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SURFACES OF REVOLUTION WITH PRESCRIBED MEAN CURVATURE

Dedicated to Professor Shiing-shen Chern

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In this paper we study a surface of revolution in the Euclidean three space $\boldsymbol{R}^3$. The generating curve of the surface satisfies a non-linear differential equation which describes the mean curvature.

The purpose of this note is to solve the differential equation by an elementary method. Solutions are represented explicitly by generalized Fresnel's integrals which involve the mean curvature. Therefore, for a given continuous function $H(s)$, we can construct a 3-parameter family of surfaces of revolution admitting $H(s)$ as the mean curvature.

We shall remark that these computations are different from the one in Delaunay [1]. About 140 years ago, he solved the differential equation under the constancy of the mean curvature and gave a method of geometric constructions for such surfaces. For the proof, he first obtains a solution of an evolute of the generating curve. By making use of this solution, he found a representation formula of the generating curve. Therefore these solutions hold only on some intervals on which the evolute can be defined. It seems not to be simple to obtain a global solution from his method.

Our calculations will easily find global solutions and the corollary of the main theorem of this note describes all complete surfaces of revolution with constant mean curvature.

During this research I stayed in Köln and received many nice advices from Professors Peter Dombrowski and Helmut Reckziegel. By their suggestions, my original computations could be simplified and generalized. And also I shall mention that Reckziegel got interested in drawing graphs of these generating curves by making use of the computer and obtained many beautiful pictures. He permitted me with favor to include some of the graphs by his programming in this paper. The author wishes to express his deep gratitude to both of them for helpful

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conversations and kindness.

1. Generating curve of a surface of revolution. Let $(x(s), y(s))$, $s \in I$, be any C^2-curve which is parametrized by the arc length and the domain of definition I is any open interval of real numbers including zero. We define a surface of revolution in R^3 by $M = (x(s), y(s)\cos\theta, y(s)\sin\theta)$, $s \in I$, $0 \leqq \theta \leqq 2\pi$. Then the first and second fundamental forms of M are $ds^2 + y(s)^2 d\theta^2$ and $(x''(s)y'(s) - x'(s)y''(s))ds^2 + x'(s)y(s)d\theta^2$, respectively. By the regularity of the surface we may assume $y(s) > 0$ on I. The mean curvature $H(s)$, by definition, satisfies

$$2H(s)y(s) - x'(s) - x''(s)y(s)y'(s) + x'(s)y(s)y''(s) = 0\,, \quad s \in I\,. \tag{1}$$

We shall study this differential equation under the condition

$$x'(s)^2 + y'(s)^2 = 1\,, \quad s \in I\,. \tag{2}$$

Multiplying (1) by $x'(s)$, it becomes $2H(s)y(s)x'(s) + x'(s)^2(y(s)y''(s) - 1) - x'(s)x''(s)(y(s)y'(s)) = 0$. By making use of (2) and its differentiated formula, we get

$$2H(s)(y(s)x'(s)) + (y(s)y'(s))' - 1 = 0\,. \tag{3}$$

On the other hand we consider another formula obtained from (1) by multiplication of $y'(s)$: $2H(s)y(s)y'(s) - x'(s)y'(s) - x''(s)y(s)(1 - x'(s)^2) + x'(s)y(s)(-x'(s)x''(s)) = 0$, which gives,

$$2H(s)(y(s)y'(s)) - (y(s)x'(s))' = 0\,. \tag{4}$$

Combining (3) and (4) we obtain a first order complex linear differential equation

$$Z'(s) - 2iH(s)Z(s) - 1 = 0\,, \quad s \in I\,, \tag{5}$$

where we put $Z(s) = y(s)y'(s) + iy(s)x'(s)$.

It can be easily solved by an elementary calculus. A general solution of (5) is

$$\begin{aligned} Z(s) = &\left\{\int_0^s \exp\left(-2i\int_0^t H(u)du\right)dt\right\} \exp\left(2i\int_0^s H(t)dt\right) \\ &+ C\exp\left(2i\int_0^s H(t)dt\right), \end{aligned}$$

where C is a complex constant. It is convenient to introduce the following functions:

$$F(s) = \int_0^s \sin\left(2\int_0^u H(t)dt\right)du\,, \qquad G(s) = \int_0^s \cos\left(2\int_0^u H(t)dt\right)du\,.$$

Then the general solution of (5) is represented by

$$Z(s) = \{(F(s) - c_1) + i(G(s) + c_2)\}(F'(s) - iG'(s)) , \tag{6}$$

where we put $iC = -c_1 + ic_2$. Since we have $|Z(s)|^2 = y(s)^2$, we obtain, for some constant C,

$$y(s) = \{(F(s) - c_1)^2 + (G(s) + c_2)^2\}^{1/2} , \quad s \in I . \tag{7}$$

Combining (6), (7) and $Z(s) - \overline{Z(s)} = 2iy(s)x'(s)$, we get

$$x'(s) = \frac{(G(s) + c_2)F'(s) - (F(s) - c_1)G'(s)}{\{(F(s) - c_1)^2 + (G(s) + c_2)^2\}^{1/2}} , \quad s \in I .$$

Thus we obtained a family of generating curves which is denoted by

$$\begin{aligned} &X(s; H(s), c_1, c_2, c_3) \\ &= \left(\int_0^s \frac{(G(t) + c_2)F'(t) - (F(t) - c_1)G'(t)}{\{(F(t) - c_1)^2 + (G(t) + c_2)^2\}^{1/2}} dt + c_3, \{(F(s) - c_1)^2 \right. \\ &\quad \left. + (G(s) + c_2)^2\}^{1/2}\right) , \quad s \in I , \end{aligned} \tag{8}$$

where c_1, c_2 and c_3 are any integral constants. As the geometric meaning of these constants we have

$$X(0; H(s), c_1, c_2, c_3) = (c_3, \{c_1^2 + c_2^2\}^{1/2}) , \tag{9}$$

$$X'(0; H(s), c_1, c_2, c_3) = (c_1(c_1^2 + c_2^2)^{-1/2}, c_2(c_1^2 + c_2^2)^{-1/2}) . \tag{10}$$

Since the integrand of the x-component of (8) converges to zero at a point s_0 satisfying $F(s_0) - c_1 = 0$ and $G(s_0) + c_2 = 0$, the curve is continuous on I. Conversely for any continuous function $H(s)$, $s \in I$, there exists a subset T of R^2 such that for a $(c_1, c_2) \in T$, we have $(F(s) - c_1)^2 + (G(s) + c_2)^2 > 0$, $s \in I$, because we may take T as the set complementary to the regular curve $\{(F(s), -G(s)), s \in I\}$ in R^2. For any vector $(c_1, c_2, c_3) \in T \times R$ and given continuous function $H(s)$, we define a curve $X(s; H(s), c_1, c_2, c_3)$ by (8). Then it is directly verified that the curve is parametrized by the arc length and satisfies (1). Summarizing up these results, we have proved

THEOREM. *Let $(x(s), y(s))$, $s \in I$, be the generating curve, parametrized by the arc length, of a surface of revolution whose mean curvature at the point $(x(s), y(s), 0)$ is given by $H(s)$. Then for some constants c_1, c_2 and c_3 we have $(x(s), y(s)) = X(s; H(s), c_1, c_2, c_3)$, $s \in I$. Conversely for any given continuous function $H(s)$, $s \in I$, we take a vector $(c_1, c_2, c_3) \in T \times R$. Then we can construct a surface of revolution by means of* (8) *in such a way that the mean curvature is $H(s)$ and the initial data are given by* (9) *and* (10).

REMARK 1 (Reckziegel). We have a nice relation between the curvatures of the curves $(F(s), G(s))$ and $(x(s), y(s))$: The curvature $k(s)$ of the curve $(F(s), G(s))$ is given by $-2H(s)$. Conversely for any continuous function $k(s)$, the curve $\left(\int_0^s \sin\left(\int_0^u k(t)dt\right)du,\ \int_0^s \cos\left(\int_0^u k(t)dt\right)du\right)$ has the curvature $-k(s)$. From this curve we can construct the generating curve of a surface of revolution whose mean curvature is $k(s)/2$.

2. Surfaces of revolution with constant mean curvature. We assume that the mean curvature is a constant function. If the constant is zero, then we have $F(s) = 0$ and $G(s) = s$, which gives

$$X(s; 0, c_1, c_2, c_3) = \left(\int_0^s c_1\{c_1^2 + (t + c_2)^2\}^{-1/2}dt + c_3,\ \{c_1^2 + (s + c_2)^2\}^{1/2}\right).$$

This is a catenary for each $c_1 \neq 0$. The corresponding surface of revolution is a catenoid.

The case of the non-zero constant function is interesting. If we put $H(s) = H(\neq 0)$, then (8) gives, after some simplifications and parallel translations of the x-axis and the arc length,

$$X(s; H, B) = \left(\int_0^s \frac{1 + B\sin 2Ht}{\{1 + B^2 + 2B\sin 2Ht\}^{1/2}}dt\,,\right.$$
$$\left.\frac{1}{2|H|}\{1 + B^2 + 2B\sin 2Hs\}^{1/2}\right),\quad s \in R\,, \tag{11}$$

where B is any constant. It is easily verified that (a) $X(s; -H, B) = X(s; H, -B)$, (b) $X(s; H, -B) = X(s - \pi/2H; H, B) +$ a constant vector, (c) $X(s; \lambda H, B) = (1/\lambda)X(\lambda s; H, B)$, $\lambda > 0$. Therefore it is enough to consider only the cases of $B \geqq 0$ and $H > 0$.

$X(s; H, 0)$ is clearly a generating line for a circular cylinder. $X(s; H, 1)$ is the only continuous curve and represents a sequence of continuous half circles over the x-axis which have the same radii. The corresponding surface is a sequence of continuous spheres which have the same radii. According as $0 < B < 1$ or $B > 1$, the smooth curves $X(s; H, B)$ have different figures. If we assume $0 < B < 1$, then the function $x(s)$ is monotone increasing as the parameter s goes to the positive infinity but in the case of $B > 1$, it is not monotone. Nevertheless we know $\lim x(s) = \infty$ $(s \to \infty)$ in both cases, because $X(s; H, B)$ is periodic and has the period π/H, which are proved in the following way: We have

$$X(s + \pi/H; H, B) = X(s; H, B) + \left(\frac{1}{2H}\int_0^\pi g(t)dt,\ 0\right),$$

where we will define, for each $B \geqq 0$,

$$g(t) = (1 - B\sin t)(1 + B^2 - 2B\sin t)^{-1/2} + (1 + B\sin t)(1 + B^2 + 2B\sin t)^{-1/2}\,, \quad 0 \leqq t \leqq \pi\,.$$

This satisfies $g((\pi/2) + t) = g((\pi/2) - t)$. In the case of $0 \leqq B \leqq 1$, we have clearly $\int_0^\pi g(t)dt > 0$. If B is greater than one, then by simple calculations we can show $g(t) > 0$ on $(0, \pi/2)$, which also implies $\int_0^\pi g(t)dt > 0$. Therefore we obtained the 1-parameter family of complete surfaces of revolution with the same constant mean curvature. This is Delaunay's theorem for complete metrics.

COROLLARY. *Any complete surface of revolution with constant mean curvature is a sphere, a catenoid, or a surface whose generating curve is given by $X(s; H, B)$ for some B.*

REMARK 2. By (11) we have $y^2(s) - (1/H)y(s)x'(s) + (1 - B^2)\,4H^2 = 0$. This differential equation was studied by Sturm [1]. He derived this equation from the Euler equation of some variational problem. By his results, one can see that the curve $X(s; H, B)$ has the geometric characterization by Delaunay: In order to see the generating curve of a surface of revolution with constant mean curvature, let us roll along the axis an ellipse or a hyperbola of which the major axis is equal to $1/H$. Then the focus will describe the generating curve which we seek.

REMARK 3. In [2] Eells explains the work of Delaunay in connection with the theory of harmonic mappings. I came to know the paper in a conversation with Professor D. Ferus in Berlin. Professor S. S. Chern kindly also informed me of the paper.

In Figure 1, which was drawn by the programming of Reckziegel and the computer of Universität zu Köln, any curve represents the

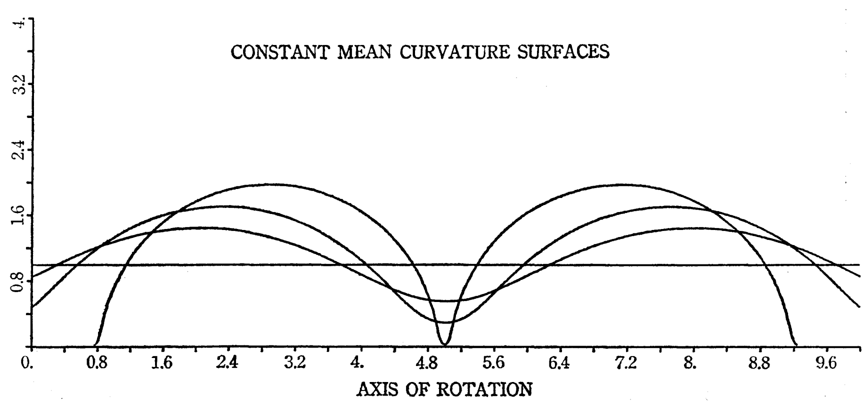

FIGURE 1

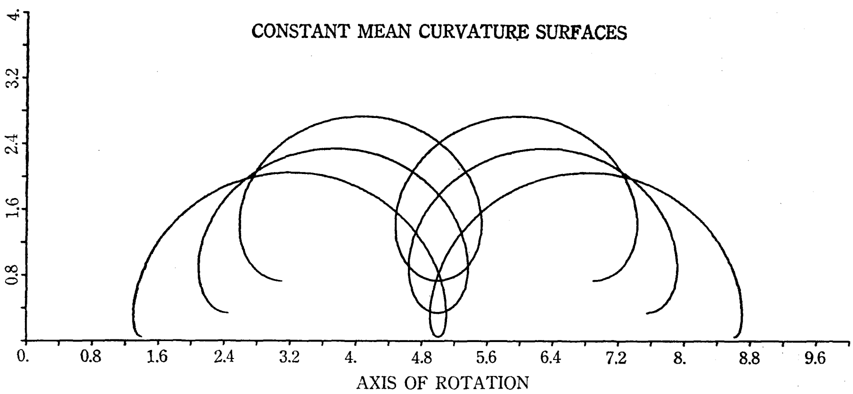

FIGURE 2

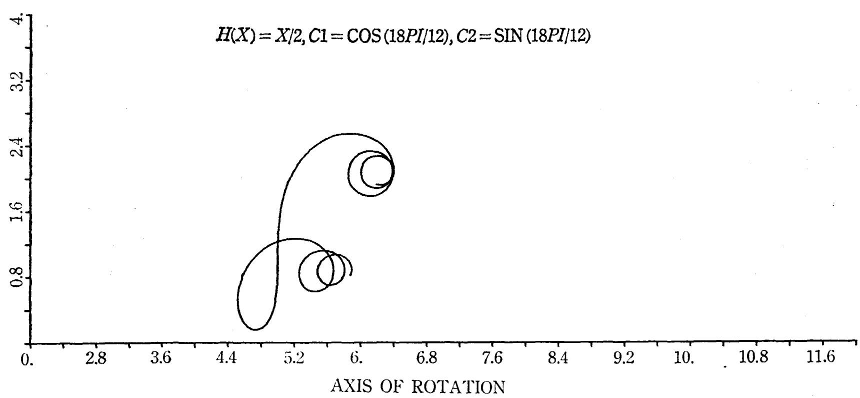

FIGURE 3

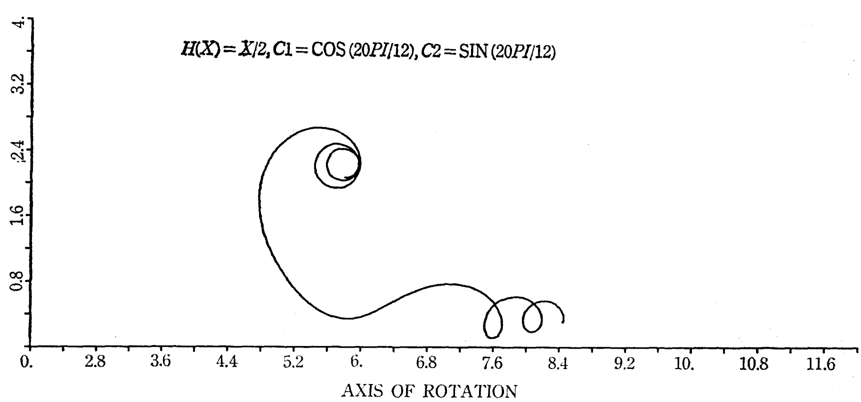

FIGURE 4

generating curve of a surface of revolution whose mean curvature is $1/2$ and $0 \leqq B < 1$. Figure 2 means the case of $H = 1/2$ and $B > 1$. We shall remark that in Figures 1 and 2, all curves are arranged in such a way that they have the same tangent vectors $(1, 0)$ and $(-1, 0)$, respectively, at $(5, |1 - B|)$.

3. Surfaces of revolution with $H(s) = s/2$. If we take $H(s) = s/2$ as the mean curvature of a surface of revolution, then we have

$$(F(s), G(s)) = \left(\int_0^s \sin(u^2/2)du\ ,\ \int_0^s \cos(u^2/2)du\right),$$

which is the spiral of Cornu and each component functions are Fresnel's integrals. From this famous curve, we can construct many surfaces of revolution admitting $s/2$ as the mean curvature. Figures 3 and 4 represent two of those generating curves whose initial data are $c_1 = 0$, $c_2 = -1$ and $c_3 = 5$ and $c_1 = 1/2$, $c_2 = -\sqrt{3}/2$ and $c_3 = 5$, respectively.

REFERENCES

[1] C. DELAUNAY, Sur la surface de révolution dont la courbure moyenne est constante, J. Math. Pures. Appl. Ser. 1, 6 (1841), 309–320. With a note appended by M. Sturm.
[2] J. EELLS, On the surfaces of Delaunay and their Gauss maps, to appear.

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