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LEAVES WITH NON-EXACT POLYNOMIAL GROWTH

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1. Introduction. Let $\mathscr{F}$ be a codimension one foliation of class C^r $(r \geqq 0)$ of a compact manifold M which is tangent to the boundary. Let $M = \bigcup_{i=0}^{m} U_i$ be a finite cover of M by regular distinguished charts. Let F be a leaf and $P_0 \subset F \cap U_0$ a plaque contained in F. The *growth function* of F at P_0, $f: \boldsymbol{Z}^+ \to \boldsymbol{Z}^+$ is defined by

$f(n) =$ the number of distinct plaques which can be reached from the initial plaque P_0 by a plaque chain of length at most n (see [5], [6]).

DEFINITION. F has *polynomial growth* if the growth function f of F is dominated by a polynomial. In this case we define the *upper growth* and *lower growth* of F, denoted $u.gr(F)$ and $l.gr(F)$ respectively, as follows:

$$u.gr(F) = \inf \{d \in \boldsymbol{R}^+ \mid f \text{ is dominated by the polynomial } g(n) = n^d\}$$
$$l.gr(F) = \sup \{d \in \boldsymbol{R}^+ \mid f \text{ dominates the polynomial } g(n) = n^d\} .$$

Finally, we say F has *exact polynomial growth* of degree d if the growth function of F dominates a polynomial of degree d and is dominated by a polynomial of degree d.

It is easy to see that the upper and lower growth of a leaf depend neither on the choice of regular distinguished charts nor on the choice of the initial plaque ([5]).

In [4] Hector posed the following problem: if F has polynomial growth, then does F have exact polynomial growth of an integer degree? In this paper we give two partial answers to this problem. We remark that the answer is affirmative if the foliation is transversely analytic or it is almost without holonomy ([7]).

THEOREM A. *Let $\mathscr{F}$ be a codimension one foliation of class C^r $(r \geqq 2)$ of a compact manifold M which is tangent to the boundary. Let F be a leaf of $\mathscr{F}$. Assume that the growth function of F is dominated by a polynomial of degree* 2. *Then F has exact polynomial growth of degree* 0, 1 *or* 2.

THEOREM B. *There exists a codimension one foliation of class C^0 of a closed three manifold with the following properties. For each integer $d \geqq 2$, there are leaves F_d and F'_d such that*

$$d < l.gr(F_d) = u.gr(F_d) < d + 1$$

and

$$d < l.gr(F'_d) < u.gr(F'_d) < d + 1 .$$

In the second section we prove Theorem A. The proof depends heavily on our preceding results ([6], [7]). In the third section we prove Theorem B.

2. Leaves with linear or quadratic growth. We recall from [7] the definition of the *proper depth* of a leaf. Let F_1 and F_2 be leaves of a foliation. By $F_1 > F_2$, we mean that F_2 is contained in the limit set of F_1 and $F_1 \neq F_2$. The *proper depth* of a leaf F, denoted $pd(F)$, is defined as follows:

$$pd(F) = \sup\{k \mid \text{there exists a sequence } F_0, F_1, \cdots, F_k \text{ of leaves such that } F_0 < F_1 < \cdots < F_k = F \text{ and } F_i \text{ is proper for } i < k\} .$$

The following theorem follows from the proof of [6, Theorem 2].

THEOREM 2.1. *Let $\mathscr{F}$ be a codimension one foliation of class C^r $(r \geqq 1)$ of a compact manifold. Then for each leaf F of $\mathscr{F}$, the growth function of F dominates a polynomial of degree $pd(F)$.*

Let U be a connected open saturated subset of a foliated manifold. We say U is *nice* if the closed saturated set $\bar{U} - U$ consists of finitely many proper leaves and each leaf in U has trivial holonomy group ([7]). A nice saturated set U is *minimal* if U contains no non-empty proper relatively closed saturated subset. In [1], Cantwell and Conlon proved the following theorem.

THEOREM 2.2 (Cantwell-Conlon). *Let F be a leaf with polynomial growth in a transversely orientable codimension one foliation of class C^r $(r \geqq 2)$ of a compact manifold. Then one of the following conditions is satisfied.*

(1) *All leaves contained in the closure of F are proper.*

(2) *The closure of F is the closure of a minimal nice saturated set.*

After these preliminaries we are ready to prove Theorem A.

PROOF OF THEOREM A. Since the growth of a leaf is unchanged when we pass to a finite cover of M, we may assume $\mathscr{F}$ is transversely orientable. If $u.gr(F) \leqq 1$, then we have proved in [7, (7.2)] that F has exact polynomial growth of degree 0 or 1. So we assume $1 < u.gr(F) \leqq 2$. Then $pd(F) = 0$, 1 or 2 by Theorem 2.1.

If $pd(F) = 2$, then the growth function of F dominates a polynomial of degree 2 by (2.1), and F has exact quadratic growth.

Assume $pd(F) = 1$. We show F is non-proper. In fact, if F is proper, then by the Theorem of Cantwell-Conlon (2.2), the limit set of F consists of compact leaves. Then by [7, Theorem 3], F has linear growth. This contradiction shows that F is a non-proper leaf. Then by (2.2), the closure of F is the closure of a minimal nice saturated set. Since $pd(F)=1$, each proper leaf contained in the closure of F is compact. From [7, (5.7)], it follows that each leaf contained in the closure of F has abelian holonomy group on the side approached by F. So we can apply [7, Theorem 2] and conclude that F has exact quadratic growth.

Finally assume $pd(F) = 0$. Then $\mathscr{F}$ is without holonomy again by the Theorem of Cantwell-Conlon. By [7, Theorem 2], every leaf of $\mathscr{F}$ has exact quadratic growth. This completes the proof of Theorem A.

3. Leaves with non-exact polynomial growth. Let S denote the closed orientable surface of genus 2 and let $d \geqq 2$ be an integer. We shall construct a foliation of $S \times [-1, 1]$ which contains a leaf F such that $d < l.gr(F) \leqq u.gr(F) < d + 1$.

If f and g are two order preserving homeomorphisms of $[-1, 1]$, there exists a representation of the fundamental group of S into the group of homeomorphisms of $[-1, 1]$ whose image is the group generated by f and g. Using the construction of [2, (1.8)], we get a C^0 foliation of the trivial bundle $S \times [-1, 1]$ transverse to the fibres.

Let f be a diffeomorphism of $[-1, 1]$ such that $f(x) > x$ for each $x \in (-1, 1)$. Let $a_0 = 0$ and $a_n = f^n(0)$, for each $n \in \boldsymbol{Z}$. Let h be a diffeomorphism of $[-1, 1]$ whose support is $[a_0, a_1]$, $h(x) > x$ for $x \in (a_0, a_1)$ and h is embedded in a one-parameter subgroup of diffeomorphisms of $[-1, 1]$. So we have h^α for each $\alpha \in \boldsymbol{R}$. Choose real numbers $\alpha_1, \alpha_2, \cdots, \alpha_{d-2}$ such that $1, \alpha_1, \cdots, \alpha_{d-2}$ are linearly independent over the rationals. Finally choose an increasing sequence $2 \leqq N(1) < N(2) < N(3) < \cdots$ of positive integers.

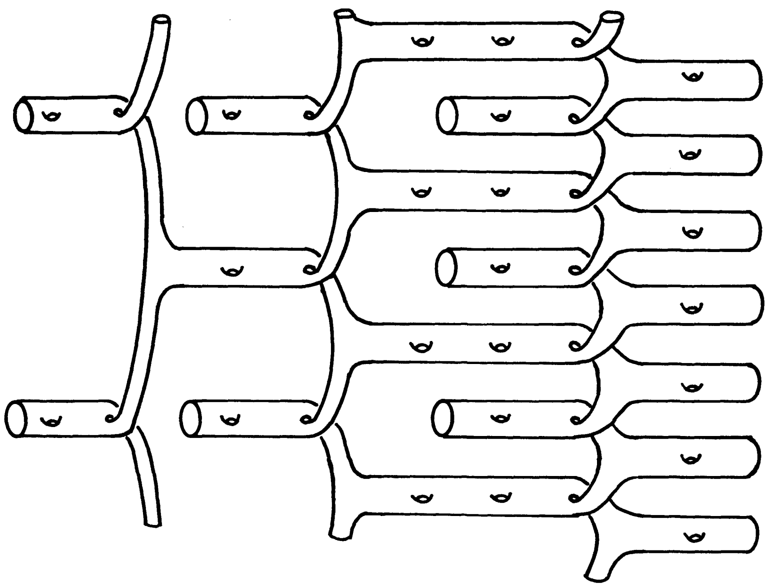
Using these data, we define a homeomorphism g of $[-1, 1]$ as follows:

$$g(x) = \begin{cases} f^{-i} \circ h^{\alpha_i} \circ f^i(x) & \text{for } x \in [a_{-i}, a_{-i+1}],\ i = 1, \cdots, d-2 \\ h(x) & \text{for } x \in [a_0, a_1] \\ f^{N(k)} \circ h^{2^{-k}} \circ f^{-N(k)}(x) & \text{for } x \in [a_{N(k)}, a_{N(k)+1}],\ k \geqq 1 \\ x & \text{otherwise}. \end{cases}$$

It is easy to see that g is a homeomorphism of $[-1, 1]$ and is smooth on $(-1, 1)$.

Let $\mathscr{F}$ denote the resulting foliation of $S \times [-1, 1]$. We identify $[-1, 1]$ with a fibre over a base point of S. Note that the saturation of (a_0, a_1) is a minimal nice saturated set.

Let $x \in (a_0, a_1)$ and let F be the leaf through x. We shall study the growth type of F (see Figure). Let T denote the set $F \cap [-1, 1]$ and



FIGURE

G the subgroup of the group of homeomorphisms of $[-1, 1]$ generated by f and g. The length of elements of G with respect to the generating set $\{f, g\}$ induces a natural metric δ on T. Let $x' \in T$. Following Hector [3], we say an element c of G is a short-cut from x to x' if $x' = c(x)$ and length$(c) = \delta(x, x')$.

For a given $x' \in T$, we shall see that there exists a canonical form of short-cuts from x to x'. First, there exists uniquely an integer r such that $f^{-r}(x')$ is contained in (a_0, a_1). Secondly, since $f^{-r}(x') \in G \cdot x$, there exist uniquely an integer q with $0 \leqq q \leqq d-2$, sequence of integers $(\beta_1, \beta_2, \cdots, \beta_q)$ with $1 \leqq \beta_1 < \beta_2 < \cdots \beta_q \leqq d-2$, $(r_1, r_2, \cdots, r_q) \in \boldsymbol{Z}^q$ and a rational number s such that for $r' = r_1 \cdot \alpha_{\beta_1} + r_2 \cdot \alpha_{\beta_2} + \cdots + r_q \cdot \alpha_{\beta_q} + s$ we have $f^{-r}(x') = h^{r'}(x)$.

Let $[s]$ denote the integer part of s. Then there exist uniquely a positive integer p and a sequence of positive integers $(i_1, i_2, \cdots, i_p)$, $i_1 < i_2 < \cdots < i_p$, such that

$$s = [s] + \operatorname{sign}(s) \cdot \{2^{-i_1} + 2^{-i_2} + \cdots + 2^{-i_p}\} .$$

We call the sequence $\{r; (\beta_1, \cdots, \beta_q), (r_1, \cdots, r_q); s, p, (i_1, \cdots, i_p)\}$ the *G-coordinate* of the point x'.

LEMMA 3.1. *Let $x' \in T$. If the G-coordinate of x' is*

$$\{r; (\beta_1, \cdots, \beta_q), (r_1, \cdots, r_q); s, p, (i_1, \cdots, i_p)\} ,$$

then the element $c_{x'}$ of G defined by the following formula is a short-cut from x to x':

$$\begin{aligned} c_{x'} = f^r \circ g^{\varepsilon} \circ f^{N(i_p)-N(i_{p-1})} \circ g^{\varepsilon} \circ f^{N(i_{p-1})-N(i_{p-2})} \circ g^{\varepsilon} \circ \cdots \circ g^{\varepsilon} \circ f^{N(i_1)} \\ \circ g^{[s]} \circ f^{\beta_1} \circ g^{r_1} \circ f^{-\beta_1+\beta_2} \circ g^{r_2} \circ \cdots \circ f^{-\beta_{q-1}+\beta_q} \circ g^{r_q} \circ f^{-\beta_q} , \end{aligned}$$

where $\varepsilon = \operatorname{sign}(s)$.

The proof is easy but the details are long and tedious so we omit it.

A short-cut in the form of (3.1) will be called a canonical short-cut, and the number p in (3.1) will be called the rank of a canonical short-cut.

Let $\gamma(m)$ (resp. $\gamma_p(m)$) denote the number of distinct points in T that can be reached from x by the application of canonical short-cuts (resp. canonical short-cuts of rank p) of length at most m. If $N(p) \leqq m < N(p+1)$, then $\gamma(m) = \gamma_0(m) + \gamma_1(m) + \cdots + \gamma_p(m)$. It is well-known that the growth type of F can be calculated in terms of the growth function $\gamma(m)$ of T (see e.g., [7, (2.1)]).

Let $\alpha(m)$ denote the growth function of the free abelian group of rank d with respect to the canonical generators. It is easy to see that there exist positive numbers C, C' and m_0 such that $C \cdot m^d \leqq \alpha(m) \leqq C' \cdot m^d$ for $m \geqq m_0$ (see e.g., [8]).

LEMMA 3.2. *If $N(p) \leqq m < N(p+1)$, then we have*

$$2^{p-1} \cdot \alpha(m - 2d + 4 - N(p-1) - 2p) \leqq \gamma(m) \leqq 2^p \cdot \alpha(m) .$$

PROOF. By the explicit forms of canonical short-cuts in (3.1), it is easy to see that we have, for each r, $0 \leqq r \leqq p$,

$$\begin{aligned} \sum_{i=r}^{p} \binom{i-1}{r-1} \cdot \alpha(m - 2d + 4 - N(i) - r) \leqq \gamma_r(m) \\ \leqq \sum_{i=r}^{p} \binom{i-1}{r-1} \cdot \alpha(m - N(i) - r) . \end{aligned}$$

The desired inequality is easily obtained by adding these inequalities for $r = 0, 1, \cdots, p$.

PROPOSITION 3.3. *Let $A > 2$ be an integer. If $N(i) = A^i$ for each i, then we have $l.gr(F) = u.gr(F) = d + \log 2/\log A$.*

PROOF. If m is sufficiently large and $N(p) \leqq m < N(p+1)$, then from the inequality in (3.2) it follows that $\gamma(m) \geqq \gamma(N(p)) \geqq 2^{p-1} \cdot \alpha(N(p) - N(p-1) - 2d + 4 - p) \geqq C \cdot 2^{p-1} \cdot A^{(p-1)d} \geqq C \cdot m^{(d+\log 2/\log A)(p-1)/(p+1)}$. Thus we have proved $l.gr(F) = l.gr(\gamma) \geqq d + \log 2/\log A$. Similarly, we can prove $u.gr(\gamma) \leqq d + \log 2/\log A$.

PROPOSITION 3.4. *Let A and B be positive integers such that $\log((A+1)/A) < \log 2/2d$ and $\log B > 2\log(A+1)$. Assume that the sequence $N(i)$ satisfies the following conditions:*

(1) $(A+1)^i \leqq N(i) \leqq B^i$,

(2) $N(i) - N(i-1) \geqq A^{i-1} + 2d - 4 + i$ *for each i, and*

(3) *there exists an infinite sequence $\{p_i\}$ of positive integers such that $N(p_i) = (A+1)^{p_i}$ and $N(p_{i+1}) = B^{p_i+1}$.*

Then we have

$$d < l.gr(F) = d + \log 2/\log B < d\log A/\log(A+1) + \log 2/\log(A+1) \leqq u.gr(F) \leqq d + \log 2/\log A < d + 1 .$$

PROOF. Note that the conditions on A and B imply $d + \log 2/\log B < d\log A/\log(A+1) + \log 2/\log(A+1) < d + \log 2/\log A < d + 1$. By the condition (1), it is easy to see that $d + \log 2/\log B \leqq l.gr(F) \leqq u.gr(F) \leqq d + \log 2/\log A$. We prove $l.gr(F) = d + \log 2/\log B$. From the condition (3) and the inequality in (3.2), it follows that

$$\gamma(B^{p_i+1}) \leqq C' B^{(p_i+1)(d+\log 2/\log B)}, \qquad \text{for each } i .$$

This shows $l.gr(F) \leqq d + \log 2/\log B$. Thus we have proved $l.gr(F) = d + \log 2/\log B$.

Finally we prove $u.gr(F) \geqq d\log A/\log(A+1) + \log 2/\log(A+1)$. From conditions (2), (3) and the inequality in (3.2), we have, for large i,

$$\gamma((A+1)^{p_i}) \geqq 2^{p_i-1} \cdot \alpha(N(p_i) - N(p_{i-1}) - 2p_i) \geqq C \cdot \{(A+1)^{p_i}\}^{(p_i-1)/p_i\{d\log A/\log(A+1)+\log 2/\log(A+1)\}} .$$

This shows $u.gr(F) \geqq d\log A/\log(A+1) + \log 2/\log(A+1)$. Thus the proposition is proved.

PROOF OF THEOREM B. By (3.3) and (3.4), there exist, for each $d \geqq 2$, foliations $\mathscr{F}_d$ and $\mathscr{F}'_d$ of $S \times [-1, 1]$ and leaves F_d and F'_d of $\mathscr{F}_d$ and $\mathscr{F}'_d$, respectively, such that $d < l.gr(F_d) = u.gr(F_d) < d + 1$ and

$d < l.gr(F'_d) < u.gr(F'_d) < d+1$. The foliation mentioned in Theorem B is easily constructed by pasting the foliations $\mathscr{F}_d$, $\mathscr{F}'_d$, $d = 2, 3, \cdots$. This completes the proof.

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