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ON REPRESENTATIONS OF NON-TYPE I GROUPS

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For a unitary representation $g \to U_g$ of a separable locally compact group G on a Hilbert space $\mathscr{H}$, let M be the smallest W^*-algebra on $\mathscr{H}$ generated by $\{U_g \mid g \in G\}$.

When M is a type I (resp. type II, type III) W^*-algebra, we say that the representation $g \to U_g$ is of type I (resp. type II, type III).

A separable locally compact group is called a type I group if all its unitary representations are of type I. For example, commutative groups, compact groups, connected semi-simple Lie groups, connected nilpotent Lie groups and solvable Lie groups of exponential type are type I groups ([4], [10]).

In this paper, by a *non-type I group* we mean a group which is not of type I.

For unitary representations of non-type I groups, the following is known ([5]). A separable locally compact group G has a faithful type II unitary representation if and only if it has a faithful type III unitary representation. Therefore, non-type I groups have type II as well as type III unitary representations.

In this paper, we shall construct type II and type III factors associated with unitary representations of some concrete semi-direct product groups.

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1. **Preliminaries.** 1. Let the locally compact group G admit a commutative closed normal subgroup N and let G also contain a closed subgroup H such that $N \cap H = \{e\}$, where e is the identity of G, and $NH = G$. Then H is isomorphic to G/N and every element in G is uniquely expressed as a product nh where $n \in N$ and $h \in H$. One says that G is a *semi-direct product group* of N and H and denotes it by $G = N \times_s H$.

Let $\boldsymbol{R}$ denote the additive group of real numbers and $\boldsymbol{C}^2$ the product of two copies of the field of complex numbers. If (z_1, z_2) denotes an element of C^2, we will define the action of $t \in R$ on C^2 by

$$t(z_1, z_2) = (e^{it}z_1, e^{i\alpha t}z_2) ,$$

where α is an irrational number and $i = \sqrt{-1}$.

Let M_5 denote the semi-direct product $R \times_s C^2$, where R acts on C^2 as above. M_5 is called the *Mautner group*. For $\zeta = (t; z_1, z_2)$, $\zeta' = (t'; z_1', z_2') \in R \times_s C^2$,

$$\zeta \circ \zeta' = (t + t'; z_1 + e^{it}z_1', z_2 + e^{i\alpha t}z_2') .$$

Let $M_{11} = R^3 \times_s C^4$, where the action of R^3 on C^4 is given by the matrix below relative to a basis (z_1, z_2, z_3, z_4) of C^4 and (t_1, t_2, t_3) of R^3

$$\begin{pmatrix} e^{it_1}e^{it_2} & 0 & 0 & 0 \\ 0 & e^{it_3}e^{i\alpha t_2} & 0 & 0 \\ 0 & 0 & e^{it_1}e^{it_3} & 0 \\ 0 & 0 & 0 & e^{it_3} \end{pmatrix}$$

with α an irrational number. That is, for $\zeta = ((t_1, t_2, t_3); z_1, z_2, z_3, z_4)$, $\zeta' = ((t_1', t_2', t_3'); z_1', z_2', z_3', z_4') \in M_{11} = R^3 \times_s C^4$,

$$\zeta \circ \zeta' = ((t_1 + t_1', t_2 + t_2', t_3 + t_3'); z_1 + e^{i(t_1+t_2)}z_1', z_2 + e^{i(t_3+\alpha t_2)}z_2' ,$$
$$z_3 + e^{i(t_1+t_3)}z_3', z_4 + e^{it_3}z_4') .$$

M_{11} is called an *extended Mautner group*.

It is known that M_5 and M_{11} are non-type I solvable Lie groups ([1]).

2. Let G be a separable locally compact group. Let E be a locally compact space on which G acts on the right such that (1) $(x)g_1g_2 = (xg_1)g_2$, (2) $xe = x$, where e is the identity of G, (3) $(x, g) \rightarrow xg$ is a continuous mapping from $E \times G$ to E.

For a positive Radon measure μ on E, let μ_g be the measure on E defined by $\mu_g(F) = \mu(Fg)$ for each measurable subset of E.

Let μ be a positive Radon measure on E which is quasi-invariant under the action of G, i.e., μ_g and μ are absolutely continuous. The triple (G, E, μ) is called a *dynamical system*.

Let $L = L_{E \times G}$ be the set of all continuous complex valued functions on $E \times G$ with compact support.

A non-negative continuous function $\rho(x, \alpha)$ on $E \times G$ is called a *multiplier* if it satisfies

$$\rho(x, \alpha\beta) = \rho(x\alpha, \beta)\rho(x, \alpha)$$

for each $x \in E$, $\alpha, \beta \in G$.

Suppose that there exists a positive continuous function $\gamma(x, \alpha)$ on

$E\times G$ such that $d\mu(x\alpha)=\gamma(x,\alpha)d\mu(x)$. Then $\gamma(x,\alpha)$ is a multiplier. Indeed,

$$\begin{aligned} d\mu(x\alpha\beta) = \gamma(x, \alpha\beta)d\mu(x) &= \gamma(x\alpha, \beta)d\mu(x\alpha) \\ &= \gamma(x\alpha, \beta)\gamma(x, \alpha)d\mu(x) . \end{aligned}$$

For $f, g \in L$, define

$$(f, g) = \iint f(x, \alpha)\overline{g(x, \alpha)}\chi(\alpha)d\alpha d\mu(x) ,$$

where $\chi(\alpha)$ is a non-negative continuous function on G such that $\chi(\alpha\beta)=\chi(\alpha)\chi(\beta)$.

Then L is a prehilbert space. Let $\mathscr{H}$ be the Hilbert space which is the completion of L.

Let ψ be a complex valued, measurable and essentially bounded function on (E, μ). For each $f\in\mathscr{H}$, define

$$L_\psi f(x, \alpha) = \psi(x\alpha)f(x, \alpha)$$
$$(\text{resp. } L'_\psi f(x, \alpha) = \psi(x)f(x, \alpha)) .$$

Then L_ψ (resp. L'_ψ) is a bounded operator on $\mathscr{H}$.

Next, we shall define a unitary operator U_α (resp. U'_α), for each $\alpha\in G$, on $\mathscr{H}$ by

$$U_{\alpha_0}f(x, \alpha) = \Delta(\alpha_0)^{-1/2}\chi(\alpha_0)^{-1/2}f(x, \alpha_0^{-1}\alpha)$$
$$(\text{resp. } U'_{\alpha_0}f(x, \alpha) = \gamma(x, \alpha_0)^{1/2}\chi(\alpha_0)^{-1/2}f(x\alpha_0, \alpha\alpha_0))$$

where Δ is the modular function of G.

Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a W^*-subalgebra of $B(\mathscr{H})$ generated by $\{L_\psi | \psi\in L^\infty(E, \mu)\}$ and $\{U_\alpha | \alpha\in G\}$ (resp. $\{L'_\psi | \psi\in L^\infty(E, \mu)\}$ and $\{U'_\alpha | \alpha\in G\}$).

DEFINITION 1. (1) A dynamical system (G, E, μ) is called *free* if for any $\alpha\in G$ $(\alpha\neq e)$, the set of points satisfying the condition $x = x\alpha$ $(x\in E)$ is of μ-measure 0.

(2) A dynamical system (G, E, μ) is called *ergodic* if $Fg = F$ for a measurable set F and for every $g\in G$ implies either $\mu(F)=0$ or $\mu(E\setminus F)=0$.

LEMMA 1 ([2, Theorem 6]). *Let (G, E, μ) be free. $\mathscr{F}$ (and $\mathscr{F}'$) is a factor if and only if (G, E, μ) is ergodic.*

DEFINITION 2. A dynamical system (G, E, μ) is called *measurable* if there exists a positive measurable function ψ on (E, μ) such that

$$\psi(x\alpha)\psi(x)^{-1} = \Delta(\alpha)\gamma(x, \alpha)^{-1} , \qquad \alpha\in G .$$

LEMMA 2. *A dynamical system (G, E, μ) is measurable if and only if there exists a σ-finite positive measure ν which is equivalent to μ and $d\nu(x\alpha) = \Delta(\alpha)d\nu(x)$, $\alpha \in G$, that is, ν is invariant under G.*

PROOF. For any integrable set A, put

$$\nu(A) = \int_A \psi(x)d\mu(x) .$$

Then ν is a σ-finite positive measure and equivalent to μ. Moreover, for each $\alpha \in G$

$$\begin{aligned}\nu(A\alpha) &= \int_A \psi(x\alpha)d\mu(x\alpha) = \int_A \psi(x)\Delta(\alpha)\gamma(x, \alpha)^{-1}\gamma(x, \alpha)d\mu(x) \\ &= \Delta(\alpha)\nu(A) .\end{aligned}$$

Therefore,

$$d\nu(x\alpha) = \Delta(\alpha)d\nu(x) .$$

Conversely, let ν be a σ-finite positive measure which is equivalent to μ and $d\nu(x\alpha) = \Delta(\alpha)d\nu(x)$. Then, by Lebesgue-Nikodym's theorem, there exists a measurable function $\psi(x)$ on (E, μ) such that $0 < \psi(x) < \infty$ and $\nu(A) = \int_A \psi(x)d\mu(x)$ for all integrable set A. Therefore,

$$\begin{aligned}\Delta(\alpha)\int_A \psi(x)d\mu(x) &= \Delta(\alpha)\nu(A) = \nu(A\alpha) = \int_{A\alpha} \psi(x)d\mu(x) \\ &= \int_A \psi(x\alpha)\gamma(x, \alpha)d\mu(x) .\end{aligned}$$

Consequencely,

$$\Delta(\alpha)\psi(x) = \psi(x\alpha)\gamma(x, \alpha) .$$

LEMMA 3 ([2, Proposition 12]). *If (G, E, μ) is free, ergodic and measurable, then $\mathscr{F}$ (and $\mathscr{F}'$) is a type* I *or type* II *factor. In particular, if G is not a discrete group, then $\mathscr{F}$ (and $\mathscr{F}'$) is a type* I_∞ *or type* II_∞ *factor.*

LEMMA 4 ([2, Theorem 7]). *If (G, E, μ) is free, ergodic and non-measurable, then $\mathscr{F}$ (and $\mathscr{F}'$) is a type* III *factor.*

REMARK. $\mathscr{F}$ is considered as a continuous crossed product $G\times_{u_\alpha}L^\infty(E, \mu)$ ([11]).

2. Direct integrals of irreducible representations. A *topological transformation group* (G, Ω) is a topological group G together with a locally compact space Ω and a continuous mapping: $(g, \omega) \rightarrow g\omega$ of $G\times\Omega$

into Ω such that $(gh)\omega = g(h\omega)$, and if e is the identity of G, $e\omega = \omega$ for all $g, h \in G$, and $\omega \in \Omega$.

(G, Ω) is *polonais* if G and Ω are polonais, i.e., they are separable and metrizable by a complete metric. (G, Ω) satisfies the *condition*$(*)$ if each neighborhood U of e in G contains a neighborhood W of e such that for all ω in Ω, $\mathrm{Cl}[W\omega] \subseteqq U\omega$. $\mathrm{Cl}[X]$ indicates the closure of X. If G is locally compact and Ω is Hausdorff, (G, Ω) satisfies the condition$(*)$, as one may take W to be any compact neighborhood of e with $W \subseteqq U$.

A Borel measure μ on Ω is *non-trivially ergodic* if it is not concentrated at an orbit.

LEMMA 5 ([3]). *Let (G, Ω) be a polonais transformation group satisfying the condition$(*)$. Then the orbit space Ω/G is a T_0-space if and only if Ω has no non-trivially ergodic measure.*

For a separable locally compact group G, let G^{ir} be the standard Borel space of all irreducible representations of G ([7]).

PROPOSITION 1. *Let (Γ, Σ) be a polonais transformation group satisfying the condition$(*)$, μ an ergodic Borel measure in Σ, and $\sigma \to \Pi^\sigma$ a Borel function from Σ to G^{ir}. If for any $\sigma \in \Sigma$, Π^σ is equivalent to $\Pi^{\gamma(\sigma)}$ for all $\gamma \in \Gamma$, then the direct integral $K = \int_\Sigma \Pi^\sigma d\mu(\sigma)$ is a factor representation. Moreover, if μ is non-trivially ergodic, then K is a non-type* I *factor representation.*

PROOF. Suppose K is not a factor representation. Then, there exists a projection $E \neq 0, I$ in $K(G)' \cap K(G)''$, where $K(G)$ is the algebra generated by the representation $\{K(g) | g \in G\}$ and $K(G)'$ is the commutant of $K(G)$.

Since Π^σ, $\sigma \in \Sigma$, are irreducible, the Boolean algebra of projections associated with the direct integral is maximal in $K(G)'$, and must contain E. Let B be a Borel set with $0 \neq \mu(B) \neq \mu(\Sigma)$ and $E = \int \chi_B(\sigma) I d\mu(\sigma)$, where $\chi_B(\sigma)$ is the characteristic function of B. K is proper because each Π^σ is irreducible. By the double commutant theorem, the unit ball of $K(G)$ is strongly dense in the unit ball of $K(G)''$. The latter being metrizable in the strong topology, there is a sequence $\{g_n\} \subset G$ with $K(g_n) \to E$ strongly i.e., $\int \Pi^\sigma(g_n) d\mu(\sigma) \to \int \chi_B(\sigma) I d\mu(\sigma)$. There is a subsequence $\{g_{n_k}\}$ and a null set N of Σ such that $\Pi^\sigma(g_{n_k}) \to \chi_B(\sigma) I$ strongly for all $\sigma \in \Sigma \setminus N$. Changing notation, we assume that $\Pi^\sigma(g_n) \to \chi_B(\sigma) I$ strongly for all $\sigma \in \Sigma \setminus N$.

Let $A = \Gamma(B \setminus N)$. As $B \setminus N$ is Borel, A is analytic (it is the image of $\Gamma\times(B\setminus N)$ under a continuous map), and hence measurable. A is invariant, and we claim that $\mu(A)=\mu(B)$. As $\mu(N)=0$ and $B\setminus N\subseteqq A$, it suffices to show $A\setminus N\subseteqq B$. For any $\sigma\in A\setminus N$, there are $\gamma\in\Gamma$ and $\beta\in B\setminus N$ such that $\sigma=\gamma(\beta)$, i.e., $\Pi^{\sigma}=\Pi^{\gamma(\beta)}$. Then $\Pi^{\sigma}(g_n)\to\chi_B(\sigma)I$ implies $\Pi^{\gamma^{-1}(\sigma)}(g_n)\to\chi_B(\sigma)I$. But $\Pi^{\gamma^{-1}(\sigma)}(g_n)=\Pi^{\beta}(g_n)\to\chi_B(\beta)I=I$, hence $\chi_B(\sigma)=I$ and $\sigma\in B$. Thus μ is not ergodic.

If K is a factor representation of type I, then μ-almost all the representations Π^{σ} are unitary equivalent, i.e., μ is trivially ergodic under Γ ([8]). Therefore if μ is non-trivially ergodic, then K is a non-type I factor representation.

3. On a construction of non-measurable dynamical system. In this section, we shall construct a non-measurable dynamical system (G, X, μ) by means of a given topological transformation group (G, Ω) with some conditions.

The idea is due to [3] and [6].

Hereafter, we assume that (G, Ω) *is a topological transformation group which is polonais and satisfies the condition*$(*)$ *and the orbit space* $\Omega/G=\{G\omega\,|\,\omega\in\Omega\}$ *is not a* T_0*-space.*

The latter assumption is essential in our study.

We may select points p and q in Ω/G with $p\neq q$, $q\in \mathrm{Cl}[\{p\}]$ and $p\in\mathrm{Cl}[\{q\}]$.

Let Π be the canonical mapping of Ω onto Ω/G. Let $X=\Pi^{-1}(\mathrm{Cl}[\{p\}])$. (G, X) is a polonais transformation group satisfying the condition$(*)$.

REMARK. $\Pi^{-1}(\mathrm{Cl}[\{p\}])=\mathrm{Cl}[\Pi^{-1}(\{p\})]$.

LEMMA 6 ([3]). *There is a neighborhood* W *of* e *in* G *such that* $W=W^{-1}$, *and if* $\{Q_m\}$ *is a decreasing basis of open sets at an arbitrary point* y *in* X, $\mathrm{Cl}[\bigcap_{m=1}^{\infty}WQ_m]\subseteqq Gy$.

The following lemma is essential for our study.

LEMMA 7 ([3]). *Let* (G, X) *be the topological transformation group defined as above. We can inductively define, for each integer* $n\geqq 0$ *and element* $g(n)$ *in* G *and for each* n*-tuple* $(i_1,\cdots,i_n)$ *with* $i_k=0$ *or* 1, *an open set* $P(i_1,\cdots,i_n)$ *in* X *satisfying the following properties*:

$(1)_n$ $x\in P(0_n)$,

$(2)_n$ *if* $(i_1,\cdots,i_n)\neq(j_1,\cdots,j_n)$, *then* $WP(i_1,\cdots,i_n)\cap P(j_1,\cdots,j_n)=\varnothing$, *where* W *is as in Lemma* 6,

$(3)_n$ $\mathrm{Cl}[P(i_1,\cdots,i_n)]\subseteqq P(i_1,\cdots,i_{n-1})$ $(n\geqq 1)$,

$(4)_n$ diameter $P(i_1,\cdots,i_n)<1/n$ $(n\geqq 1)$,

$(5)_n$ $g(k)P(0_k, i_{k+1},\cdots,i_n)=P(0_{k-1}, 1, i_{k+1},\cdots,i_n)$ $(n\geqq 1)$, *where* 0_n *is the family of* n *zeros.*

LEMMA 8 ([9]). *Let* M_n $(n = 1, 2, \cdots)$ *be copies of the group* $\boldsymbol{Z}/2\boldsymbol{Z}$, *the additive group of integers* mod 2. *Let* μ_n *be a Radon measure on* M_n *with* $\mu_n\{(0)\} = p$ *and* $\mu_n\{(1)\} = q$ *with* $0 < p < 1$ *and* $q = 1 - p$. *Then* (M_n, μ_n) *is a measure space* $(n = 1, 2, \cdots)$. *Let* (M, μ) *be the infinite product measure space of* (M_n, μ_n). *Let* $\mathscr{G}$ *be the set of those* $a = (a_n | n = 1, 2, \cdots)$ *in* M *for which* $a_n \neq 0$ *occurs for a finite number of* n *only.* ($\mathscr{G}$ *is a countable group which acts on* M). *Then*

(1) $\mathscr{G}$ *is free, ergodic and measurable if* $p = q = 1/2$.

(2) $\mathscr{G}$ *is free, ergodic and non-measurable if* $p \neq q \neq 1/2$.

For each $i = \{i_1, i_2, \cdots, i_n, \cdots\} \in M$, the set $\bigcap_{n=1}^{\infty} P(i_1, \cdots, i_n)$ has precisely one element, say $\Theta(i)$. This is due to $(3)_n$, $(4)_n$ and the completeness of X. It is easily verified that Θ is a one-to-one mapping of M into X by $(2)_n$. Let $M(i_1, \cdots, i_n) = \{i_1\} \times \cdots \times \{i_n\} \times \{0, 1\} \times \cdots$, where $i_1, \cdots, i_n \in \{0, 1\}$. Then

$$\Theta(M(i_1, \cdots, i_n)) = \Theta(M) \cap P(i_1, \cdots, i_n) .$$

Hence Θ is a homeomorphism because the sets $M(i_1, \cdots, i_n)$ form an open basis. Therefore we may identify M with $\Theta(M)$.

Thus we can define a measure λ on X by the formulas

$$\lambda(M(i_1, \cdots, i_n)) = p^r q^{n-r}$$

and

$$\lambda(X \setminus M) = 0 ,$$

where r is the number of 0's in $(i_1, \cdots, i_n)$.

Each point has measure zero with respect to this measure.

Let ν be a finite measure on G, equivalent to a right Haar measure. We shall define the convolution product measure $\nu * \lambda$ of ν and λ as follows: If B is a Borel subset of X, let

$$\nu * \lambda(B) = \int_G \lambda(hB) d\nu(h) . \qquad (*)$$

We have to show that the integral in $(*)$ exists. The proof was given by Glimm [6].

Denote this convolution product measure $\nu * \lambda$ by β.

LEMMA 9. *The measure* β *is quasi-invariant under* G.

PROOF. For any Borel set B of X, $\beta(B) = 0$ if and only if $\lambda(hB) = 0$ for almost every h. On the other hand, for all $g \in G$, $\lambda(hB) = 0$ (for almost every h) if and only if $\lambda(hgB) = 0$ (for almost every h). Therefore, for any Borel set B, $\beta(B)=0$ if and only if $\beta(gB)=0$ for all $g\in G$.

LEMMA 10. *If* (G, X) *is effective, then* $gM\cap M=\emptyset$ *for all* $W\ni g\neq e$, *where* W *is as in Lemma* 6.

PROOF. If $y\in gM\cap M$, $g\neq e$, then there exists an element $z=\bigcap_{n=1}^{\infty}P(i_1,\cdots,i_n)\cap M$ such that $y=gz\in Wz\cap M$. However, $Wz\cap M\subseteqq \bigcap_{n=1}^{\infty}WP(i_1,\cdots,i_n)\cap M=\bigcap_{n=1}^{\infty}P(i_1,\cdots,i_n)=z$ by $(2)_n$ of Lemma 7. Hence, $gz=z$, a contradiction.

Let $\boldsymbol{P}_n(0)=\bigcup P(i_1,\cdots,i_{n-1},0)$ and $\boldsymbol{P}_n(1)=\bigcup P(i_1,\cdots,i_{n-1},1)$, where the union is taken over all $i_j=0$ or 1, $j=1,\cdots,n-1$. Define $\overline{g(n)}$, $n=0,1,2,\cdots$, by $\overline{g(n)}\boldsymbol{P}_n(0)=\boldsymbol{P}_n(1)$ and $\overline{g(n)}^{-1}\boldsymbol{P}_n(1)=\boldsymbol{P}_n(0)$. Let $\mathscr{K}$ be the countable free abelian group generated by $\{\overline{g(n)}\}$. Then we can consider $\mathscr{K}$ as acting on M and the dynamical system $\{\mathscr{K}, M, \lambda\}$ is non-measurable.

PROPOSITION 2. *If* (G, X) *is effective and* G *is abelian, the dynamical system* (G, X, β) *is free and non-measurable.*

PROOF. It is obvious that (G, X, β) is free. Let β_0 be a σ-finite positive measure on X which is invariant under G. Define a measure λ_0 on M by $\lambda_0(K)=\beta_0(WK)$ for each Borel set K, where W is as in Lemma 6. Then $\lambda_0(\overline{g(n)}K)=\lambda_0(g(n)(K\cap\boldsymbol{P}_n(0)))+\lambda_0(g(n)^{-1}(K\cap\boldsymbol{P}_n(1)))=\beta_0(g(n)W(K\cap\boldsymbol{P}_n(0)))+\beta_0(g(n)^{-1}W(K\cap\boldsymbol{P}_n(1)))=\beta_0(W(K\cap\boldsymbol{P}_n(0)))+\beta_0(W(K\cap\boldsymbol{P}_n(1)))=\beta_0(WK)=\lambda_0(K)$ for each $g(n)$, where $g(n), n=0,1,2,\cdots$, are elements in G which are chosen in Lemma 7. Hence λ_0 is a σ-finite positive measure which is invariant under $\mathscr{K}$. Since $(\mathscr{K}, M, \lambda)$ is non-measurable, λ_0 is non-equivalent to λ. Therefore, there exists a Borel set B of M such that $\lambda_0(B)=0$ and $\lambda(B)\neq 0$ (resp. $\lambda_0(B)\neq 0$ and $\lambda(B)=0$). Thus by Lemma 10 we have $\beta_0(WB)=0$ and $\beta(WB)=\nu(W)\lambda(B)=0$ (resp. $\beta_0(WB)\neq 0$ and $\beta(WB)=0$). However, WB is a Borel set of X. Hence β_0 is non-equivalent to β. Consequently, β is non-measurable.

4. The type of factors associated with unitary representations of semi-direct product groups of Mautner type. Define the action of $\boldsymbol{R}$ on $\boldsymbol{C}^2$ by

$$t(z_1, z_2)=(e^{it}z_1, e^{i\alpha t}z_2)\ ,\quad (z_1, z_2)\in\boldsymbol{C}^2\ ,\quad t\in\boldsymbol{R}\ ,$$

where α is an irrational number.

Then the topological transformation group $(\boldsymbol{R}, \boldsymbol{C}^2)$ is polonais and satisfies the condition$(*)$. Moreover the orbit space $\boldsymbol{C}^2/\boldsymbol{R}$ is not a T_0-space.

The closure of the orbit $\{t(z_1, z_2)\,|\,t\in\boldsymbol{R}\}$ through a point $(z_1, z_2)\in\boldsymbol{C}^2$ is the two-dimensional torus $\boldsymbol{T}^2$ if $z_1 \neq 0, z_2 \neq 0$. $\boldsymbol{T}^2$ is a compact Hausdorff space satisfying the second countability axiom. Let μ be the Lebesgue measure on $\boldsymbol{T}^2$.

LEMMA 11. *The dynamical system* $(\boldsymbol{R}, \boldsymbol{T}^2, \mu)$ *is free, ergodic and measurable.*

The proof is well known in the theory of dynamical system.

LEMMA 12. *There exists a measure* β *on* $\boldsymbol{T}^2$ *such that the dynamical system* $(\boldsymbol{R}, \boldsymbol{T}^2, \beta)$ *is free, ergodic and non-measurable.*

PROOF. Since $(\boldsymbol{R}, \boldsymbol{T}^2)$ is effective and $\boldsymbol{R}$ is abelian, we can construct a measure β such that the dynamical system $(\boldsymbol{R}, \boldsymbol{T}^2, \beta)$ is free and non-measurable by Proposition 2.

The ergodicity is trivial because $\boldsymbol{T}^2$ is the orbit closure.

Now we define a unitary representation Π of $\boldsymbol{R} \times_s \boldsymbol{C}^2$ in $\mathscr{H} = L^2(\boldsymbol{T}^2 \times \boldsymbol{R}, \mu \times \nu)$, where ν is the Lebesgue measure on $\boldsymbol{R}$, in the following manner.

For each $(t;\ z_1, z_2) \in \boldsymbol{R} \times_s \boldsymbol{C}^2$, define

$$\begin{aligned}(t;\ z_1, z_2) \to \Pi_{(t;z_1,z_2)} f((\zeta_1, \zeta_2), p) \\ = \exp(-i(\mathrm{Re}(e^{-it}\zeta_1 z_1 + e^{-i\alpha t}\zeta_2 z_2)) f((\zeta_1, \zeta_2), p - t))\end{aligned}$$

$$\begin{aligned}(\text{resp. } (t;\ z_1, z_2) \to \Pi'_{(t;z_1,z_2)} f((\zeta_1, \zeta_2), p) \\ = \exp(i(\mathrm{Re}(z_1\zeta_1 + z_2\zeta_2)) f(e^{it}\zeta_1, e^{i\alpha t}\zeta_2), t + p)),\end{aligned}$$

where Re denotes the real part.

REMARK. For each $(z_1, z_2) \in \boldsymbol{C}^2$, define

$$f_{(z_1,z_2)}(\zeta, \xi) = \exp(i(\mathrm{Re}(z_1\zeta + z_2\xi)))\ ,$$

$(\zeta, \xi) \in \boldsymbol{T}^2$. Then $f_{(z_1,z_2)} \in L^\infty(\boldsymbol{T}^2, \mu)$ and the closed linear hull of $\{f_{(z_1,z_2)} | (z_1, z_2) \in \boldsymbol{C}^2\}$ is $L^\infty(\boldsymbol{T}^2, \mu)$.

By Lemma 3 and Lemma 11, we have

LEMMA 13. *The W^*-subalgebra* $\mathscr{F}$ *(resp.* $\mathscr{F}'$*) of* $\boldsymbol{B}(\mathscr{H})$ *associated with the unitary representation* Π *(resp.* Π'*) is a type* I_∞ *or type* II_∞ *factor.*

LEMMA 14. *For each* $(\zeta_1, \zeta_2) \in \boldsymbol{T}^2$, *define*

$$\Pi^{(\zeta_1\ \zeta_2)}_{(t;z_1\ z_2)} \phi(p) = \exp(-i(\mathrm{Re}(e^{-it}\zeta_1 z_2 + e^{-i\alpha t}\zeta_2 z_2)) \phi(p - t))\ ,$$

for $(t;\ z_1, z_2) \in \boldsymbol{R} \times_s \boldsymbol{C}^2$, $\phi(p) \in L^2(\boldsymbol{R}, \nu)$. *Then*

(1) $\Pi^{(\zeta_1,\zeta_2)}$, $(\zeta_1, \zeta_2) \in \boldsymbol{T}^2$, *are irreducible unitary representations of* $\boldsymbol{R} \times_s \boldsymbol{C}^2$ *in* $L^2(\boldsymbol{R}, \nu)$.

(2) *For any* $(\zeta_1, \zeta_2) \in \boldsymbol{T}^2$, $\Pi^{(\zeta_1,\zeta_2)}$ *is equivalent to* $\Pi^{t(\zeta_1,\zeta_2)}$ *for all* $t \in \boldsymbol{R}$.

PROOF. (1) Every operator commuting with the operators $\Pi^{(\zeta_1,\zeta_2)}_{(0;z_1,z_2)}$ is the multiplication by a function.

(2) The intertwining operator for $\Pi^{(\zeta_1,\zeta_2)}$ and $\Pi^{(\zeta_1',\zeta_2')}$ can only be the translation by t.

Therefore, by Proposition 1, Lemma 13 and Lemma 14 we have the following theorem.

THEOREM 1. *The W^*-subalgebra $\mathscr{F}$ (resp. $\mathscr{F}'$) of $\boldsymbol{B}(\mathscr{H})$ associated with the unitary representation*

$$
\begin{aligned}
(t;\ z_1, z_2) \to \Pi_{(t;z_1,z_2)} &f((\zeta_1, \zeta_2), p) \\
&= \exp(-i(\mathrm{Re}(e^{-it}\zeta_1 z_1 + e^{-i\alpha t}\zeta_2 z_2)) f((\zeta_1, \zeta_2), p - t))
\end{aligned}
$$

$$
\begin{aligned}
(\textit{resp. } (t;\ z_1, z_2) \to \Pi'_{(t;z_1,z_2)} &f((\zeta_1, \zeta_2), p) \\
&= \exp(i(\mathrm{Re}(z_1\zeta_1 + z_2\zeta_2)) f(e^{it}\zeta_1, e^{i\alpha t}\zeta_2), t + p))
\end{aligned}
$$

of $\boldsymbol{R}\times_s \boldsymbol{C}^2$ in $\mathscr{H} = L^2(\boldsymbol{T}^2\times \boldsymbol{R}, \mu\times\nu)$ with the Lebesgue measure ν on $\boldsymbol{R}$, is a type II_∞ *factor. Namely, Π and Π' are type* II_∞ *factor representations of $\boldsymbol{R}\times_s \boldsymbol{C}^2$.*

On the other hand, by Lemma 4 and Lemma 12, we have the following theorem.

THEOREM 2. *The W^*-subalgebra $\mathscr{B}$ (resp. $\mathscr{B}'$) of $\boldsymbol{B}(\mathscr{H})$ associated with the unitary representation*

$$
\begin{aligned}
(t;\ z_1, z_2) \to \Phi_{(t;z_1,z_2)} &f((\zeta_1, \zeta_2), p) \\
&= \exp(-i(\mathrm{Re}(e^{-it}\zeta_1 z_1 + e^{-i\alpha t}\zeta_2 z_2)) f((\zeta_1, \zeta_2), p - t))
\end{aligned}
$$

$$
\begin{aligned}
(\textit{resp. } (t;\ z_1, z_2) \to \Phi'_{(t;z_1,z_2)} &f((\zeta_1, \zeta_2), p) \\
&= \exp(i(\mathrm{Re}(z_1\zeta_1 + z_2\zeta_2))\rho((\zeta_1, \zeta_2), t) f((e^{it}\zeta_1, e^{i\alpha t}\zeta_2), t + p)\ ,
\end{aligned}
$$

where $\rho((\zeta_1, \zeta_2), t)$ is a multiplier) of $\boldsymbol{R}\times_s \boldsymbol{C}^2$ in $\mathscr{H} = L^2(\boldsymbol{T}^2\times \boldsymbol{R}, \beta\times\nu)$ with the Lebesgue measure ν on $\boldsymbol{R}$, is a type III *factor. Namely, Φ and Φ' are type* III *factor representations of $\boldsymbol{R}\times_s \boldsymbol{C}^2$.*

For the same reason as in Theorem 1 and Theorem 2, we have the following theorems.

THEOREM 1′. *The W^*-subalgebra $\mathscr{F}$ (resp. $\mathscr{F}'$) of $\boldsymbol{B}(\mathscr{H})$ associated with the unitary representation*

$$
\begin{aligned}
&((t_1, t_2, t_3);\ z_1, z_2, z_3, z_4) \\
&\quad\to \Pi_{((t_1,t_2,t_3),z_1,z_2,z_3,z_4)} f((\zeta_1, \zeta_2, \zeta_3, \zeta_4), (p_1, p_2, p_3)) \\
&\qquad = \exp(-i(\mathrm{Re}(e^{-i(t_1+t_2)}\zeta_1 z_1 + e^{-i(t_3+\alpha t_2)}\zeta_2 z_2
\end{aligned}
$$

$$+ e^{-i(t_1+t_3)}\zeta_3 z_3 + e^{-it_3}\zeta_4 z_4)))$$
$$\times f((\zeta_1, \zeta_2, \zeta_3, \zeta_4), (p_1 - t_1, p_2 - t_2, p_3 - t_3))$$

(resp. $((t_1, t_2, t_3); z_1, z_2, z_3, z_4)$

$$\to \Pi'_{((t_1,t_2,t_3);z_1,z_2,z_3,z_4)} f((\zeta_1, \zeta_2, \zeta_3, \zeta_4), (p_1, p_2, p_3)) = \exp(i(\textstyle\sum_{i=1}^4 z_i\zeta_i))$$
$$\times f((e^{i(t_1+t_2)}\zeta_1, e^{i(t_3+\alpha t_2)}\zeta_2, e^{i(t_1+t_3)}\zeta_3, e^{it_3}\zeta_4), t_1 + p_1, t_2 + p_2, t_3 + p_3)$$

of $M_{11} = \boldsymbol{R}^3 \times_s \boldsymbol{C}^4$ *in* $\mathscr{H} = L^2(\boldsymbol{T}^4 \times \boldsymbol{R}^3, \mu \times \nu)$ *with the Lebesgue measure* μ *(resp.* ν*) on* $\boldsymbol{T}^4$ *(resp.* $\boldsymbol{R}^3$*), is a type* II_∞ *factor. Namely,* Π *and* Π' *are type* II_∞ *factor representations of* $\boldsymbol{R}^3 \times_s \boldsymbol{C}^4$.

THEOREM 2′. *The* W^**-subalgebra* $\mathscr{B}$ *(resp.* $\mathscr{B}'$*) of* $\boldsymbol{B}(\mathscr{H})$ *associated with the unitary representation*

$$((t_1, t_2, t_3); z_1, z_2, z_3, z_4) \to \Phi_{((t_1,t_2,t_3);z_1,z_2,z_3,z_4)} f((\zeta_1, \zeta_2, \zeta_3, \zeta_4), (p_1, p_2, p_3))$$
$$= \exp(-i(\mathrm{Re}(e^{-i(t_1+t_2)}\zeta_1 z_1 + e^{-i(t_3+\alpha t_2)}\zeta_2 z_2 + e^{-i(t_1+t_3)}\zeta_3 z_3$$
$$+ e^{-it_3}\zeta_4 z_4)) \times f((\zeta_1, \zeta_2, \zeta_3, \zeta_4), (p_1 - t_1, p_2 - t_2, p_3 - t_3))$$

(resp. $((t_1, t_2, t_3); z_1, z_2, z_3, z_4) \to \Phi'_{((t_1,t_2,t_3);z_1,z_2,z_3,z_4)} f((\zeta_1, \zeta_2, \zeta_3, \zeta_4), (p_1, p_2, p_3))$

$$= \exp\Big(i\Big(\mathrm{Re}\Big(\sum_{i=1}^4 z_i\zeta_i\Big)\Big)\Big)\rho((\zeta_1, \zeta_2, \zeta_3, \zeta_4), (t_1, t_2, t_3))$$
$$\times f((e^{i(t_1+t_2)}\zeta_1, e^{i(t_3+\alpha t_2)}\zeta_2, e^{i(t_1+t_3)}\zeta_3, e^{it_3}\zeta_4), (t_1 + p_1, t_2 + p_2, t_3 + p_3)),$$

where $\rho((\zeta_1, \zeta_2, \zeta_3, \zeta_4), (t_1, t_2, t_3))$ *is a multiplier) of* $M_{11} = \boldsymbol{R}^3 \times_s \boldsymbol{C}^4$ *in* $\mathscr{H} = L^2(\boldsymbol{T}^4 \times \boldsymbol{R}^3, \beta \times \nu)$ *is a type* III *factor, where* β *is a measure such that the dynamical system* $(\boldsymbol{R}^3, \boldsymbol{T}^4, \beta)$ *is free, ergodic and non-measurable, and* ν *is the Lebesgue measure on* $\boldsymbol{R}^3$. *Namely,* Φ *and* Φ' *are type* III *factor representations of* $\boldsymbol{R}^3 \times_s \boldsymbol{C}^4$.

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