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THE FIRST EIGENVALUE OF THE LAPLACIAN ON EVEN DIMENSIONAL SPHERES

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1. Introduction. Let (M^n, g) be an n-dimensional compact connected Riemannian manifold. The Laplacian acting on smooth functions on M has a discrete spectrum with finite multiplicities. Hersch [6] showed that for any Riemannian metric g on the two dimensional sphere S^2,

$$\lambda_1(g)\ \mathrm{vol}\,(S^2, g) \leqq 8\pi$$

where $\lambda_1(g)$ denotes the first eigenvalue of the Laplacian with respect to g. The equality holds if and only if g is the canonical metric (up to a constant multiple).

This implies an affirmative answer to the Blaschke conjecture on S^2 and gives another proof of Green's theorem [5] (cf. [3]). In connection with this result, Berger [1] posed a problem: Does there exist a constant $k(M)$ satisfying

$$\lambda_1(g)\mathrm{vol}(M^n, g)^{2/n} \leqq k(M)$$

for any Riemannian metric g on M? When M is a sphere, can one characterize the canonical metric up to a constant multiple by the above equality?

If this problem is affirmatively answered for an n-dimensional sphere S^n, the Blaschke conjecture is affirmatively answered for S^n (cf. [3]). And it is interesting to know some relations between the spectrum theory and differential geometry. It is known (cf [1], [9]) that the answer to this problem is affirmative when M is a flat torus. But Urakawa [8] gave a counterexample when M is a compact Lie group with the nontrivial commutator subgroup, in particular, S^3. Tanno [7] also answered the problem negatively when M is $S^{2n+1} (n \geqq 1)$. Urakawa and Muto [10] showed that there are many counterexamples when M has Euler number zero.

In this paper, we give a negative answer also when M is S^{2n} $(n\geqq 2)$.

THEOREM. *There exists a continuous deformation* g_t $(0 \leqq t < \infty)$ *of the canonical metric* g_0 *on* S^{2n} $(n \geqq 2)$ *such that*

$$\lambda_1(g_t)\mathrm{vol}(S^{2n}, g_t)^{1/n} \to \infty \quad (t \to \infty)\ .$$

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2. Construction of the deformation g_t. Let $(u^0, u^1, \cdots, u^{2n})$ be the canonical coordinate system on $\boldsymbol{R}^{2n+1}$, $N=(1, 0, \cdots, 0)$ and $S=(-1, 0, \cdots, 0)$ $(n \geqq 2)$. Let S^{2n} be the unit sphere in $\boldsymbol{R}^{2n+1}$ and $g_0(2n)$ the canonical metric on S^{2n} induced by the Euclidean structure on $\boldsymbol{R}^{2n+1}$. Let $S^{2n-1}=\{(0, u^1, \cdots, u^{2n}) \in S^{2n}\}$. Let (r, x), $r \in (0, \pi)$, $x \in S^{2n-1}$, be a geodesic polar coordinate system around N on $S^{2n}-\{N, S\}$ with respect to $g_0(2n)$, that is, $x=(x^1, \cdots, x^{2n-1})$ is a local coordinate on S^{2n-1} and r is the distance from the north pole N. Let $g_0(2n-1)$ be the metric on S^{2n-1} induced by $g_0(2n)$. Then its metric on S^{2n-1} has constant curvature 1. Let η be a contact form on S^{2n-1}, that is, η is a unit Killing form on $(S^{2n-1}, g_0(2n-1))$. Then there exists a 1-form $\tilde{\eta}$ on $(S^{2n}, g_0(2n))$ such that

$$\tilde{\eta}_{(r,x)} = (\sin r)^2 \eta_x \quad \text{on} \quad S^{2n}-\{N, S\}\,,$$
$$\tilde{\eta}_N = 0\,, \quad \text{and} \quad \tilde{\eta}_S = 0\,.$$

Here we regard η_x as a covector at (r, x) in S^{2n} via the geodesic polar coordinate.

DEFINITION 2.1. We define a deformation $g_t(2n)$ $(0 \leqq t < \infty)$ of $g_0(2n)$ as follows:

$$g_t(2n) = g_0(2n) + t\tilde{\eta} \otimes \tilde{\eta}\,, \quad (0 \leqq t < \infty)\,. \tag{2.1}$$

In particular, on $S^{2n}-\{N, S\}$,

$$g_t(2n) = (dr)^2 + (\sin r)^2 (g_0(2n-1) + t(\sin r)^2 \eta \otimes \eta)\,.$$

We notice here that $g_0(2n-1) = \eta \otimes \eta + \pi^* h(n-1)$, where π is the Hopf fibering $S^{2n-1} \to \boldsymbol{CP}^{n-1}$ and $h(n-1)$ is the canonical metric on $\boldsymbol{CP}^{n-1}$. Therefore on $S^{2n}-\{N, S\}$, we have

$$\{\det g_t(2n)\}_{(r,x)} = (1+t(\sin r)^2)\{\det g_0(2n)\}_{(r,x)}\,, \tag{2.2}$$

where we denote by $g_t(2n)$ the coefficient matrix of $g_t(2n)$ with respect to the coordinate (r, x) for any $t \in [0, \infty)$. Let $\xi=(\xi^i)$ be the dual vector field of η on $(S^{2n-1}, g_0(2n-1))$. Then ξ is a unit Killing vector field on S^{2n-1}. Therefore the inverse matrix $g_t(2n)^{-1}$ of $g_t(2n)$ with respect to the coordinate (r, x) is of the following form on $S^{2n}-\{N, S\}$:

$$g_t(2n)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (\sin r)^{-2} g_0^{jk}(2n-1) - t(1+t(\sin r)^2)^{-1}\xi^j \xi^k \end{pmatrix}. \tag{2.3}$$

LEMMA 2.2. *Let ${}^{(t)}\Delta_{S^{2n}}$ be the Laplacian on S^{2n} defined by $g_t(2n)$ and $\Delta_{S^{2n-1}}$ the Laplacian on S^{2n-1} defined by $g_0(2n-1)$. Then, on $S^{2n}-\{N, S\}$,*

$$
\begin{aligned}
{}^{(t)}\Delta_{S^{2n}} = (\partial^2/\partial r^2) &+ [(2n-1)(\cos r)(\sin r)^{-1} \\
&+ t(\sin r)(\cos r)\{1 + t(\sin r)^2\}^{-1}](\partial/\partial r) \\
&+ (\sin r)^{-2}\Delta_{S^{2n-1}} - t\{1 + t(\sin r)^2\}^{-1}\mathscr{L}_\xi\mathscr{L}_\xi ,
\end{aligned}
$$

where ξ is a unit Killing vector field on S^{2n-1} and $\mathscr{L}_\xi$ is the Lie derivation with respect to ξ.

PROOF. We denote the geodesic polar coordinate $(r, x^{-1}, \cdots, x^{2n-1})$by $(v^1, \cdots, v^{2n})$ and set $\theta = (\det g_t(2n))^{1/2}$ with respect to $(v^1, \cdots, v^{2n})$. Then

$$
{}^{(t)}\Delta_{S^{2n}} = \theta^{-1}(\partial/\partial v^j)(\theta g_t^{jk}(2n)(\partial/\partial v^k)) .
$$

Therefore by (2.2) and (2.3), we have

$$
\begin{aligned}
(2.4)\quad {}^{(t)}\Delta_{S^{2n}} = (\partial^2/\partial r^2) &+ [(2n-1)(\cos r)(\sin r)^{-1} \\
&+ t(\sin r)(\cos r)\{1 + t(\sin r)^2\}^{-1}](\partial/\partial r) + (\sin r)^{-2}\Delta_{S^{2n-1}} \\
&- t\{1 + t(\sin r)^2\}^{-1}(\det g_0(2n-1))^{-1/2} \\
&\quad \times(\partial/\partial x^i)\{(\det g_0(2n-1))^{1/2}\xi^i\xi^j(\partial/\partial x^j)\} .
\end{aligned}
$$

As η is a coclosed form on $(S^{2n-1}, g_0(2n-1))$, we have $0 = -\delta\eta = \Gamma^k_{ki}\xi^i + (\partial\xi^i/\partial x^i)$, where δ is the co-differentiation of $(S^{2n-1}, g_0(2n-1))$ and Γ^i_{jk} is the Christoffel's symbol on $(S^{2n-1}, g_0(2n-1))$. Therefore the last term on the right hand side of (2.4) coincides with $-t(1+t(\sin r)^2)^{-1}\mathscr{L}_\xi\mathscr{L}_\xi$. q.e.d.

3. The estimate of the first eigenvalue. We first consider the eigenfunctions of Δ_{S^m}. Let λ_k be the k-th eigenvalue of Δ_{S^m} and V_k be the vector space of eigenfunctions corresponding to λ_k. Then on $(S^m, g_0(m))$ (cf [2]),

$$
\begin{aligned}
&\lambda_k = k(k + m - 1) , \quad k \geqq 0 , \\
&\dim V_k = {}_{m+k}C_k - {}_{m+k-2}C_{k-2} , \quad k \geqq 2 , \\
&\dim V_0 = 1 , \quad \dim V_1 = m + 1 .
\end{aligned}
$$

As ξ is a unit Killing vector field on S^{2n-1} $(n \geqq 2)$, $\mathscr{L}_\xi$ commutes with $\Delta_{S^{2n-1}}$ and induces a linear endomorphism on V_k. We define an inner product $\langle\ ,\ \rangle$ on smooth functions on S^m as follows:

$$
\langle f, g\rangle = \int_{S^m} fg\,d\mathrm{vol}\,(S^m, g_0(m)) ,
$$

for any $f, g \in C^\infty(S^m)$, where $d\mathrm{vol}(S^m, g_0(m))$ is the volume element with respect to $g_0(m)$. By Stokes' theorem, $\mathscr{L}_\xi$ induces a skew-symmetric linear endomorphism on V_k with respect to the above inner product. Tanno [7] gave a decomposition of V_k with respect to the action of $\mathscr{L}_\xi\mathscr{L}_\xi$.

LEMMA 3.1 (Tanno [7]). *On* $(S^{2n-1}, g_0(2n-1))$, $(n \geqq 2)$, *we have*

$$V_k = V_{k,0} + V_{k,1} + \cdots + V_{k,[k/2]} ,$$

for any integer $k \geqq 0$, *where* $[k/2]$ *is the integer part of* $k/2$, *and for any* $f \in V_{k,p}$, $0 \leqq p \leqq [k/2]$, $\mathscr{L}_\xi \mathscr{L}_\xi f + (k-2p)^2 f = 0$.

Now let f be a non-zero eigenfunction of ${}^{(t)}\Delta_{S^{2n}}$ corresponding to λ. Then we can regard f as $f(r, x) \in C^\infty((0, \pi) \times S^{2n-1})$. Let $\{\varphi^i_{k,p}(k \geqq 0,\ 0 \leqq p \leqq [k/2],\ 1 \leqq i \leqq \dim V_{k,p})\}$ be a complete orthonormal basis on the space of square integrable functions on S^{2n-1} with respect to $g_0(2n-1)$,where $\varphi^i_{k\ p} \in V_{k,p}$. We set

$$a^i_{k,p}(r) = \int_{S^{2n-1}} f(r, x)\varphi^i_{k,p}(x) d\mathrm{vol}(S^{2n-1}, g_0(2n-1)) .$$

Then $a^i_{k,p} \in C^2([0, \pi])$. Note that there exist some k, p, i such that $a^i_{k,p} \not\equiv 0$.

Now as $\Delta_{S^{2n-1}}$ and $\mathscr{L}_\xi \mathscr{L}_\xi$ are self-adjoint with respect to $\langle\ ,\ \rangle$, $a^i_{k,p}(r)$ must satisfy the following equation:

$$\begin{aligned}(3.1)\quad & [(d^2/dr^2) + [(2n-1)(\cos r)(\sin r)^{-1} \\ & + t(\sin r)(\cos r)\{1 + t(\sin r)^2\}^{-1}](d/dr) + [\lambda - k(k+2n-2)(\sin r)^{-2} \\ & + t(k-2p)^2\{1 + t(\sin r)^2\}^{-1}]\varphi = 0 , \quad \text{on} \quad (0, \pi) .\end{aligned}$$

LEMMA 3.2. *When* $\lambda < 2n-2$ *and* $k \geqq 1$, (3.1) *has no nontrivial solution in* $C^2([0, \pi])$ *for any* p, $0 \leqq p \leqq [k/2]$, *and* $t \geqq 0$.

PROOF. By $\lambda < 2n-2$ and $k \geqq 1$, we see that on $(0, \pi)$,

$$\lambda - k(k+2n-2)(\sin r)^{-2} + t(k-2p)^2\{1 + t(\sin r)^2\} < 0 .$$

Let $\varphi \in C^2([0, \pi])$ be a solution of (3.1). Multiply both sides of (3.1) by $(\sin r)^2$ and take the limits as $r \to 0$ and $r \to \pi$. Then $\varphi(0) = \varphi(\pi) = 0$. Therefore by Rolle's theorem, there exists $r_0 \in (0, \pi)$ such that $(d\varphi/dr)(r_0) = 0$. For any $r_0 \in (0, \pi)$ satisfying $(d\varphi/dr)(r_0) = 0$, we have

$$(d^2\varphi/dr^2)(r_0) = -[\lambda - k(k+2n-2)(\sin r_0)^{-2} + t(k-2p)^2\{1 + t(\sin r_0)^2\}^{-1}]\varphi(r_0) .$$

If we assume φ is a non-trivial solution, then by the uniqueness of a solution for an initial condition, $\varphi(r_0) \neq 0$. So $(d^2\varphi/dr^2)(r_0) > 0$ if $\varphi(r_0) > 0$ and $(d^2\varphi/dr^2)(r_0) < 0$ if $\varphi(r_0) < 0$. This contradicts the fact $\varphi(0) = \varphi(\pi) = 0$. q.e.d.

Next we consider the case of $k = 0$ in (3.1). Set $z = \cos r$. If $y(\cos r)$ is a solution of (3.1), then the function $y(z)$ must be in $C^2(-1, 1)$ and satisfy the following equation (3.1′):

$$(3.1')\quad (1 - z^2)y'' - [2n + t(1 - z^2)\{1 + t(1 - z^2)\}^{-1}]zy' + \lambda y = 0 \quad \text{on} \quad (-1, 1) ,$$

where $y'(z)$ (resp. $y''(z)$) denotes $(dy/dz)(z)$ (resp. $(d^2y/dz^2)(z)$). Set $y(z) = \sum_{j\geqq 0} a_j z^j$ formally. Then we obtain $2a_2 = -\lambda a_0$, $6(1+t)a_3 = \{(2n-\lambda) + (2n+1-\lambda)t\}a_1$ and

$$(3.2)\quad (1+t)((j+2)(j+1)a_{j+2} - t\{(j+2)^2 + (2n-4)(j+2) - 2(2n-2) - \lambda\}a_j$$
$$= (1+t)j(j-1)a_j - t\{j^2 + (2n-4)j - 2(2n-2) - \lambda\}a_{j-2}$$
$$+ (2nj-\lambda)a_j\,, \quad j \geqq 2\,.$$

The function y is well-defined by (3.2), that is, $\sum_{j\geqq 0} a_j z^j$ is absolutely convergent on $(-1, 1)$. It is classical that (3.1) is equivalent to (3.1$'$). By (3.2), we can choose $y_1 = \sum_{j\geqq 0} a_{2j} z^{2j}$ and $y_2 = \sum_{j\geqq 1} a_{2j-1} z^{2j-1}$ as a fundamental system of (3.1$'$).

LEMMA 3.3. *Let* $a_0 = -1$ *and* $a_1 = 1$. *Then* $a_j > 0$ $(j \geqq 1)$ *if* $0 < \lambda < 2n$.

PROOF. We first consider a_{2j}. By $a_0 = -1$ and $a_2 = \lambda/2$, we have $12(1+t)a_4 - t\{4^2 + 4(2n-4) - 2(2n-2) - \lambda\}a_2 = 2a_2 + (4n-\lambda)a_2 > 0$. Therefore $a_4 > 0$. We assume $a_j > 0$ for any even integer j, $4 \leqq j \leqq m$ for some even integer m. Set $b_j = (1+t)(j+2)(j+1)a_{j+2} - t\{(j+2)^2 + (2n-4)(j+2) - 2(2n-2) - \lambda\}a_j$. Then by (3.2), $b_j = b_{j-2} + (2nj-\lambda)a_j$. By our assumption, $b_m = b_{m-2} + (2nm-\lambda)a_m > b_{m-2} > \cdots > b_2 > 0$. Thus $a_{m+2} > 0$.

Next we consider a_{2j-1}. By $a_1 = 1$ and $a_3 = 6^{-1}[\{2n + (2n+1)t\}(1+t)^{-1} - \lambda] > 0$, we have $b_3 = (2n-\lambda) + (6n-\lambda)a_3 > 0$. In the same way as in the case of a_{2j}, we obtain $a_j > 0$ for any odd integer $j > 0$. q.e.d.

LEMMA 3.4. *When* $0 < \lambda < n$, (3.1$'$) *has no nontrivial bounded solution in* $C^2(-1, 1)$ *for any* $t \geqq 0$.

PROOF. We first consider y_1. By (3.2),

$$a_{2j+2} = t(1+t)^{-1}\{(2j+2)^2 + (2n-4)(2j+2) - 2(2n-2) - \lambda\}$$
$$\times\{(2j+2)(2j+1)\}^{-1}a_{2j} + \{(1+t)(2j+2)(2j+1)\}^{-1}$$
$$\times\left\{2a_2 + \sum_{i=1}^{j}(4ni-\lambda)a_{2i}\right\}\,.$$

When $0 < \lambda < n$ and $1 \leqq i \leqq j$, we have $4ni - \lambda > 3ni$. When $0 < \lambda < n$, $n \geqq 2$ and $j \geqq 3$, we have

$$\{(2j+2)^2 + (2n-4)(2j+2) - 2(2n-2) - \lambda\}\{(2j+2)(2j+1)\}^{-1}$$
$$= (2j/2j+2)[1 + (4nj - 2j - \lambda)\{2j(2j+1)\}^{-1}] > (2j/2j+2)\,.$$

By Lemma 3.3, we have $a_j > 0$ $(j \geqq 1)$ when $0 < \lambda < n$, $a_0 = -1$ and $a_1 = 1$. Thus there exists a positive constant K such that $a_2 > (K/2)a_2$, $a_4 > (K/4)a_2$ and $a_6 > (K/6)a_2$. We assume $a_{2j} > (K/2j)a_2$ for $3 \leqq j \leqq m$.

Then as $n \geqq 2$ and $m \geqq 3$, we have

$$a_{2m+2} > (t/1+t)(2m/2m+2)(K/2m)a_2 + \{(1+t)(2m+2)(2m+1)\}^{-1} \times \sum_{j=1}^{m}(3nj/2j)Ka_2$$
$$> (t/1+t)(K/2m+2)a_2 + (1/1+t)(K/2m+2)a_2 = (K/2m+2)a_2 .$$

Therefore $y_1(z) > -1 + (K/2)a_2\{\log(1-z^2)^{-1}\}$, when $z \neq 0$. Thus $y_1(z)$ is unbounded on $(-1, 1)$. Similarly we can show that $y_2(z)$ is unbounded on $(-1, 1)$. Since $\{y_1, y_2\}$ give a fundamental system of (3.1′), we obtain the desired result. q.e.d.

THEOREM 3.5. *There exists a continuous deformation g_t $(0 \leqq t < \infty)$ of the canonical metric g_0 on S^{2n} $(n \geqq 2)$ such that*

$$\lambda_1(g_t)\mathrm{vol}(S^{2n}, g_t)^{1/n} \to \infty \quad (t \to \infty) .$$

PROOF. Set $g_t = g_t(2n)$. Then Lemmas 3.2 and 3.4 imply $\lambda_1(g_t) \geqq n$ for any $t \geqq 0$. By (2.2), we have

$$\mathrm{vol}(S^{2n}, g_t) = \mathrm{vol}(S^{2n-1}, g_0(2n-1))\int_0^{\pi}(1+t(\sin r)^2)^{1/2}(\sin r)^{2n-1}dr$$
$$\to \infty \quad (t \to \infty) .$$

q.e.d.

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