_n}^{1/2}(r_n v) D_W^{1/2}(v) + 2D_W(v) .$$

This in view of (13) implies that $\int_{W_n} v^2 dr_n \wedge *dr_n = \mathcal{O}(1)$. Substituting this into (19) and then combining (18) and (19) with (17), we get (16).

12. From (16) and the monotone convergence theorem we deduce that

$$\int_W v^2(r - \delta)\delta P < +\infty .$$

We substitute the expression for $r - \delta$ from (12) into this and apply Fubini's theorem to obtain

$$\int_{W\times W} v^2(z) g_W(z, \zeta)\delta(z)\delta(\zeta)P(z)P(\zeta) < +\infty .$$

By the Schwarz inequality we see that

$$\langle \delta v, \delta v\rangle_W^P < +\infty .$$

Since $\delta | W > \alpha > 0$, we conclude that

(20) $$\langle v, v\rangle_W^P < +\infty .$$

13. We arrive at the final stage of the proof of our theorem. Let $\{R_n\}$ be an exhaustion of R by regular regions. Let $s_n \in \tilde{M}(R)$ such that $s_n | R \setminus (R_n \cap W) = v$ and $d * ds_n = s_n P$ on $R_n \cap W$. Then $0 \leqq s_n \leqq v$ and hence $s_{n+1} \leqq s_n$. By the Harnack principle $s = C\text{-}\lim s_n$ exists on W. Since $v | R \setminus W = 0$, it is easily seen that actually $s = C\text{-}\lim s_n$ on R and $s | R \setminus W = 0$. We estimate $D_W(s_n)$ using (2) and that the fact that $s_n \leqq v$:

$$\begin{aligned} D_W(s_n) &= D_{R_n \cap W}(s_n) + D_{W\setminus (R_n \cap W)}(v) \\ &= D_W(v) + \langle s_n, s_n\rangle_{W\cap R_n}^P \leqq D_W(v) + \langle v, v\rangle_W^P . \end{aligned}$$

In view of (20) and Fatou's lemma we obtain $D_W(s) < +\infty$, i.e., $s \in PD(W; \partial W)$.

We shall now show that also $s = D\text{-}\lim s_n$. To this end note that

$$D_{W\cap R_n}(s_n - s, s_n) = -\int_{W\cap R_n} (s_n - s)s_n P \leqq 0 .$$

Consequently,

$$0 \leqq D_{W\cap R_n}(s - s_n) \leqq D_{W\cap R_n}(s) - D_{W\cap R_n}(s_n) .$$

Thus by Fatou's lemma we arrive at

(21) $$\begin{aligned} 0 \leqq \limsup D_{W\cap R_n}(s - s_n) &\leqq D_W(s) - \liminf D_{W\cap R_n}(s_n) \\ &\leqq D_W(s) - D_W(s) = 0 . \end{aligned}$$

Furthermore,

$$0 \leqq \liminf D_W(s - s_n) \leqq \limsup D_W(s - s_n)$$
$$\leqq \lim D_{W \setminus (W \cap R_n)}(s - v) + \limsup D_{W \cap R_n}(s - s_n) .$$

The first term on the right is 0 because $D_W(s - v) < +\infty$ and by (21) also the second term is 0. We have established $s = CD\text{-}\lim s_n$.

Note that also $v - s = CD\text{-}\lim (v - s_n)$ and $v - s_n \in M_0(R)$. Thus $v - s|\Delta = 0$. The function s is a nonnegative subsolution in $\widetilde{M}(R)$ and hence $u = \Pi^P s$ exists. We have established that $u|\Delta = s|\Delta = v|\Delta = h|\Delta$ and the proof of the sufficiency is complete.

REFERENCES

[1] M. GLASNEAR AND M. NAKAI, The roles of nondensity points, Duke Math. J. 43 (1976), 579–595.

[2] M. GLASNER AND M. NAKAI, Images of reduction operators, Arch. Rational Mech. Anal. 76 (1980).

[3] M. NAKAI, Order comparisons on canonical isomorphisms, Nagoya Math. J. 50 (1973), 67–87.

[4] M. NAKAI, Extremizations and Dirichlet integrals on Riemann surfaces, J. Math. Soc. Japan 28 (1976), 581–603.

[5] L. SARIO AND M. NAKAI, Classification theory of Riemann surfaces, Springer-Verlag, 1970.

[6] I. SINGER, Dirichlet finite solutions of $\Delta u = Pu$, Proc. Amer. Math. Soc. 32 (1972), 464–468.

[7] I. SINGER, Boundary isomorphism between Dirichlet finite solutions of $\Delta u = Pu$ and harmonic functions, Nagoya Math. J. 50 (1973), 7–20.

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