Tôhoku Math. Journ.
32 (1980), 217-224.

INVARIANCE, ASYMPTOTIC BEHAVIOR AND STABILITY PROPERTIES FOR ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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(Received April 11, 1979, revised July 21, 1979)

1. Introduction. The main objective of this paper is to find conditions under which $(x(t), \dot{x}(t)) \to (\eta, 0)$, as $t \to \infty$, for every solution of a suitable nonautonomous second order differential equation. Here $(\eta, 0)$ will be an equilibrium point of a certain autonomous equation. We are also interested in studying the stability properties of a class of equilibrium points of the above mentioned second order differential equation.

Theorem 1 is a generalization of a result of Yoshizawa obtained by Onuchic et al. in [4]. Theorem 2 is a modified but closely related version of Yoshizawa's Theorem 3 in [6]. Theorem 2 can also be seen as a special case of Miller's Theorem 1 in [1]. The basic tool used here in attacking the above problems is provided by Theorems 1 and 2.

2. ω-limit set and invariance. Consider a system of ordinary differential equations, defined on a region $Q \subset R^n$,

$$\dot{x} = H(x) , \tag{1}$$

where $H(x)$ is continuous on Q. Here R^n denotes a normed, real n-dimensional vector space with any convenient norm $|\cdot|$.

If M is a subset of Q then M is called semi-invariant with respect to (1) if, and only if, for each $x_0 \in M$, there is at least one solution $x(t)$ of (1), with $x(0) = x_0$, such that $x(t)$ exists and remains in M for all real t. If in addition some uniqueness condition with respect to the initial value problem holds for (1) then M is called invariant with respect to (1).

Let $x(t)$ be a continuous function defined in the future, that is, for all $t \geqq$ some real t_0. A point $p \in R^n$ is said to be an ω-limit point of $x(t)$ if there exists a sequence $\{t_m\}$, $t_m \to \infty$ as $m \to \infty$, such that $x(t_m) \to p$ as $m \to \infty$. The set of all ω-limit points of $x(t)$ is denoted by Ω and is called the ω-limit set of $x(t)$. If $x(t)$ is bounded in the future, that is, $x(t)$ is bounded on some interval $[a, \infty)$, $a > -\infty$, it is easily seen that Ω is a nonempty, connected and compact set with $x(t) \to \Omega$ as $t \to$

∞, that is, $\mathrm{dist}(x(t), \Omega) \to 0$ as $t \to \infty$.

Consider the differential system defined on $[0, \infty) \times Q$

$$\dot{x} = F(t, x) + G(t, x) , \tag{2}$$

where $F(t, x)$ and $G(t, x)$ are continuous for $t \geqq 0$ and $x \in Q$.

THEOREM 1. *Suppose that the following hypotheses hold with respect to system* (2):

(i) *$F(t, x)$ is bounded for all $t \geqq 0$ when x belongs to an arbitrary compact subset of Q;*

(ii) *For every compact subset B of Q and every continuous function $z(t) \in B$, defined on $[0, \infty)$, it follows that*

$$\int_s^{s+t} G(\tau, z(\tau))d\tau \to 0 \quad as \quad s \to \infty \tag{3}$$

uniformly for t on $[0, 1]$;

(iii) *There are real-valued nonnegative functions $V(t, x)$ and $W(t, x)$ satisfying the following conditions*:

(a) *$V(t, x)$ is continuous and locally Lipschitzian with respect to x, for $t \geqq 0$, $x \in Q$;*

(b) *$W(t, x)$ is a continuous function of x for each fixed t where the continuity in x is uniform for t on $[0, \infty)$;*

(c) *There is $x_0 \in Q$ such that $W(t, x_0)$ is bounded on $[0, \infty)$;*

(d) $\dot{V}_{(2)}(t, x) = \limsup_{h \to 0^+} [V(t+h, x + hF(t, x) + hG(t, x)) - V(t, x)]/h \leqq - W(t, x)$, $t \geqq 0$, $x \in Q$.

Let $x(t)$ be a solution of (2) *defined in the future, with $x(t) \in K$ for $t \geqq$ some t_0, where K is a compact subset of Q. Then $\Omega \subset E \cap K$, where $E = \{x \in Q \mid \liminf_{t \to \infty} W(t, x) = 0\}$ and Ω is the ω-limit set of $x(t)$.*

NOTE. This theorem is more general than Yoshizawa's result [6, Theorem 5]. Yoshizawa considers the case in which $W(t, x)$ does not depend on t and he also assumes condition $\int^{\infty} |G(t, z(t))| dt < \infty$ which is stronger than the one given by (3), but the ideas contained in the proof of Theorem 1 are closely related to the ones in Yoshizawa's mentioned result.

A sufficient condition for (3) is given as follows: For every compact subset B of Q there corresponds a scalar function $\sigma_B(t)$ defined for $t \geqq 0$ so that $|G(t, x)| \leqq \sigma_B(t)$ for all $t \geqq 0$, $x \in B$ and

$$\int_t^{t+1} \sigma_B(s)ds \to 0 \quad \text{as} \quad t \to \infty . \tag{4}$$

The example $G = (t \sin t^3, t \cos t^3)$ considered in [5] shows that condition

(4) is not implied by condition (3). A proof of Theorem 1 can be found in [4, Theorem 1].

Consider the differential system difined on $[0, \infty) \times Q$:

$$\dot{x} = H(x) + S(t, x) + G(t, x) , \tag{5}$$

where $H(x)$, $S(t, x)$ and $G(t, x)$ are assumed to be continuous on $[0, \infty) \times Q$. Let A and K be subsets of Q, K compact. Assume that $S(t, x)$ satisfies the following property with respect to A: For each $\varepsilon > 0$ there correspond $\delta = \delta(\varepsilon) > 0$ and $T = T(\varepsilon)$ such that $t \geqq T(\varepsilon)$, $x \in K$ and $\operatorname{dist}(x, A) < \delta$ imply

$$|S(t, x)| < \varepsilon . \tag{6}$$

THEOREM 2. *Let hypotheses* (3) *and* (6) *hold. Let $x(t)$ be a solution of* (5) *such that $x(t) \in K$, $t \geqq$ some T and K is a compact subset of Q with $x(t) \to A$ as $t \to \infty$. Then the ω-limit set Ω of $x(t)$ is a nonempty, connected, compact and semi-invariant set with respect to* (1).

NOTE. As observed, this theorem is a version of Yoshizawa's Theorem 3 in [6]. See also Miller's Theorem 1 in [1].

Let

$$\dot{x} = f(t, x) , \qquad t \geqq 0 , \quad x \in Q , \tag{7}$$

where $f(t, x) \in R^n$ is continuous on $[0, \infty) \times Q$. Let $\psi(t)$ be a solution of (7) defined on $[0, \infty)$ such that $|\psi(t)| \leqq H^*$, $H^* < H$, for all $t \geqq 0$.

LEMMA 1. *Suppose that*

(i) *$\psi(t)$ is uniformly stable;*

(ii) *There is $\rho > 0$ such that, for every $t_0 \geqq 0$, $|x_0 - \psi(t_0)| < \rho$ implies $|x(t; t_0, x_0) - \psi(t)| \to 0$ as $t \to \infty$.*

Then $\psi(t)$ is equiasymptotically stable, that is, there exists a $\delta_0 > 0$ and, for each $\varepsilon > 0$, a $T(\varepsilon) > 0$ such that $|x_0 - \psi(0)| < \delta_0$ implies $|x(t; 0, x_0) - \psi(t)| < \varepsilon$ for $t \geqq T(\varepsilon)$.

The proof of Lemma 1 can be done by an argument similar to that in the proof of [7, Theorem 7.6].

3. Applications. The main objective of this section is to apply the results of Section 2 to obtain sufficient conditions under which, for every solution $x(t)$ of the second order differential equation

$$\ddot{x} + h(t, x, \dot{x})\dot{x} + f(x) + g(t, x, \dot{x}) + p(t, x, \dot{x}) = 0 , \tag{8}$$

we can guarantee that $x(t)$ tends to some η satisfying $f(\eta) = 0$ and $\dot{x}(t) \to 0$ as $t \to \infty$. We wish also to study the stability properties of a certain class of equilibrium points of (8′) which is defined by

$$(8') \qquad \begin{cases} \dot{x} = y \\ \dot{y} + h(t, x, y)y + f(x) + g(t, x, y) + p(t, x, y) = 0 \,, \end{cases}$$

where $t \geqq 0$ and $(x, y) \in R^2$. Consider also

$$(9) \qquad \ddot{x} + f(x) = 0 \,,$$

or equivalently

$$(9') \qquad \begin{cases} \dot{x} = y \\ \dot{y} + f(x) = 0 \,. \end{cases}$$

Let $D = \{(\xi, 0) \in R^2 \mid$ there is a positive $\rho = \rho(\xi)$ such that $(x - \xi)\cdot f(x) > 0$ for all $0 < |x - \xi| \leqq \rho\}$.

Suppose that some uniqueness condition with respect to the initial value problem holds for (8′). Consider also the following set of assumptions:

(H_1) $f(x)$ is a continuous function on R so that

(i) D is nonempty;

(ii) $\int_0^x f(s)ds \to \infty$ as $|x| \to \infty$;

(iii) there is no interval $[a, b]$, $b > a$, such that $f(x) = 0$ on $[a, b]$.

A sufficient condition for (H_1) is given as follows:

(H_1') $f(x)$ is a C^1 function on R so that $x\cdot f(x) > 0$ for all $x \neq 0$ and $\int_0^x f(s)ds \to \infty$ as $|x| \to \infty$.

In this case $D = \{(0, 0)\}$.

(H_2) $p(t, x, y)$ is continuous and $|p(t, x, y)| \leqq \beta(t)$ for all $t \geqq 0$, x, y in R, where $\beta(t)$ is continuous with $\int_0^\infty \beta(t)dt < \infty$.

(H_3) $g(t, x, y)$ is continuous and $y\cdot g(t, x, y) \geqq 0$ for all $t \geqq 0$, x, y in R.

(H_4) For every compact B of R and all continuous functions $x(t)$ and $y(t)$, defined on $[0, \infty)$ with values in B, we have that $\int_s^{s+t} g(\tau, x(\tau), y(\tau))d\tau \to 0$, as $s \to \infty$, uniformly for $t \in [0, 1]$.

A sufficient condition for (H_4) is given as follows:

(H_4') For every compact B of R there corresponds a real valued continuous function $\sigma_B(t)$, $t \geqq 0$, such that $\int_t^{t+1} \sigma_B(s)ds \to 0$ as $t \to \infty$ and $|g(t, x, y)| \leqq \sigma_B(t)$ for all $t \geqq 0$, x, y in B.

(H_5) $h(t, x, y)$ is a continuous nonnegative function on $[0, \infty) \times R^2$, where the continuity in x, y is uniform for t on $[0, \infty)$. Also $h(t, 0, 0)$ is bounded on $[0, \infty)$.

REMARK. We observe that condition (H_5) implies that $h(t, x, y)$ is bounded on $[0, \infty) \times B$ for every compact B of R^2.

(H_6) For all x, y such that $\liminf_{t\to\infty} h(t, x, y) = 0$ we have that $\lim_{t\to\infty} h(t, x, y)$ exists.

(H_7) For every bounded orbit γ of (9′), γ not being an equilibrium point of (9′), there is at least one point $(x, y) \in \gamma$ such that $h^*(x, y) \neq 0$ where $h^*(x, y) = \liminf_{t\to\infty} h(t, x, y)$.

A sufficient condition for (H_7) is given as follows:

(H_7') (H_1') is satisfied and for every orbit γ of (9′), $\gamma \neq (0, 0)$, there is at least one point $(x, y) \in \gamma$ such that $h^*(x, y) \neq 0$.

LEMMA 2. *Suppose that some uniqueness condition with respect to the initial value problem holds for* (8′). *Let $h(t, x, y)$ be nonnegative and continuous for $t \geqq 0, x, y$ in R. Let hypotheses* (H_1-i), (H_2) *and* (H_3) *hold. Let $(\xi, 0) \in D$ and $p(t, \xi, 0) = 0$, $t \geqq 0$. Then the equilibrium point $(\xi, 0)$ of* (8′) *is uniformly stable.*

The proof follows from [3, Lemma 2 and Corollary 1].

LEMMA 3. *Let $h(t, x, y)$ be nonnegative and continuous for $t \geqq 0$, $x, y \in R$. Let hypotheses* (H_1-ii), (H_2) *and* (H_3) *hold. Then every solution of* (8′) *is bounded in the future.*

PROOF. Let

$$V(t, x, y) = \left[y^2 + 2\int_0^x f(s)ds + M\right]^{1/2} + \int_t^\infty \beta(s)ds$$

where M is chosen so that $2\int_0^x f(s)ds + M > 0$ for all x. It is easy to see that $\dot V_{(8')}(t, x, y) \leqq 0$ for all $t \geqq 0, x, y$ in R. Then, as $W(x, y) = \left[y^2 + 2\int_0^x f(s)ds + M\right]^{1/2} \leqq V(t, x, y)$ and $W(x, y) \to \infty$ as $|x| + |y| \to \infty$, it follows that every solution of (8′) is bounded in the future.

THEOREM 3. *Let hypotheses* (H_1)-(H_7) *hold. Then for every solution $x(t)$ of* (8) *we have that $x(t) \to \eta$ and $\dot x(t) \to 0$ as $t \to \infty$, where η is a real number satisfying $f(\eta) = 0$.*

PROOF. Let $(x(t), y(t))$ be any solution of (8′). Lemma 3 implies that this solution is bounded in the future. Then the ω-limit set Ω of $(x(t), y(t))$ is a nonempty, connected and compact set with $(x(t), y(t)) \to \Omega$ as $t \to \infty$. We must have $\Omega \cap R_x \neq \varnothing$ because otherwise it would follow $|x(t)| \to \infty$ as $t \to \infty$, a contradiction. Define

$$V(t, x, y) = \left[\, y^2 + 2\int_0^x f(s)ds + M\right]^{1/2} + \int_t^\infty \beta(s)ds$$

where $M > 0$ is chosen such that $2\int_0^x f(s)ds + M > 0$ for every $x \in R$.

Define

$$W(t, x, y) = y^2 h(t, x, y)\left[y^2 + 2\int_0^x f(s)ds + M\right]^{-1/2}$$

and rewrite (8′) as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F + G\,, \tag{10}$$

where

$$F = \begin{pmatrix} y \\ -yh(t, x, y) - f(x) \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 \\ -g(t, x, y) - p(t, x, y) \end{pmatrix}.$$

An easy computation shows that $\dot{V}_{(10)}(t, x, y) \leqq -W(t, x, y)$. Let us apply Theorem 1 with respect to (10), $Q = R^2$. It is not hard to see that the hypotheses (i), (ii) and (iii) with $V(t, x, y)$ and $W(t, x, y)$ defined as above are satisfied. Then the solution $(x(t), y(t))$ of (10), or equivalently of (8′), is contained in some compact set $K \subset R^2$, for $t \geqq$ some t_0. We can say also that $\Omega \subset E \cap K$ where $E = \{(x, y) \in R^2 \mid \liminf_{t\to\infty} W(t, x, y) = 0\}$ and Ω is the ω-limit set of $(x(t), y(t))$. Write now (8′) as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = H + S + G\,, \tag{11}$$

where

$$H = \begin{pmatrix} y \\ -f(x) \end{pmatrix}, \quad S = \begin{pmatrix} 0 \\ -yh(t, x, y) \end{pmatrix} \quad \text{and}$$
$$G = \begin{pmatrix} 0 \\ -g(t, x, y) - p(t, x, y) \end{pmatrix}.$$

Let us apply Theorem 2 with respect to (11), $Q = R^2$ and $A = E \cap K$. As a consequence of hypotheses (H_2) and (H_4) it follows that G satisfies condition (3). Let us show that condition (6) is satisfied with respect to

$$S = \begin{pmatrix} 0 \\ -yh(t, x, y) \end{pmatrix} \quad \text{and} \quad A = E \cap K\,,$$

where $E = \{(x, y) \in R^2 \mid yh^*(x, y) = 0\}$, $h^*(x, y) = \liminf_{t\to\infty} h(t, x, y)$. For every $(x_0, y_0) \in A$ we have $y_0 h^*(x_0, y_0) = 0$ and from (H_6) it follows that $\lim_{t\to\infty} y_0 h(t, x_0, y_0) = 0$. Given a positive ε it is easy to see, as a consequence of (H_5), that there are $T = T(\varepsilon, x_0, y_0)$ and $\delta = \delta(\varepsilon, x_0, y_0)$ such that $|yh(t, x, y)| < \varepsilon$ for $t \geqq T$, $|x - x_0| + |y - y_0| < \delta$, $(x, y) \in K$. As A is compact it follows the existence of $T = T(\varepsilon)$ and $\delta = \delta(\varepsilon)$ so that for

$t \geqq T(\varepsilon)$, $(x, y) \in K$ and $\operatorname{dist}((x, y), A) < \delta$ imply $|S(t, x, y)| < \varepsilon$. Then (6) is satisfied with respect to $S = S(t, x, y)$ and $A = E \cap K$.

It follows from Theorem 1 that the solution $(x(t), y(t))$ of (10) is contained in K, $t \geqq$ some t_0, and that $(x(t), y(t)) \to A = E \cap K$ as $t \to \infty$ since $\Omega \subset E \cap K$. Consequently, by applying Theorem 2, we have that Ω is a nonempty, connected, compact and invariant set with respect to (9′).

We claim that $\Omega \subset R_x$. Suppose that this is not true. Then there exists $(x_0, y_0) \in \Omega$ with $y_0 \neq 0$. As Ω is compact and invariant with respect to (9′) it follows that the orbit γ of (9′), defined by $(x_0, y_0) \in \Omega$ with $y_0 \neq 0$, remains in Ω for all $t \in R$ and is bounded. One can see also that γ is not an equilibrium point of (9′). Then $\gamma \subset \Omega \subset E \cap K$ and consequently $\gamma \subset E = \{(x, y) \in R^2 \mid yh^*(x, y) = 0\}$. Hence, for every $(x, y) \in \gamma$, we have that $yh^*(x, y) = 0$ or, equivalently, $h^*(x, y) = 0$ for every $(x, y) \in \gamma$. By taking into account condition (H_7) we have a contradiction. Then $\Omega \subset R_x$.

As Ω is connected and invariant with respect to (9′) and by considering condition $(\mathrm{H}_1\text{-iii})$, it follows the existence of a real number η such that $\Omega = \{(\eta, 0)\}$. Therefore $x(t) \to \eta$ and $\dot{x}(t) \to 0$, as $t \to \infty$, where η is a real number satisfying $f(\eta) = 0$.

Remark. By considering another set of assumptions for system (8′), we can also show that for every solution $x(t)$ of (8) we have that $x(t) \to$ some point η satisfying $f(\eta) = 0$ and $\dot{x}(t) \to 0$ as $t \to \infty$. [3, Theorem 3]. The basic tool used in [3, Theorem 3] is closely related to the one provided by Theorems 1 and 2. Several results on the subject under consideration can be found in [2, 3].

Theorem 4. *Suppose that some uniqueness condition with respect to the initial value problem holds for* (8′). *Let hypotheses* (H_1)-(H_7) *hold. Let* $p(t, \xi, 0) = 0$, $t \geqq 0$, *where* $(\xi, 0) \in D$. *Then the equilibrium point* $(\xi, 0)$ *of* (8′) *is*

(a) *uniformly stable*

and

(b) *equiasymptotically stable.*

Proof. From Lemma 2 it follows that (a) is satisfied. Therefore, condition (i) of Lemma 1 holds with respect to system (8′) and $\psi(t) = (\xi, 0)$. Let us show that condition (ii) of Lemma 1 is satisfied with respect to system (8′) and $\psi(t) = (\xi, 0)$. To this end it is enough to show that there is a positive $\sigma = \sigma(\xi)$ such that $t_0 \geqq 0$ and $|x_0 - \xi| + |y_0| < \sigma$ imply $(x(t), y(t)) \to (\xi, 0)$ as $t \to \infty$, where $(x(t), y(t))$ is the solution of (8′) satisfying $x(t_0) = x_0$, $y(t_0) = y_0$. Let $\rho = \rho(\xi) > 0$, given by the definition of D, such that $(x-\xi)\cdot f(x)>0$ for $0<|x-\xi|\leqq\rho$. As $(\xi, 0)$ is uniformly stable there is $\delta=\delta(\rho)$ such that $|x_0-\xi|+|y_0|<\delta$ implies $|x(t)-\xi|+|y(t)|<\rho/2$ for $t\geqq t_0$. Theorem 3 implies that there is η, $f(\eta)=0$, such that $x(t)\to\eta$ and $y(t)\to 0$ as $t\to\infty$. Then $|\eta-\xi|\leqq\rho/2$. But this is satisfied only for $\eta=\xi$. Hence $|x_0-\xi|+|y_0|<\delta$ implies $x(t)\to\xi$ and $y(t)\to 0$. Thus, condition (ii) of Lemma 1 is satisfied with respect to system (8′) and $\psi(t)=(\xi, 0)$. Then, by using Lemma 1 with respect to system (8′) and $\psi(t)=(\xi, 0)$, we see that the equilibrium point $(\xi, 0)$ of (8′) is equiasymptotically stable, that is, (b) is satisfied. The proof is complete.

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