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ON CYCLOTOMIC Z_2-EXTENSIONS OF IMAGINARY QUADRATIC FIELDS

YÛJI KIDA

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Let $K = \boldsymbol{Q}(\sqrt{-m})$ for a positive square-free integer m. For each $n \geqq 0$, let $\boldsymbol{B}_n$ be the maximal real subfield of the cyclotomic field of the 2^{n+2}-th roots of unity. Let $\boldsymbol{B}_\infty = \bigcup_{n=0}^{\infty} \boldsymbol{B}_n$ and let $K_\infty = \boldsymbol{B}_\infty \cdot K$. Then the extension K_∞/K is called a cyclotomic $\boldsymbol{Z}_2$-extension. Let h_n be the class number of $K_n = \boldsymbol{B}_n \cdot K$ and let 2^{e_n} be the exact power of 2 dividing h_n. Iwasawa proved, in [2] and [3], that there exist an integer $n_0 \geqq 0$ and an integer c such that

$$e_n = \lambda n + c \quad \text{for all} \quad n \geqq n_0 , \tag{1}$$

where λ is the invariant of this $\boldsymbol{Z}_2$-extension.

The group-theoretic meaning of this invariant λ is as follows. Let A_n be the 2-Sylow subgroup of the ideal class group of K_n. For each $m \geqq n \geqq 0$, the norm map from K_m to K_n defines a morphism from A_m to A_n. Let X be the limit of this projective system, then as an abelian group

$$X \cong \boldsymbol{Z}_2^{\lambda} \oplus T , \tag{2}$$

where T is a finite abelian 2-group. This integer λ coincides with that of (1).

We always define the natural action of $\Gamma = \mathrm{Gal}(K_\infty/K)$ on X and call X the Iwasawa module for K_∞/K as a Γ-module. The action of Γ will be used in Section 4.

In this paper, we shall determine the right hand side of (2), especially the invariant λ, and find a value of n_0 satisfying (1).

Finally, the author would like to express his hearty thanks to Professor K. Uchida for his kind encouragement and guidance.

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After this paper was accepted for publication, the author received the preprint by B. Ferrero entitled "The cyclotomic $\boldsymbol{Z}_2$-extension of imaginary quadratic fields" in which he proves the same formula for the invariant λ by a purely algebraic method. Moreover, his Theorem 5 c) and f) implies the torsion subgroup T in our Theorem 1 is in fact of order 2.

1. It is clear that $\boldsymbol{Q}(\sqrt{-m})\cdot\boldsymbol{B}_n = \boldsymbol{Q}(\sqrt{-2m})\cdot\boldsymbol{B}_n$ for all $n \geqq 1$, and in the case of $m = 1$ or 3, $\lambda = c = 0$ is well-known, so it may be sufficient to treat only the case that m is an odd integer bigger than 3. In this case, every $K_n (n \geqq 0)$ contains no roots of unity other than ± 1. For simplicity, we shall use the following notations.

$e^{(2)}N$: the exponent of the exact power of 2 dividing a natural number N.

$d^{(2)}A$: the 2-rank of a compact abelian group A.

LEMMA 1. *The class number of $\boldsymbol{B}_n$ in the narrow sense, hence also in the wide sense, is odd for all $n \geqq 0$.*

PROOF. It is well-known in the wide sense, and the proof is almost the same (see Iwasawa [1] and [2]).

Let $a(K_n)$ be the number of the ambiguous ideal classes in $K_n/\boldsymbol{B}_n$, and let s_n be the number of the ramified prime ideals in $K_n/\boldsymbol{B}_n$. Then a well-known formula states that

$$(3) \qquad a(K_n) = h(\boldsymbol{B}_n)\cdot\frac{2^{s_n+2^n-1}}{[E_n : E_n \cap \mathcal{N} K_n]} \quad \text{for all} \quad n \geqq 0 ,$$

where $h(\boldsymbol{B}_n)$ is the class number of $\boldsymbol{B}_n$, E_n is the unit group of $\boldsymbol{B}_n$ and $\mathcal{N}$ is the norm map from K_n to $\boldsymbol{B}_n$. The following lemma is essential to our theorems, which was suggested to the author by Professor K. Uchida.

LEMMA 2. $[E_n : E_n \cap \mathcal{N} K_n] = 2^{2^n}$ *for all* $n \geqq 0$.

PROOF. Let $\boldsymbol{B}_n^*$ be the multiplicative group of all non-zero elements of $\boldsymbol{B}_n$, and let $\boldsymbol{B}_{n,+}^*$ be its subgroup of totally positive elements. Let P be the principal ideal group of $\boldsymbol{B}_n$, and let P_+ be its subgroup of ideals generated by $\boldsymbol{B}_{n,+}^*$, then it holds that

$$P/P_+ \cong \boldsymbol{B}_n^*/E_n\cdot\boldsymbol{B}_{n,+}^* .$$

But by Lemma 1, the left hand side vanishes, therefore

$$\boldsymbol{B}_n^* = E_n\cdot\boldsymbol{B}_{n,+}^* .$$

In any finite algebraic number field, there exist elements with an arbitrary signature, so the above equality states that there exist in E_n elements with an arbitrary signature. On the other hand, since K_n is an imaginary abelian field, any element of $\mathcal{N} K_n$ is totally positive. Therefore

$$[E_n : E_n \cap \mathcal{N} K_n] \geqq 2^{2^n} .$$

Conversely, it is clear that

$$E_n \cap \mathcal{N} K_n \supset E_n^2 ,$$

then by Dirichlet's unit theorem,

$$[E_n \colon E_n \cap \mathcal{N} K_n] \leqq [E_n \colon E_n^2] = 2^{2^n} .$$

This completes the proof.

LEMMA 3. $\quad e^{(2)}a(K_n) = s_n - 1 \qquad$ *for all* $\quad n \geqq 0$.

PROOF. Apply $e^{(2)}$ to both sides of (3), then the lemma follows at once from Lemmas 1 and 2.

LEMMA 4. $\quad e^{(2)}a(K_n) = d^{(2)}A_n \qquad$ *for all* $\quad n \geqq 0$.

PROOF. Let J be the generator of Gal $(K_n/\boldsymbol{B}_n)$. Then for any element c of A_n, c^{1+J} is the natural image of an ideal class of $\boldsymbol{B}_n$. But by Lemma 1, c^{1+J} is of odd order, so c^{1+J} must be 1. Therefore $c^2 = 1$ if and only if $c = c^J$.

Combining this with Lemma 3, we have the following proposition.

PROPOSITION 1. $\quad d^{(2)}A_n = s_n - 1 \qquad$ *for all* $\quad n \geqq 0$.

As K_∞/K is a cyclotomic $\boldsymbol{Z}_2$-extension, it is well-known that s_n is constant for all sufficiently large n (the exact value of this constant will be given later). Thus we have another proof of the vanishing of the invariant μ of this $\boldsymbol{Z}_2$-extension (see Iwasawa [3]).

2. As $K_{n+1}/\boldsymbol{B}_n$ is an abelian extension of type (2, 2), there exists an intermediate field L_n of degree 2 over $\boldsymbol{B}_n$ different from K_n and $\boldsymbol{B}_{n+1}$. For an abelian field k, let $h(k)$, $R(k)$, $W(k)$, and $\hat{k}$ be its class number, regulator, the number of roots of unity contained in k and the group of Dirichlet characters, respectively. For a Dirichlet character θ, let f_θ be its conductor and let $M(\theta) = L(1, \theta)\cdot\sqrt{f_\theta}$, where $L(s, \theta)$ is of course usual Dirichlet's L-function. Then a class number formula states that

$$h(k)R(k) = \frac{W(k)}{2^i(2\pi)^j} \prod_\theta M(\theta) ,$$

where θ ranges over all non-principal elements of $\hat{k}$, and i and $2j$ are the numbers of real and complex conjugate fields of k, respectively. Applying this formula to each K_n, we get

$$\frac{h(K_{n+1})R(K_{n+1})}{h(K_n)R(K_n)} = (2\pi)^{-2^n} \prod_\theta M(\theta) \qquad \text{for all} \quad n \geqq 0 ,$$

where θ ranges over all the elements of $\hat{K}_{n+1}$ not contained in $\hat{K}_n$.

The right hand side $= (2\pi)^{-2^n}\{\prod_{\theta_1} M(\theta_1)\}\{\prod_{\theta_2} M(\theta_2)\}$,

$$= \frac{h(L_n)h(\boldsymbol{B}_{n+1})R(L_n)R(\boldsymbol{B}_{n+1})}{h(\boldsymbol{B}_n)^2R(\boldsymbol{B}_n)^2},$$

where θ_1(resp. θ_2) ranges over all the elements of $\hat{L}_n$ (resp. $\hat{\boldsymbol{B}}_{n+1}$) not contained in $\hat{\boldsymbol{B}}_n$. Hence we get

$$\frac{h(K_{n+1})}{h(K_n)} = \frac{h(L_n)h(\boldsymbol{B}_{n+1})}{h(\boldsymbol{B}_n)^2} \cdot \frac{R(L_n)R(\boldsymbol{B}_{n+1})R(K_n)}{R(\boldsymbol{B}_n)^2R(K_{n+1})}.$$

By our assumption, $W(K_n) = W(L_n) = 2$ and some prime ideal of $\boldsymbol{B}_n$ not dividing 2 must be ramified in K_n and L_n for all $n \geqq 0$. Thus the unit index is 1 in each case, that is,

$$R(K_n)/R(\boldsymbol{B}_n) = R(L_n)/R(\boldsymbol{B}_n) = 2^{2^n-1} \qquad \text{for all} \quad n \geqq 0 .$$

Therefore

$$h(K_{n+1})/h(K_n) = h(L_n)h(\boldsymbol{B}_{n+1})/2h(\boldsymbol{B}_n)^2 \qquad \text{for all} \quad n \geqq 0 . \tag{4}$$

Since $h(\boldsymbol{B}_n)$ is prime to 2 for all $n \geqq 0$ by Lemma 1, we get the following lemma by applying $e^{(2)}$ to both sides of (4).

LEMMA 5. $e_{n+1} - e_n = -1 + e^{(2)}h(L_n)$ *for all* $n \geqq 0$.

Let $a(L_n)$ be the number of the ambiguous ideal classes in $L_n/\boldsymbol{B}_n$, and let t_n be the number of the ramified prime ideals in $L_n/\boldsymbol{B}_n$. Then the same argument as in the preceding section shows that $e^{(2)}a(L_n) = t_n - 1$ for all $n \geqq 0$. Hence we get the following proposition.

PROPOSITION 2. $e_{n+1} - e_n \geqq t_n - 2$ *for all* $n \geqq 0$,
especially $\lambda \geqq t_n - 2$ *for all* $n \geqq n_0$.

3. If 2 is ramified in $K/\boldsymbol{Q}$, every prime ideal of $\boldsymbol{B}_n$ is ramified in K_n and L_n at the same time. If 2 is not ramified in $K/\boldsymbol{Q}$, a unique prime ideal of $\boldsymbol{B}_n$ dividing 2 is not ramified in K_n, but is ramified in L_n. Every other prime ideal of $\boldsymbol{B}_n$ is ramified in K_n and L_n at the same time. Hence we get the following proposition.

PROPOSITION 3.

$$t_n = \begin{cases} s_n + 1 & \textit{if} \quad m \equiv 3 \pmod 4 . \\ s_n & \textit{if} \quad m \equiv 1 \pmod 4 . \end{cases}$$

Combining this with Propositions 1 and 2, we get that for all $n \geqq n_0$

$$s_n - 1 = d^{(2)}A_n \geqq \lambda \geqq s_n - 1 \qquad \text{if} \quad m \equiv 3 \pmod 4 ,$$
$$s_n - 1 = d^{(2)}A_n \geqq \lambda \geqq s_n - 2 \qquad \text{if} \quad m \equiv 1 \pmod 4 .$$

As we stated before, s_n is constant for all sufficiently large n. If we denote this constant by s_∞, then $d^{(2)}A_n = s_\infty - 1$ for all sufficiently large n. Therefore $d^{(2)}X = s_\infty - 1$ by the properties of projective limits. When $m \equiv 1(\mathrm{mod}\, 4)$, the cases $\lambda = s_\infty - 1$ and $s_\infty - 2$ are possible. But the former means X is torsion-free as an abelian group, a contradiction to the following lemma. Hence $\lambda = s_\infty - 2$ must hold.

LEMMA 6. *If $m \equiv 1(\mathrm{mod}\, 4)$, $c^* = (c(2_0), \cdots, c(2_n), \cdots)$ is of order 2 in X, where 2_n is a unique prime ideal of K_n dividing 2 and $c(2_n)$ is the ideal class of K_n containing 2_n for each $n \geqq 0$.*

PROOF. Clearly c^* is contained in X and 2_n^2 is principal even in B_n for all $n \geqq 0$. It is easily shown that 2_0 is not principal in K_0. Hence 2_n cannot be principal in K_n for any $n \geqq 0$.

Finally we must find the exact value of s_∞. It is clear that 2 has a unique prime divisor in $\boldsymbol{B}_n$ for all $n \geqq 0$. For odd primes, the theory of the cyclotomic fields shows the following.

LEMMA 7. *An odd prime p is completely decomposed in $\boldsymbol{B}_{n(p)}/\boldsymbol{Q}$ and is not decomposed in $\boldsymbol{B}_\infty/\boldsymbol{B}_{n(p)}$, where $n(p) = -3 + e^{(2)}(p^2 - 1)$.*

If a prime p is ramified in $K/\boldsymbol{Q}$, every prime ideal of $\boldsymbol{B}_n$ dividing p must be ramified in $K_n/\boldsymbol{B}_n$, and conversely. Therefore s_∞ is the sum of the decomposition numbers in $\boldsymbol{B}_\infty/\boldsymbol{Q}$ of all the ramified primes in $K/\boldsymbol{Q}$, that is,

$$s_\infty = \begin{cases} \sum_{p|m} 2^{n(p)} & \text{if} \quad m \equiv 3 \pmod 4 . \\ 1 + \sum_{p|m} 2^{n(p)} & \text{if} \quad m \equiv 1 \pmod 4 . \end{cases}$$

Consequently we get the following theorem.

THEOREM 1. *Let $K = \boldsymbol{Q}(\sqrt{-m})$ or $\boldsymbol{Q}(\sqrt{-2m})$, where m is a square-free odd integer bigger than 3, and let X be the Iwasawa module for the cyclotomic $\boldsymbol{Z}_2$-extension of K. Then as an abelian group*

$$X \cong \begin{cases} \boldsymbol{Z}_2^\lambda & \textit{if} \quad m \equiv 3 \pmod 4 , \\ \boldsymbol{Z}_2^\lambda \oplus T & \textit{if} \quad m \equiv 1 \pmod 4 , \end{cases}$$

where T is a non-trivial finite cyclic 2-group and in both cases, $\lambda = -1 + \sum_{p|m} 2^{n(p)}$, where $n(p) = -3 + e^{(2)}(p^2 - 1)$.

4. Next we shall find a value of n_0 satisfying (1). As usual, let $\Gamma = \mathrm{Gal}\,(K_\infty/K)$ and let γ be a fixed topological generator of Γ, and put $\gamma_n = \gamma^{2^n}$, $\omega_n = 1 - \gamma_n$ for each $n \geqq 0$.

When $m \equiv 3(\mathrm{mod}\, 4)$, X is torsion-free as an abelian group. Iwasawa's argument in [2] Sections 3-2, 7-4 and 7-5 really shows that we can take n_0 to be $s+1$ if $2X \supset \omega_s X$. In general, $d^{(2)}X \geqq d^{(2)}\{X/\omega_n X\} \geqq d^{(2)}A_n$ for all $n \geqq 0$. Thus if $d^{(2)}X = d^{(2)}A_s$, $[X: 2X] = d^{(2)}\{X/\omega_s X\} = [X: 2X]/[\omega_s X: \omega_s X \cap 2X]$, that is $2X \supset \omega_s X$. From the preceding section, $d^{(2)}A_s = d^{(2)}X$ if $s \geqq \max_{p|m} n(p)$. Therefore we can take n_0 to be $1 + \max_{p|m} n(p)$.

When $m \equiv 1 \pmod 4$, X has torsion as an abelian group, but is strictly-finite as a Γ-module.

LEMMA 8. *The projection from $\prod_{n=0}^{\infty} A_n$ to A_0 induces an injection from T to A_0.*

PROOF. Let c be an element of T of order 2^i. Then since T is cyclic, $2^{i-1}c$ coincides with c^* of Lemma 6. Taking the 0-th factors, we get $2^{i-1}c_0 = c(2_0)$. We are done since $c(2_0)$ is of order 2 in A_0.

By this lemma, we have $(T + \omega_n X)/\omega_n X \cong T$ for all $n \geqq 0$. Let $X^* = X/T$, then it holds that $2^{e_n} = [X: \omega_n X] = [X^*: \omega_n X^*][T]$ for all $n \geqq 0$. Applying Iwasawa's method to X^*, we can get the same result as that in the preceding case.

THEOREM 2. *Let the notations be as in the introduction and Theorem 1. Then it holds that*

$$e_n = \lambda n + c \qquad \text{for all} \quad n \geqq 1 + \max_{p|m} n(p).$$

REFERENCES

[1] K. IWASAWA, A note on class numbers of algebraic number fields, Abh. Math. Sem. Univ. Hamburg 20 (1956), 257-258.

[2] K. IWASAWA, On Γ-extensions of algebraic number fields, Bull. Amer. Math. Soc., 65 (1959), 183-226.

[3] K. IWASAWA, On the μ-invariants of $\mathbf{Z}_l$-extensions, In: Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, 1-11.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
YAMAGATA UNIVERSITY
YAMAGATA, 990 JAPAN