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CONSTRUCTIBILITY AND GEOMETRIC FINITENESS OF KLEINIAN GROUPS

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1. Preliminaries. Let $g = (a, b; c, d)$ $(ad - bc = 1)$ denote a Moebius transformation acting on $\bar{H} = \{(z, t); z \in \boldsymbol{C}, t \geqq 0\} \cup \{\infty\}$ in the manner

$$g: (z, t) \mapsto \left(\frac{ac(|z|^2 + t^2) + a\bar{d}z + b\bar{c}\bar{z} + b\bar{d}}{|cz + d|^2 + c^2t^2}, \frac{t}{|cz + d|^2 + |c|^2t^2}\right).$$

Obviously g keeps the upper half space $H = \{(z, t); z \in \boldsymbol{C}, t > 0\}$ and the boundary $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$ of H invariant, that is, $\{g(\zeta); \zeta \in H\} = H$ and $\{g(\zeta); \zeta \in \hat{\boldsymbol{C}}\} = \hat{\boldsymbol{C}}$. Let SL' be the group of all such Moebius transformations. A Kleinian group G is a subgroup of SL' which acts discontinuously at some point of $\hat{\boldsymbol{C}}$. A Kleinian group acts discontinuously in H. The set of all points in $\hat{C}$ at which G acts discontinuously is denoted by $\Omega(G)$, and $\Lambda(G) = \hat{\boldsymbol{C}} - \Omega(G)$ is called the limit set of G.

An elementary group G is a Kleinian group such that $\Lambda(G)$ is a finite set. A finitely generated Kleinian group G is quasi-Fuchsian, if $\Omega(G)$ consists of two invariant components. A finitely generated Kleinian group G is a web group, if each component of $\Omega(G)$ is the image of an open disc under a quasi-conformal automorphism of $\hat{\boldsymbol{C}}$. Quasi-Fuchsian groups give the simplest examples of web groups. A finitely generated Kleinian group is totally degenerate if $\Omega(G)$ is connected and simply connected. A finitely generated Kleinian group with an invariant component is a function group. For a set S in $\bar{H}$, we denote by $\bar{S}$ the closure of S in $\bar{H}$. The points ζ and ζ' in $\bar{H}$ are equivalent under G if there exists an element $g \in G$ which transforms ζ into ζ'. Let S be an invariant subset of $\bar{H} - \Lambda(G)$ under G. A subset D of S is a fundamental region for G in S, if no pair of distinct points in D are equivalent under G and if $G\bar{D} = \{g(\zeta); g \in G, \zeta \in \bar{D}\}$ covers S.

A set S in H is convex, if any two points in S can be joined by a part of a line or a circle in S orthogonal to $\boldsymbol{C}$. A Kleinian group G is geometrically finite if there is a convex fundamental region for G in H surrounded by a finite number of hyperbolic planes, that is, hemispheres or half planes orthogonal to $\boldsymbol{C}$.

Beardon-Maskit [4] proved that any convex fundamental region for a geometrically finite group is surrounded by a finite number of hyperbolic planes. It is known that elementary groups and quasi-Fuchsian groups are geometrically finite and it is also known that totally degenerate groups are not (Greenberg [7]). In his paper [16], Maskit showed the existence of geometrically finite web groups and that of geometrically infinite web groups.

For $g = (a, b; c, d) \in \mathrm{SL}'$, $c \neq 0$, we define the isometric sphere $I(g)$ of g as $\{(z, t) \in \bar{H}; |z - p(g)|^2 + t^2 = r(g)^2\}$, where $p(g) = g^{-1}(\infty)$ and $r(g) = |c|^{-1}$. As is well known, g can be decomposed into a product $g = u_3 \circ u_2 \circ u_1$ of motions, where u_1 is the inversion in $I(g)$, u_2 is the reflection about the plane $\{\zeta \in \bar{H}; |\zeta - p(g)| = |\zeta - p(g^{-1})|\}$ and u_3 is the rotation with the line $\{(z, t) \in \bar{H}; z = p(g^{-1})\}$. Since u_2 and u_3 are Euclidean motions, the Jacobian $J_g(\zeta)$ of g at the point $\zeta \in \bar{H}$ is more than, equal to or less than 1, respectively, if ζ is in the bounded component $\operatorname{int} I(g)$ of $\bar{H} - I(g)$, on $I(g)$ or in the unbounded component $\operatorname{ext} I(g)$ of $\bar{H} - I(g)$.

Let H be a subgroup of a Kleinian group G. A set S in $\hat{C}$ is precisely invariant under H in G if $HS = S$ and if $(G - H)S \cap S = \emptyset$. For a cyclic subgroup H, a precisely invariant disc B for H is a Jordan subdomain of $\hat{C}$ such that $\bar{B} - \Lambda(H)$ is precisely invariant under H, and $(\bar{B} - \Lambda(H)) \subset \Omega(G)$. For later use we state Maskit's combination theorems in the following form.

COMBINATION THEOREM I (Maskit [9], [10]). *For $i = 1, 2$, let B_i be a precisely invariant disc for H, a finite or parabolic cyclic subgroup of both G_1 and G_2. Assume that B_1 and B_2 have the common boundary γ and $B_1 \cap B_2 = \emptyset$. Let G be the group generated by G_1 and G_2. Then*

(1) *G is Kleinian and $(\gamma - \Lambda(H)) \subset \Omega(G)$,*

(2) *G is the free product of G_1 and G_2 with the amalgamated subgroup H, and*

(3) *$\Omega(G)/G = (\Omega(G_1)/G_1 - B_1/H) \cup (\Omega(G_2)/G_2 - B_2/H)$, where $(\Omega(G_1)/G_1 - B_1/H) \cap (\Omega(G_2)/G_2 - B_2/H) = (\gamma \cap \Omega(H))/H$.*

In this case, G is said to be constructed from G_1 and G_2 via Combination theorem I.

COMBINATION THEOREM II (Maskit [9], [11]). *Let G_1 be a Kleinian group. For $i = 1, 2$, let B_i be a precisely invariant disc for a finite or parabolic cyclic subgroup H_i and let γ_i be the boundary of B_i. Assume that $G_1(\bar{B}_1 - \Lambda(H_1)) \cap (\bar{B}_2 - \Lambda(H_2)) = \emptyset$. Let F be a cyclic group generated by $f \in \mathrm{SL}'$, where $f\gamma_1 = \gamma_2$, $fB_1 \cap B_2 = \emptyset$, and $fH_1f^{-1} = H_2$. Let G be the group generated by G_1 and F. Then*

(1) *G is Kleinian and* $(\gamma_i - \Lambda(H_i)) \subset \Omega(G)$, $i = 1, 2$,

(2) *every relation in G is a consequence of the relations in G_1 and the relation* $fH_1f^{-1} = H_2$, *and*

(3) $\Omega(G)/G = \Omega(G_1)/G_1 - (B_1/H_1 \cup B_2/H_2)$, *where* $(\gamma_1 \cap \Omega(G))/H_1$ *is identified in* $\Omega(G)/G$ *with* $(\gamma_2 \cap \Omega(G))/H_2$.

In this case, G is said to be constructed from G_1 and f via Combination theorem II. The loops γ and γ_1 appearing in Combination theorems are called structure loops.

Recently Abikoff and Maskit proved the following basic decomposition theorem of finitely generated Kleinian groups.

THEOREM A (Maskit [13], [14], Abikoff-Maskit [2]). *Every finitely generated Kleinian group can be constructed from elementary groups, totally degenerate groups and web groups by a finite number of applications of Combination theorems* I *and* II.

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2. Statement of results. The main purpose of this paper is to prove the following.

THEOREM 1. *A Kleinian group is geometrically finite if and only if it is constructed from elementary groups and geometrically finite web groups by a finite number of applications of Combination theorems* I *and* II.

The following corollary to Theorem 1 was proved by Maskit, whose proof was different from ours.

COROLLARY (Maskit [15]). *A function group is geometrically finite if and only if it is constructed from elementary groups and quasi-Fuchsian groups by a finite number of applications of Combination theorems* I *and* II.

The proof of Theorem 1 is immediately obtained by Theorem A and Lemmas 3 through 7. We note that these lemmas are valid, even if the groups are not finitely generated.

Lemmas 3 through 7 together with (1) and (2) in Combination theorems I and II proved in Maskit [9], give a simpler proof of (3) in Combination theorems proved in Maskit [10], [11], if we regard the Kleinian groups as those acting only on $\hat{C}$.

The following Theorem 2 is an immediate corollary to Theorem 1 and to the fact that totally degenerate groups are geometrically infinite. This theorem was proved originally by Abikoff, whose proof was based on instability of totally degenerate groups.

THEOREM 2 (Abikoff [1]). *Totally degenerate groups cannot be constructed from elementary groups by a finite number of applications of Combination theorems.*

Finally, we can find a proof of the following generalization of a classical theorem on the method of constructing a fundamental region in Lemma 4 (see Ford [6, p. 45]).

THEOREM 3. *Let G be a Kleinian group such that $\infty \in \Omega(G)$ or such that ∞ is fixed by a parabolic element of G. Let P be a convex fundamental region for the stabilizer subgroup $G_\infty = \{g \in G; g(\infty) = \infty\}$ of $\{\infty\}$ in G. Then $P \cap (\bigcap_{g \in G - G_\infty} \operatorname{ext} I(g))$ is a fundamental region for G in $\bar{H} - \Omega(G)$.*

Thus, in the remainder of this paper, we shall only give the proof of Theorem 1.

3. Reduction. Let $A \subset \hat{C}$ be a domain whose boundary consists of more than two points. Then we can define the Poincaré metric $\lambda_A(z)|dz|$ with the constant negative curvature -1 in A. We denote by $l(\alpha, A)$ the length of a curve α in A measured by $\lambda_A(z)|dz|$. As is well known, $l(g(\alpha), g(A)) = l(\alpha, A)$ for each $g \in \mathrm{SL}'$, and $l(\alpha, A) \leqq l(\alpha, A')$ for sets A and A' $(\subset A)$. For a Kleinian group G, G' denotes G with the identity removed. Set $\Omega'(G) = \Omega(G) - \{z \in \hat{C}; g(z) = z$ for some $g \in G'\}$ and $\Sigma(G) = \hat{C} - \Omega'(G)$.

A parabolic fixed point $\zeta \in \Lambda(G)$ is a cusped parabolic fixed point, if there is an open set which is precisely invariant under $G_\zeta = \{g \in G; g(\zeta) = \zeta\}$ in G and which consists of two disjoint nonempty discs, or if G_ζ is not a finite extension of a cyclic group. A point $\zeta \in \Lambda(G)$ is a point of approximation, if there is a point $z \in \Omega(G)$, a constant M_1 and a sequence $\{g_j\}_{j=1}^\infty \subset G$ such that $|g_j(\zeta) - g_j(z)| > M_1$. The following lemma plays a central role in the proof of Theorem 1.

LEMMA 1. *Let w be a quasi-conformal automorphism of $\hat{C}$ compatible with G, that is, wGw^{-1} is again Kleinian. Then wGw^{-1} is geometrically finite if and only if G is.*

PROOF. Without loss of generality we may assume that both $\Lambda(G)$ and $\Lambda(wGw^{-1})$ are bounded. Note that $\Lambda(wGw^{-1}) = w\Lambda(G)$. If G is geometrically finite, then every point of $\Lambda(G)$ is either a cusped parabolic fixed point or a point of approximation (Beardon-Maskit [4]). If $\zeta \in \Lambda(G)$ is the former, then clearly so is $w(\zeta) \in \Lambda(wGw^{-1})$. If $\zeta \in \Lambda(G)$ is the latter, then there are a point $z \in \Omega(G)$, a constant M_1 and a sequence $\{g_j\}_{j=1}^{\infty} \subset G$ such that $|g_j(\zeta) - g_j(z)| > M_1$. Therefore we have

$$\begin{aligned} |w \circ g_j \circ w^{-1}(w(\zeta)) - w \circ g_j \circ w^{-1}(w(z))| &= |w(g_j(\zeta)) - w(g_j(z))| \\ &\geqq ((1/M_2)|g_j(\zeta) - g_j(z)|)^K \geqq (M_1/M_2)^K , \end{aligned}$$

where the first inequality is immediate from Ahlfors [3, p. 51] and K is a constant depending only on w. This means that $w(\zeta) \in \Lambda(wGw^{-1})$ is a point of approximation and wGw^{-1} is geometrically finite (Beardon-Maskit [4]). We can prove the rest of the lemma in the same way as above.

We denote by $\hat{\gamma}$ (or $\hat{\gamma}_1$) a connected fundamental region for H (or H_1) in $\gamma - \Lambda(H)$ (or $\gamma_1 - \Lambda(H_1)$), where γ and γ_1 are structure loops appearing in Combination theorems. Bers [4] showed the existence of a quasi-conformal automorphism w of $\hat{C}$ compatible with G such that $l(w(\hat{\gamma}), \Omega'(wGw^{-1}))$ (or $l(w(\hat{\gamma}_1), \Omega'(wGw^{-1})))$ is sufficiently small. By Lemma 1 quasi-conformal deformations preserve both the assumption and the conclusion of Theorem 1, so, from now on,we assume that $l(\hat{\gamma}, \Omega'(G))$ (or $l(\hat{\gamma}_1, \Omega'(G))$ is sufficiently small.

Next we give a simpler proof of the following lemma due to Maskit [10].

LEMMA 2. *Let $\gamma_1, \gamma_2, \cdots$ be translates of the structure loop γ under G constructed via Combination theorems. Then the spherical diameter of γ_j tends to zero.*

PROOF. If γ is contained in $\Omega(G)$, then this lemma was proved by Maskit [12]. Therefore we need only to consider the case that $\gamma \cap \Lambda(G) \neq \emptyset$. In this case γ and the point $\xi = \gamma \cap \Lambda(G)$ are fixed by a parabolic cyclic group H. Without loss of generality, we may assume that $\infty \in \Omega(G)$, that $G_\infty = \{\mathrm{id}\}$ and that all γ_j's are contained in a bounded domain.

Suppose that the conclusion in our lemma is false. Then there exists a subsequence, again denoted by $\{\gamma_j\}_{j=1}^{\infty}$, of $\{\gamma_j\}_{j=1}^{\infty}$ such that the Euclidean diameter $\operatorname{dia}\gamma_j$ of γ_j is greater than a constant. Let g_j be an element in G with $g_j(\gamma) = \gamma_j$. Set $\gamma_j = g_j(\gamma)$. Then two cases can occure; (i) there exist infinitely many distinct ξ_j's, and (ii) otherwise.

First we consider the case (i). Let h_j be the element in $g_jHg_j^{-1}$ with $r(h_j) \geqq r(h)$ for each $h \in g_jHg_j^{-1}$. Since $r(h_j) \to 0$ (Ford [6, p. 41]), we see $\operatorname{dia}\gamma_j/r(h_j) \to \infty$. Let ζ_j be a point in $\Lambda(G)$ with $|\xi_j - \zeta_j| \geqq$

$|\xi_j - \zeta|$ for all $\zeta \in \Lambda(G)$. Let p_j be a linear transformation $C \ni z \to a_j z + b_j \in C$, $a_j, b_j \in C$, such that $p_j(\xi_j) = 0$ and $p_j(I(h_j^{-1}) \cap C) = \{z \in C; |z-1| = 1\}$. Set $\hat{\gamma}_j = \gamma_j \cap \operatorname{ext} I(h_j) \cap \operatorname{ext} I(h_j^{-1})$. Then we obtain $l(\hat{\gamma}_j, \Omega'(G)) \geqq l(\hat{\gamma}_j, \hat{C} - \{\xi_j, \zeta_j, h_j(\zeta_j)\}) = l(p_j(\hat{\gamma}_j), \hat{C} - \{0, p_j(\zeta_j), p_j(h_j(\zeta_j))\}) \to 0$, because $p_j(\zeta_j) \to \infty$, $p_j(h_j(\zeta_j)) \to 1$, $\sup_{z \in p_j(\hat{\gamma}_j)} |z| \to \infty$ and $p_j(\hat{\gamma}_j) \cap \{z \in C; |z| < 2\} \neq \emptyset$ for all j. On the other hand, $l(\hat{\gamma}_j, \Omega'(G)) = l(\hat{\gamma}_{j+1}, \Omega'(G))$, $j = 1, 2, \cdots$, which is a contradiction.

Next we consider the case (ii). In this case we may assume that $\gamma_j = \gamma_1$, $j = 1, 2, \cdots$. So $\bigcup_{j=1}^{\infty} g_j$ is a subset of an elementary group with one limit point, and our conclusion is obvious.

4. **Proof.** In this section we give a proof of Theorem 1. First we consider the case where the group is constructed via Combination theorem I.

LEMMA 3. *Let G be a Kleinian group constructed from G_1 and G_2 via Combination theorem* I, *where H is finite cyclic. Then G is geometrically finite if and only if both G_1 and G_2 are.*

PROOF. We may assume that $\Sigma(G_i) - \Sigma(H)$ contains the point $z_i = i - 1$ for $i = 1, 2$, and that γ passes through ∞. Thus we can define the isometric sphere of g for each $g \in G'$.

Since γ separates $\Sigma(G_1) - \Sigma(H)$ from $\Sigma(G_2) - \Sigma(H)$ and since $l(\gamma, \Omega'(G))$ is sufficiently small, we may assume that $(\Sigma(G_i) - \Sigma(H)) \subset D_i = \{z \in C; |z - z_i| < 1/100\}$ and $\Sigma(G) \subset (D_1 \cup D_2)$. For any $g \in G_1 - H$, we have $g(1) \in D_1$, since B_1 is precisely invariant under H in G_1 and $\Sigma(G)$ is invariant under G. Moreover, $g(0) \in g(\Sigma(G_1)) = \Sigma(G_1) \subset D_1$. If $2r(g) \geqq |p(g)|$, then

$$\begin{aligned} 1/50 > |g(0) - g(1)| &= |u_1(0) - u_1(1)| = (|1 - 0| \cdot |p(g) - u_1(0)|)/|p(g) - 1| \\ &= (r(g)^2/|p(g) - 0|)/|p(g) - 1| \geqq r(g)/2(1 + 2r(g)) . \end{aligned}$$

If $2r(g) \leqq p(g)$, then

$$1/100 + r(g) \geqq |\xi(g)| + r(g) \geqq |p(g)| \geqq 2r(g) ,$$

where $\xi(g)$ is a fixed point of g on $i(g) = I(g) \cap C$ or on the bounded subdomain of C surrounded by $i(g)$. In any case $(|z|^2 + t^2) < ((|\xi(g)| + 2r(g))^2 + r(g)^2) < 1/3$ for any $g \in G_1 - H$ and any $(z, t) \in I(g)$. So $D_1^* = \bigcap_{g \in G_1 - H} \operatorname{ext} I(g)$ contains $\{(z, t) \in \bar{H}; \operatorname{Re} z > 1/2\}$. Similarly $D_2^* = \bigcap_{g \in G_2 - H} \operatorname{ext} I(g)$ contains $\{(z, t) \in H; \operatorname{Re} z < 1/2\}$.

Set $D_H = \bigcap_{g \in H'} \operatorname{ext} I(g)$. Then $D_i = D_i^* \cap D_H$ is a convex fundamental region for G_i, $i = 1, 2$, and $D = \bigcap_{g \in G'} \operatorname{ext} I(g)$ is a convex fundamental region for G. Obviously $D_1 \cap D_2$ is surrounded by a finite number of hyperbolic planes if and only if so are both D_1 and D_2.

To complete the proof of our lemma, it suffices to show that $D_1 \cap D_2 = D$. Clearly $(D_1 \cap D_2) \supset D$. Let $\zeta \in D_1 \cap D_2$. Then obviously $\zeta \in \bigcap_{g \in H'} \operatorname{ext} I(g)$. By the conclusion (2) of Combination theorem I each $g \in G - H$ can be written in the form $g_n \circ \cdots \circ g_1$, where $g_j \in \bigcup_{i=1}^{2} (G_i - H)$, $j = 1, \cdots, n$, and both g_j and g_{j+1} are not in some $G_i - H$, $j = 1, \cdots, n-1$. We may assume that $g_1 \in G_1 - H$. Then $J_{g_1}(\zeta) < 1$ by virtue of $\zeta \in D_1 \subset \operatorname{ext} I(g_1)$. Since $g_1(\zeta) \in \operatorname{int} I(g_1) \subset \operatorname{ext} I(g_2)$, we see $J_{g_2 \circ g_1}(\zeta) = J_{g_2}(g_1(\zeta)) \cdot J_{g_1}(\zeta) < 1$. For the same reasoning, we have $J_g(\zeta) < 1$, which implies $\zeta \in \operatorname{ext} I(g)$. This means that $D_1 \cap D_2 = D$.

LEMMA 4. *Let G be a Kleinian group constructed from G_1 and G_2 via Combination theorem* I, *where H is parabolic cyclic. Then G is geometrically finite if and only if both G_1 and G_2 are.*

PROOF. We may assume that H is generated by $h = (1, 1; 0, 1)$, that γ passes through 0 and that B_1 contains $\{z \in \boldsymbol{C}; \operatorname{Im} z > y_0\}$. Since B_i is precisely invariant under H in G_i, the stabilizer subgroup $G_{i\infty} = \{g \in G_i; g(\infty) = \infty\}$ of $\{\infty\}$ in G_i is identical with H or is generated by h and by an elliptic element of order two (Ford [6, p. 142]), whose fixed points in $\hat{\boldsymbol{C}}$ are ∞ and $\xi_i \in \hat{\boldsymbol{C}} - (B_i \cup \gamma)$. Since the stabilizer subgroup G_∞ of $\{\infty\}$ in G is generated by $G_{1\infty}$ and $G_{2\infty}$, we see $J_g = 1$ for each $g \in G_\infty$. If $G_{i\infty} = H$, then we set $P_i = \{(z, t) \in \bar{H}; 0 < \operatorname{Re} z < 1\}$, $i = 1, 2$, and we also set $P_1 = \{(z, t) \in \bar{H}; 0 < \operatorname{Re} z < 1, \operatorname{Im} z > \operatorname{Im} \xi_1\}$ if $G_{1\infty}$ is not cyclic and $P_2 = \{(z, t) \in \bar{H}; 0 < \operatorname{Re} z < 1, \operatorname{Im} z < \operatorname{Im} \xi_2\}$ if $G_{2\infty}$ is not cyclic. It is clear that $P = P_1 \cap P_2$ is a fundamental region for G_∞.

We shall show that $D = P \cap (\bigcap_{g \in G - G_\infty} \operatorname{ext} I(g))$ is a fundamental region for G. Obviously no two points in D are equivalent under G. Let $\zeta \in \bar{H} - \Lambda(G)$. Then there exists a $g_0 \in G$ with $J_{g_0}(\zeta) \geqq J_g(\zeta)$ for each $g \in G$. If it were false, then there would exist a sequence $\{g_j\}_{j=1}^{\infty} \subset G$ with $J_{g_{j+1}}(\zeta) > J_{g_j}(\zeta) > J_h(\zeta) = 1$. This means $\zeta \in \operatorname{int} I(g_j)$. Then two cases can occur; (i) there exists a subsequence, again denoted by $\{g_j\}_{j=1}^{\infty}$, of $\{g_j\}_{j=1}^{\infty}$ with $\lim_{j \to \infty} r(g_j) = 0$, and (ii) otherwise. In the case (i), we see $\zeta = \lim_{j \to \infty} g_j^{-1}(\infty) \in \Lambda(G)$, which is a contradiction. In the case (ii), we can find a constant r_0 and a subsequence, again denoted $\{g_j\}_{j=1}^{\infty}$, of $\{g_j\}_{j=1}^{\infty}$ with $r(g_j) \geqq r_0$ for each j. Set $P_0 = \{(z, t) \in \bar{H}; z \in P \cap \boldsymbol{C}, t > 1\}$. Since $r(g) \leqq 1$ for each $g \in G - G_\infty$ (Kra [8, p. 51]), P_0 is contained in D. Let p_j be an element in G_∞ with $g_j(\infty) \in p_j P$. Then the Euclidean volume $\operatorname{vol} g_j^{-1} \circ p_j P_0$ of $g_j^{-1} \circ p_j P_0$ is more than a constant v_0 for each j. Since $g_j^{-1} \circ p_j P_0$ is contained in $\operatorname{int} I(g_j)$, $\bigcup_{j=1}^{\infty} g_j^{-1} \circ p_j P_0$ is contained in a bounded subset V of $\bar{H}$. Therefore we see, for each natural number N,

$$\operatorname{vol} V \geqq \operatorname{vol} \bigcup_{j=1}^{N} g_j^{-1} \circ p_j P_0 = \sum_{j=1}^{N} \operatorname{vol} g_j^{-1} \circ p_j P_0 \geqq N v_0 ,$$

which is a contradiction. Let g_0 be an element in G satisfying $J_{g_0}(\zeta) \geqq J_g(\zeta)$ for each $g \in G$. If p is an element in G with $p(g_0(\zeta)) \in \bar{P}$, then $p(g_0(\zeta))$ is in $\bar{D}$. If it were not true, then there would exist a $q \in G$ with $p(g_0(\zeta)) \in \operatorname{int} I(q)$. So $J_{q \circ p \circ g_0}(\zeta) > J_{g_0}(\zeta)$, which is a contradiction. Similarly $D_i = P_i \cap (\bigcap_{g \in G_i - G_{i\infty}} \operatorname{ext} I(g))$ is a fundamental region for G_i, $i = 1, 2$.

Since $l(\hat{\gamma}, \Omega'(G))$ is sufficiently small, where $\hat{\gamma} = \gamma \cap \{z \in \boldsymbol{C};\, 0 < \operatorname{Re} z < 1\}$, we have $\bigcup_{g \in G_1 - G_{1\infty}} p(g) \subset \Sigma(G_1) \subset \{z \in \boldsymbol{C};\, \operatorname{Im} z < -2\} \cup \{\infty\}$ and $\bigcup_{g \in G_2 - G_{2\infty}} p(g) \subset \Sigma(G_2) \subset \{z \in \boldsymbol{C};\, \operatorname{Im} z > 2\} \cup \{\infty\}$. Since $r(g) \leqq 1$ for each $g \in G - G_\infty$, $D_1 \cap D_2$ is surrounded by a finite number of hyperbolic planes if and only if so are both D_1 and D_2. As in the proof of Lemma 3, we can show that $D_1 \cap D_2 = D$ and we complete the proof of our lemma.

Next we consider the case where the group is constructed via Combination theorem II.

LEMMA 5. *Let G be a Kleinian group constructed from G_1 and f via Combination theorem* II, *where both H_1 and H_2 are trivial. Then G is geometrically finite if and only if G_1 is.*

PROOF. Without loss of generality, we may assume that $f = (a, 0; 0, a^{-1})$, $|a| > 1$ and that $1 = \inf\{|z|;\, z \in \gamma_2\}$. We note that each $g \in G_1'$ is of the form $(*, *; c, *)$, $c \neq 0$. Since $l(\gamma_2, \Omega'(G))$ is sufficiently small, the set $\Sigma(G) \cap (\hat{\boldsymbol{C}} - B_2)$ is contained in the set $\{z \in \boldsymbol{C};\, |z| < 1/100\}$. In particular, it holds that $|g(0)| < 1/100$ for each $g \in G_1$. Since $l(g(\gamma_1), \boldsymbol{C} - \{0, g(0)\}) \leqq l(g(\gamma_1), \Omega'(G))$ is sufficiently small, $g(\gamma_1)$ is contained in the set $\{z \in \boldsymbol{C};\, |z| < 1/2\}$. Set $\boldsymbol{B}_1 = \{(z, t) \in \bar{H};\, |z|^2 + t^2 < |a|^{-2}\}$ and $\boldsymbol{B}_2 = \{(z, t) \in \bar{H};\, |z|^2 + t^2 > 1\} \cup \{\infty\}$. Then B_1 is precisely invariant under H_1 in G_1, $G_1\bar{\boldsymbol{B}}_1 \cap \bar{\boldsymbol{B}}_2 = \varnothing$ and $f(\boldsymbol{\gamma}_1) = \boldsymbol{\gamma}_2$, where $\boldsymbol{\gamma}_i = \bar{\boldsymbol{B}}_i - \boldsymbol{B}_i$, $i = 1, 2$. Since both $g(0)$ and $g(\infty) = p(g^{-1})$ are in $\Sigma(G) \cap (\hat{\boldsymbol{C}} - B_2)$ for each $g \in G_1'$, we have $|u_1(0) - u_1(\infty)| = |g(0) - g(\infty)| < 1/50$. Therefore $r(g) < 1/5$, and $\operatorname{ext} I(g)$ contains $\boldsymbol{B}_2$. Now we see that $\boldsymbol{B}_2$ is precisely invariant under H_2 in G_1. These properties of B_1 and B_2 show that no pair of points in $D = (\bigcap_{g \in G_1} \operatorname{ext} I(g)) - G_1(\bar{\boldsymbol{B}}_1 \cup \bar{\boldsymbol{B}}_2)$ are equivalent under G, since each $g \in G'$ can be written in the form $f^{\alpha_{n+1}} \circ g_n \circ \cdots \circ g_1 \circ f^{\alpha_1}$, where $\alpha_k \neq 0$, $k = 2, \cdots, n$, and $g_k \in G_1'$, $k = 1, \cdots, n$.

Let F be the cyclic group generated by f. Set $\Delta_1 = G_1(\bar{D} \cup \Lambda(G_1))$, $\Delta_2 = F\Delta_1$, $\Delta_{2j-1} = G_1\Delta_{2j-2}$ and $\Delta_{2j} = F\Delta_{2j-1}$, $j = 2, 3, \cdots$. Let $\zeta \in \bar{H} - G(\bar{D} \cup \Lambda(G_1))$. Then $\zeta \in \bar{H} - \Delta_j$, $j = 1, 2, \cdots$. Let δ_j be the component of $\bar{H} - \Delta_j$ containing ζ. Since δ_j is a translate of $\boldsymbol{B}_1$ or $\boldsymbol{B}_2$ under an element of G and since $(\bar{\boldsymbol{B}}_i - \boldsymbol{B}_i) \cap \hat{\boldsymbol{C}}$ is a structure loop, $i = 1, 2$, the Euclidean diameter of δ_j tends to zero by Lemma 2. So we can find a sequence $\{g_j\}_{j=1}^{\infty} \subset G$ with $g_j(\gamma_1) \to \zeta$. Hence we see that $\bar{H} - G(\bar{D} \cup \Lambda(G_1)) \subset$

$\Lambda(G)$ and that $\bar{H} - \Lambda(G_1) \subset \Lambda(G)$ and that $\bar{H} - \Lambda(G) \subset G\bar{D}$. Therefore we have obtained a convex fundamental region D for G in $\bar{H} - \Lambda(G)$.

Next we observe that for at most finitely many g_n's of G_1, $I(g_n)$ meets $\boldsymbol{B}_1 \cup \boldsymbol{B}_2$. If it were false, then there would exist a sequence $\{g_n\}_{n=1}^{\infty} \subset G_1$ and some $\boldsymbol{B}_i$, say $\boldsymbol{B}_1$, such that $I(g_n) \cap \boldsymbol{B}_1 \neq \emptyset$, which contradicts the fact that $\bar{\boldsymbol{B}}_1 \cap \Lambda(G_1) = \emptyset$ (Ford [6]). Moreover, this means immediately that at most finitely many translates of $\bar{\boldsymbol{B}}_i$ under G_1 can intersect $\bigcap_{g \in G_1} \operatorname{ext} I(g)$, $i = 1, 2$, because $I(g) \cap \boldsymbol{B}_i \neq \emptyset$ if and only if $I(g^{-1}) \cap g(\boldsymbol{B}_i) \neq \emptyset$. Thus we complete the proof of our lemma.

LEMMA 6. *Let G be a Kleinian group constructed from G_1 and f via Combination theorem* II, *where both H_1 and H_2 are elliptic cyclic. Then G is geometically finite if and only if G_1 is.*

PROOF. Without loss of generality we may assume that f is of the form $(a, 0; 0, a^{-1})$, $|a| > 1$, and that $1 = \inf\{|z|; z \in \gamma_1\}$. Since $l(\gamma_1, \Omega'(G))$ is sufficiently small, we see $\xi_1 \in \{z \in \boldsymbol{C}; |z| \leqq 1/100\}$ and $\xi_2 \in \{z \in \boldsymbol{C}; |z| \geqq 100\} \cup \{\infty\}$, where ξ_i is the fixed point of a generator of H_i in B_i, $i = 1, 2$. Let $\tilde{B}_1 \subset B_1$ be the largest disc invariant under H_1. Then $\tilde{B}_1$ contains the disc $\{z \in \boldsymbol{C}; |z| < 1/2\}$, because $(\xi_1 - z)(\xi_2 - w)/(\xi_1 - w)(\xi_2 - z) = 1$ for any z and w on the boundary of $\tilde{B}_1$, where we understand $\infty/\infty = 1$. Set $\boldsymbol{B}_1 = \{(z, t) \in \bar{H}; |z - z_0|^2 + t^2 < r_0\}$, where $\boldsymbol{B}_1 \cap \boldsymbol{C} = \tilde{B}_1$. If we set $\boldsymbol{B}_2 = \bar{H} - \overline{f(\boldsymbol{B}_1)}$, then, as in the proof of the previous lemma, we can show that $\boldsymbol{B}_i$ is precisely invariant under H_i in G, $i = 1, 2$, and that $G_1\bar{\boldsymbol{B}}_1 \cap \bar{\boldsymbol{B}}_2 = \emptyset$. The rest of the proof of our lemma is similar to that of Lemma 5.

LEMMA 7. *Let G be a Kleinian group constructed from G_1 and f via Combination theorem* II, *where both H_1 and H_2 are parabolic cyclic. Then G is geometrically finite if and only if G_1 is.*

PROOF. We may assume that H_1 is generated by $h_1 = (1, 1; 0, 1)$ and that B_1 contains the set $\{z \in \boldsymbol{C}; \operatorname{Im} z > y_0\}$. Let $B_1^* = \{z \in \boldsymbol{C}; \operatorname{Im} z > y_1\}$ be the smallest half plane containing B_1. Set $\boldsymbol{B}_1 = \{(z, t) \in \bar{H}; z \in B_1^*\}$. Obviously $\boldsymbol{B}_2 = \bar{H} - \overline{f(\boldsymbol{B}_1)}$ is precisely invariant under H_2 in G_1. Since B_1 is precisely invariant under H_1 in G_1, the subgroup $G_{1\infty} = \{g \in G_1; g(\infty) = \infty\}$ is the cyclic group H_1 or is the group $\hat{H}_1$ generated by h_1 and by an elliptic element e of order two whose fixed points in $\hat{\boldsymbol{C}}$ are ∞ and $\eta \in \hat{\boldsymbol{C}} - \bar{B}_1$.

First we consider the case $G_{1\infty} = \hat{H}_1$. Set $\hat{\gamma}_1 = \{z \in \gamma_1; 0 < \operatorname{Re} z < 1\}$. Since $l(\hat{\gamma}_1, \Omega'(G))$ is sufficiently small, we have $\eta \in \hat{\boldsymbol{C}} - \bar{B}_1^*$. So $(G_{1\infty} - H_1)(\bar{\boldsymbol{B}}_1 - \Lambda(H_1)) \cap (\bar{\boldsymbol{B}}_1 - \Lambda(H_1)) = \emptyset$. By the same reasoning as in the proof of Lemma 4 we see $(G_1 - G_{1\infty})\bar{\boldsymbol{B}}_1 \cap \bar{\boldsymbol{B}}_1 = \emptyset$. These imply that $\bar{\boldsymbol{B}}_1 - \Lambda(H_1)$

is precisely invariant under H_1 in G_1. Since $g(\gamma_2)$ passes through the fixed point of $g\circ h_2\circ g^{-1}$ for each $g\in G_1'$, the same reasoning as in the proof of Lemma 4 yields

$$(g(\gamma_2)\cap \mathrm{ext}\, I(g\circ h_2\circ g^{-1})\cap \mathrm{ext}\, I(g\circ h_2^{-1}\circ g^{-1}))\subset(\hat{C}-\bar{B}_1^*)\ .$$

Therefore $G_1(\bar{\boldsymbol{B}}_2-\Lambda(H_2))\cap(\bar{\boldsymbol{B}}_1-\Lambda(H_1))=\varnothing$. In the case that $G_{1\infty}=H$, we can also obtain the same result as above.

The remainder of the proof of our lemma is similar to that of the previous lemmas.

It is easy to see that Theorem A and a finite number of applications of Lemmas 3 through 7 yield the proof of Theorem 1.

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