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# WEIGHT FUNCTIONS ON PROBABILITY SPACES

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**Introduction.** B. Muckenhoupt [3] proved that a nonnegative function $V \in L^1_{\mathrm{loc}}(R)$ satisfies

$$(*) \qquad \sup\left\{\int_R f^*(x)^p V(x)dx\left(\int_R |f(x)|^p V(x)dx\right)^{-1}: f \text{ is measurable}\right\} < \infty ,$$

where $1 < p < \infty$ and $f^*$ is the Hardy maximal function

$$f^*(x) = \sup\left\{(y-x)^{-1}\int_x^y |f(t)|dt: y \in R\backslash\{x\}\right\} ,$$

if and only if

$$A_p(V) = \sup\left\{(y-x)^{-p}\int_x^y V(t)dt\left(\int_x^y V(t)^{-1/(p-1)}dt\right)^{p-1}: -\infty < x < y < \infty\right\} < \infty .$$

The proof of this result consists of the following two theorems. [See also R. Coifman and C. Fefferman [1].]

THEOREM A. *Let $U$ and $V$ be nonnegative measurable functions on $R$, $p \in (1, \infty)$,*

$$A_p(U, V) = \sup\left\{(y-x)^{-p}\int_x^y U(t)dt\left(\int_x^y V(t)^{-1/(p-1)}dt\right)^{p-1}: -\infty < x < y < \infty\right\}$$

*and*

$$W_p(U, V) = \sup\left\{\lambda^p\int_{\{f^*>\lambda\}} U(t)dt\left(\int_R |f(t)|^p V(t)dt\right)^{-1}: \lambda > 0,\ f \text{ is measurable}\right\} .$$

*Then*

$$A_p(U, V) \leqq W_p(U, V) \leqq C(p)A_p(U, V) .$$

THEOREM B. *If $V$ is a nonnegative measurable function and $A_p(V) < M$ for some $p \in (1, \infty)$ and $M < \infty$, then there exist $\delta(M, p) > 0$*

*and* $N(M, p) < \infty$ *such that*

$$H_\delta(V) = \sup\left\{(y-x)^{\delta/(1+\delta)}\left(\int_x^y V(t)^{1+\delta}dt\right)^{1/(1+\delta)}\left(\int_x^y V(t)dt\right)^{-1}: -\infty < x < y < \infty\right\} < N.$$

Since $A_{p/(p-1)}(V^{-1/(p-1)}) = A_p(V)^{1/(p-1)}$, applying Theorem B to $V^{-1/(p-1)}$ we get

$$A_{p-\varepsilon}(V) \leqq (A_{p/(p-1)}(V^{-1/(p-1)})H_{\varepsilon/(p-\varepsilon-1)}(V^{-1/(p-1)}))^{p-1} < \infty$$

for some $\varepsilon > 0$. Then (*) follows from Theorem A and the Marcinkiewicz interpolation theorem. On the other hand, by Theorem A it is clear that (*) implies $A_p(V) < \infty$.

In this note we consider these theorems on a probability space with a sequence of nondecreasing sub $\sigma$-fields. The definitions of the maximal function, $A_p$, $W_p$, and $H_\delta$ on this probability space will be given in the following sections. The essential techniques are due to [1] and [3].

**1. The "weak type" problem.** Let $(\Omega, F, P)$ be a probability space with a sequence of sub $\sigma$-fields

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \subset F$$

such that $\bigvee_{n=1}^{\infty} F_n = F$. Let $V$, $U$, and $X$ be any nonnegative $F$-measurable functions. Let $\varepsilon$ and $\lambda$ be arbitrary positive numbers. We define $\circ$, $X^*$, $A_p$ and $W_p (1 \leqq p < \infty)$ as follows:

$$a \circ b = ab \qquad \text{for} \quad a, b \in [0, \infty),$$

$$a \circ \infty = \infty \circ a = \infty \qquad \text{for} \quad a \in [0, \infty],$$

$$X^* = \sup_n E[X|F_n],$$

$$A_p(U, V) = \sup_n \|E[V^{-1/(p-1)}|F_n]^{p-1}E[U|F_n]\|_\infty \qquad \text{for} \quad p \in (1, \infty),$$

$$A_1(U, V) = \sup_n \|V^{-1}E[U|F_n]\|_\infty,$$

$$W_p(U, V) = \sup_{X,\lambda} \lambda^p \int_{\{X^*>\lambda\}} UdP\left(\int X^p VdP\right)^{-1} \qquad \text{for} \quad p \in [1, \infty).$$

The above definition of $A_p$ is due to M. Izumisawa and N. Kazamaki [2]. In the case $U = V$, they proved that $A_p < \infty$ implies

$$\sup_X \int X^{*q} VdP\left(\int X^q VdP\right)^{-1} < \infty$$

for $q > p$ and conversely

$$\sup_X \int X^{*p} V dP \left( \int X^p V dP \right)^{-1} < \infty$$

implies $A_p < \infty$.

Following the theory of [1] and [3], we extend the result of [2], that is,

THEOREM 1. *$A_p = W_p$ for $p \in [1, \infty)$.*

REMARK. The fact that $W_p \leqq A_p$ has been pointed out by T. Tsuchikura.

For the proof of Theorem 1 we prove the following three lemmas.

LEMMA 1. *Set*

$$X^{**} = \sup_n E[XV|F_n] \circ E[U|F_n]^{-1} .$$

*Then* $\lambda \int_{\{X^{**} > \lambda\}} U dP \leqq \int XV dP$.

PROOF. Set

$$B_n = \{\omega \in \Omega\colon E[XV|F_n] \circ E[U|F_n]^{-1} > \lambda \quad \text{and}$$
$$E[XV|F_i] \circ E[U|F_i]^{-1} \leqq \lambda \quad \text{for} \quad i = 1, 2, \cdots, n-1\} .$$

Since $B_n \in F_n$,

$$\lambda \int_{B_n} U dP = \lambda \int_{B_n} E[U|F_n] dP \leqq \int_{B_n} E[XV|F_n] dP = \int_{B_n} XV dP .$$

Summing up for $n = 1, 2, \cdots$, we get the desired inequality.

LEMMA 2. *Let $F'$ be an arbitrary sub $\sigma$-field of $F$. Then*

$$X \leqq \lim_{n \to \infty} E[X^n | F']^{1/n} \qquad \text{a.s.}$$

PROOF. By Hölder's inequality $E[X^n|F']^{1/n}$ is monotone increasing. Set

$$B_\lambda = \{\omega \in \Omega\colon \lim_{n \to \infty} E[X^n|F']^{1/n} \leqq \lambda\} .$$

Then it suffices to show that $X \leqq \lambda$ on $B_\lambda$. Since $B_\lambda \in F'$,

$$(\lambda + \varepsilon)^n P(B_\lambda \cap \{\omega\colon X > \lambda + \varepsilon\}) \leqq \int_{B_\lambda} X^n dP$$
$$= \int_{B_\lambda} E[X^n|F'] dP \leqq \lambda^n P(B_\lambda) .$$

Thus letting $n \to \infty$, we have

$$P(B_\lambda \cap \{\omega\colon X > \lambda + \varepsilon\}) = 0$$

and we get the desired inequality.

LEMMA 3. $W_p = \lim_{\varepsilon\downarrow 0} W_{p+\varepsilon}$.

PROOF. As it is trivial that $W_p \leqq \liminf_{\varepsilon\downarrow 0} W_{p+\varepsilon}$, it suffices to prove that $W_p \geqq \limsup_{\varepsilon\downarrow 0} W_{p+\varepsilon}$. Take an arbitrary $\alpha\in(0, 1)$. Set $B_{\alpha\lambda} = \{X > \alpha\lambda\}$. Then

$$\begin{aligned}
&\lambda^{p+\varepsilon}\int_{\{X^*>\lambda\}} UdP\Big(\int X^{p+\varepsilon}VdP\Big)^{-1}\\
&\leqq \lambda^{\varepsilon}\lambda^{p}\int_{\{(I(B_{\alpha\lambda})X)^*>(1-\alpha)\lambda\}} Udp\Big(\lambda^{\varepsilon}\alpha^{\varepsilon}\int (I(B_{\alpha\lambda})X)^{p}VdP\Big)^{-1}\\
&= \alpha^{-\varepsilon}(1-\alpha)^{-p}(1-\alpha)^{p}\lambda^{p}\int_{\{(I(B_{\alpha\lambda})X)^*>(1-\alpha)\lambda\}} UdP\Big(\int (I(B_{\alpha\lambda})X)^{p}VdP\Big)^{-1}\\
&\leqq \alpha^{-\varepsilon}(1-\alpha)^{-p}W_p ,
\end{aligned}$$

where $I(B)$ is the indicator function of a measurable set $B$. Thus we get $\limsup_{\varepsilon\downarrow 0} W_{p+\varepsilon} \leqq (1-\alpha)^{-p}W_p$ for any $\alpha\in(0, 1)$, i.e., $\limsup_{\varepsilon\downarrow 0} W_{p+\varepsilon} \leqq W_p$.

PROOF OF THEOREM 1. First we consider the case $1 < p < \infty$. Since

$$\begin{aligned}
E[X|F_n] &\leqq E[X^pV|F_n]^{1/p}\circ E[V^{-1/(p-1)}|F_n]^{(p-1)/p}\\
&\leqq (A_p+\varepsilon)^{1/p}(E[X^pV|F_n]\circ E[U|F_n]^{-1})^{1/p} ,
\end{aligned}$$

by Lemma 1

$$\lambda^p\int_{\{X^*>\lambda\}} UdP \leqq \lambda^p\int_{\{(A_p+\varepsilon)(X^p)^{**}>\lambda^p\}} UdP \leqq (A_p+\varepsilon)\int X^pVdP .$$

Thus we get

$$W_p \leqq A_p . \tag{1}$$

Let $n$ be an arbitrary positive integer and $\alpha$ be an arbitrary number greater than 1. Set

$$\begin{aligned}
B_{ij} &= \{\omega: E[V^{-1/(p-1)}|F_n]\in(\alpha^i, \alpha^{i+1}], E[U|F_n]\in(\alpha^j, \alpha^{j+1}]\} ,\\
B_{i\infty} &= \{\omega: E[V^{-1/(p-1)}|F_n]\in(\alpha^i, \alpha^{i+1}], E[U|F_n]=\infty\}
\end{aligned}$$

and

$$B_\infty = \{\omega: E[V^{-1/(p-1)}|F_n]=\infty\}$$

for $i, j = 0, \pm1, \pm2, \cdots$. Let $Y = V^{-1/(p-1)}I(B_{ij})$. By $B_{ij}\in F_n$,

$$\begin{aligned}
\alpha^{ip}\alpha^{j}P(B_{ij}) &\leqq \alpha^{ip}\int_{B_{ij}} E[U|F_n]dP = \alpha^{ip}\int_{B_{ij}} UdP\\
&\leqq \alpha^{ip}\int_{\{Y^*>\alpha^i\}} UdP \leqq (W_p+\varepsilon)\int Y^pVdP \leqq (W_p+\varepsilon)\int_{B_{ij}} V^{-1/(p-1)}dP\\
&= (W_p+\varepsilon)\int_{B_{ij}} E[V^{-1/(p-1)}|F_n]dP \leqq (W_p+\varepsilon)\alpha^{i+1}P(B_{ij}) .
\end{aligned}$$

Thus if $P(B_{ij}) \neq 0$,

$$(\alpha^{-1}E[V^{-1/(p-1)}|F_n])^{p-1}(\alpha^{-1}E[U|F_n]) \leqq W_p\alpha$$

a.s. on $B_{ij}$. Using the same argument for $Y = V^{-1/(p-1)}I(B_{i\infty})$ we get

$$\alpha^{ip}\infty P(B_{i\infty}) \leqq (W_p + \varepsilon)\alpha^{i+1}P(B_{i\infty}) , \qquad \text{i.e.,} \quad P(B_{i\infty}) = 0 .$$

Let $T_j = \min(V^{-1}, j)$ and $Y = T_j^{1/(p-1)}E[T_j^{1/(p-1)}|F_n]^{-1/p}$. Then

$$\lambda^p \int_{\{E[T_j^{1/(p-1)}|F_n]>\lambda^{p/(p-1)}\}} UdP \leqq \lambda^p \int_{\{Y^*>\lambda\}} UdP \leqq (W_p + \varepsilon)\int Y^p VdP$$
$$= (W_p + \varepsilon)\int E[T_j^{p/(p-1)}V|F_n]E[T_j^{1/(p-1)}|F_n]^{-1}dP$$
$$\leqq (W_p + \varepsilon)\int E[T_j^{1/(p-1)}|F_n]E[T_j^{1/(p-1)}|F_n]^{-1}dP \leqq W_p + \varepsilon .$$

Letting $j \to \infty$, we get

$$\lambda^p \int_{B_\infty} UdP \leqq \lambda^p \int_{\{E[V^{-1/(p-1)}|F_n]>\lambda^{p/(p-1)}\}} UdP \leqq W_p .$$

Letting $\lambda \to \infty$, we have $U = 0$ a.s. on $B_\infty$. Thus

$$\Omega = (\bigcup_{\substack{-\infty<i<\infty \\ -\infty<j<\infty}} B_{ij}) \cup \{\omega : E[U|F_n] = 0 \text{ or } E[V^{-1/(p-1)}|F_n] = 0\} .$$

Therefore

$$(\alpha^{-1}E[V^{-1/(p-1)}|F_n])^{p-1}(\alpha^{-1}E[U|F_n]) \leqq W_p\alpha \quad \text{a.s. on } \Omega .$$

By (1) and the arbitrariness of $\alpha(>1)$ and $n$, we get

$$(2) \qquad A_p = W_p \qquad \text{for} \quad p \in (1, \infty) .$$

Since

$$E[V^{-1/(p-1)}|F_n]E[U|F_n]^{1/(p-1)} = E[(V^{-1}E[U|F_n])^{1/(p-1)}|F_n] \leqq A_1^{1/(p-1)} ,$$
$$\lim_{p\downarrow 1} A_p \leqq A_1 .$$

On the other hand, by Lemma 2

$$V^{-1}E[U|F_n] \leqq \lim_{m\to\infty} E[V^{-m}|F_n]^{1/m}E[U|F_n] .$$

Thus we get $A_1 = \lim_{p\downarrow 1} A_p$. Then by Lemma 3 and (2) we get $A_1 = W_1$.

**2. The reverse Hölder inequality.** Let $(\Omega, F, P)$, $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots \subset F$ and $V$ be as in Section 1. In this section further we assume $F_1 = \{\emptyset, \Omega\}$. For $p \in (1, \infty)$ set

$$A_p(V) = \sup_n \|E[V^{-1/(p-1)}|F_n]^{p-1}E[V|F_n]\|_\infty$$

and

$$H_\delta(V) = \sup_n \|E[V^{1+\delta}|F_n]^{1/(1+\delta)}E[V|F_n]^{-1}\|_\infty .$$

Recently C. Watari has pointed out the following

THEOREM C. *Let $(F_n)$ be regular, that is, each $F_n$ is atomic and there is a constant $C_0 > 0$ satisfying $P(B)/P(D) < C_0$ for any two atoms $B \in F_{n-1}$ and $D \in F_n$ with $B \supset D$. Then for each $p \in (1, \infty)$ and $M \in (1, \infty)$ there exist $\delta(p, M, C_0) > 0$ and $N(p, M, C_0) < \infty$ such that $H_\delta(V) \leqq N$ provided that $A_p(V) \leqq M$.*

Now we show that the regularity of $(F_n)$ in the above theorem is necessary.

THEOREM 2. *Assume that there exist $M, N > 1$, $p \in (1, \infty)$ and $\delta > 0$ such that $H_\delta(V) \leqq N$ provided that $A_p(V) \leqq M$. Then $(F_n)$ is regular.*

PROOF. Assume that $F_n$ is atomic. Let $B$ be an arbitrary atom of $F_n$. Let $D \in F_{n+1}$ and $D \subset B$. Set

$$V = 1 + (M-1)P(B)I(D)/P(D) .$$

Then

$$E[V|F_m]E[V^{-1/(p-1)}|F_m]^{p-1} = 1 \qquad \text{for} \quad m \geqq n+1$$

and

$$E[V|F_m]E[V^{-1/(p-1)}|F_m]^{p-1} \leqq M \qquad \text{for} \quad m \leqq n, \quad \text{i.e.,} \quad A_p(V) \leqq M .$$

Since

$$E[V^{1+\delta}|F_n] \geqq (M-1)^{1+\delta}(P(B)/P(D))^\delta \quad \text{on } B \text{ and } \quad E[V|F_n] \leqq M ,$$

by hypothesis

$$(P(B)/P(D))^{\delta/(1+\delta)} \leqq NM(M-1)^{-1} . \tag{3}$$

Thus $F_{n+1}$ is also atomic and (3) is satisfied for any two atoms $B \in F_n$ and $D \in F_{n+1}$ with $D \subset B$, that is, $(F_n)$ is regular.

Finally we add the following

THEOREM 3. *If $(F_n)$ is not regular, there exists $V$ such that $A_1(V) < \infty$ and $H_\delta(V) = \infty$ for any $\delta > 0$, where*

$$A_1(V) = \sup_n \|V^{-1}E[V|F_n]\|_\infty .$$

PROOF. First assume that $F_n$ does not consist of a finite number of atoms for some $n$. Since $F_1 = \{\emptyset, \Omega\}$, we may assume $F_{n-1}$ consists of a finite number of atoms. Let $D_k \in F_n$, $D_k \subset B$, $0 < P(D_k)/P(B) < 2^{-k^2}$

for $k = 1, 2, \cdots$, $D_k \cap D_h = \emptyset (k \neq h)$ and $B$ be an atom of $F_{n-1}$. Set

$$V = 1 + \sum_{k=1}^{\infty} 2^{-k} P(B) I(D_k)/P(D_k) .$$

Then $A_1(V) \leqq 2$ and

$$E[V^{1+\delta} | F_n] \geqq \sum_{k=1}^{\infty} 2^{-k(1+\delta)} (P(B)/P(D_k))^{\delta} = \infty$$

on $B$ for any $\delta > 0$. So, $H_\delta(V) = \infty$ for any $\delta > 0$.

Thus we may assume $F_n$ consists of a finite number of atoms. Let $\{B_n\}$ and $\{D_n\}$ be sequences of atoms of $\{F_{i(n)}\}$ and $\{F_{i(n)+1}\}$ respectively such that $D_n \subset B_n$, $\lim P(D_n)/P(B_n) = 0$ and $P(D_n) < \min(2^{-n^2}, 4^{-1}P(D_{n-1}))$.

Assume that $\inf P(B_n) = c_0 > 0$. Set

$$V = 1 + \sum_{k=1}^{\infty} 2^{-k} P(D_k)^{-1} I(D_k \cap (\bigcap_{h>k} D_h^c)) .$$

Take an arbitrary $n$ and an arbitrary atom $B$ of $F_n$. If $P(B) \geqq c_0$, then $E[V | F_n] \leqq 2c_0^{-1}$ on $B$. If $P(B) < c_0$, then $E[V | F_n] = V$ on $B$. Thus $A_1(V) \leqq 2c_0^{-1}$. On the other hand,

$$E[V^{1+\delta} | F_1] \geqq \sum 2^{-k(1+\delta)} P(D_k)^{-1-\delta} P(D_k)/2 = \infty ,$$

so $H_\delta(V) = \infty$ for any $\delta > 0$.

Assume that $\liminf_{n\to\infty} P(B_n) = 0$. In this case we may assume $i(n) + 1 < i(n + 1)$ and

$$(4) \qquad P(D_n) > 2P(B_{n+1}) .$$

By selecting a subsequence, if necessary, it suffices to consider the following two cases:

$$(5) \qquad B_1 \supset B_2 \supset \cdots ,$$

$$(6) \qquad B_h \cap B_k = \emptyset \qquad (h \neq k) .$$

In the case of (6), set

$$V = 1 + \sum_{k=1}^{\infty} P(B_k) I(D_k)/P(D_k) .$$

Then $A_1(V) \leqq 2$ and

$$E[V^{1+\delta} | F_{i(n)}]^{1/(1+\delta)} E[V | F_{i(n)}]^{-1} \geqq (P(B_n)/P(D_n))^{\delta/(1+\delta)} 2^{-1}$$

on $B_n$. Thus $H_\delta(V) = \infty$ for any $\delta > 0$.

Lastly in the case of (5), we define $V$ and $\{\alpha_k\}_{k=1}^{\infty}$ as follows. Let $V = 1$ on $B_1^c$ and $\alpha_1 = 1$. When $V$ is defined on $B_k^c$ and $\alpha_k$ is defined, let

$$V = \alpha_k \quad \text{on} \quad B_k \cap D_k^c \cap B_{k+1}^c ,$$
$$V = \alpha_k P(B_k)/P(D_k) \quad \text{on} \quad D_k \cap B_{k+1}^c ,$$
$$\alpha_{k+1} = \alpha_k \quad \text{if} \quad B_{k+1} \not\subset D_k ,$$
$$\alpha_{k+1} = \alpha_k P(B_k)/P(D_k) \quad \text{if} \quad B_{k+1} \subset D_k .$$

Then

$$\int_{B_k \setminus B_{k+1}} V dP \leqq 2\alpha_k P(B_k) \tag{7}$$

and

$$\alpha_j \leqq \alpha_k \prod_{i=k}^{j-1} (P(B_i)/P(D_i)) . \tag{8}$$

Assume that $i(k-1)+1 \leqq n \leqq i(k)$ and $B$ is an arbitrary atom of $F_n$. If $B \not\supset B_k$, $E[V|F_n] = V$ on $B$. If $B \supset B_k$, by (4), (7), and (8)

$$E[V|F_n] = P(B)^{-1}\int_B V dP \leqq P(B)^{-1}\left\{\alpha_k P(B) + \sum_{j=k}^{\infty} 2\alpha_j P(B_j)\right\}$$
$$\leqq 2P(B)^{-1}\left\{\alpha_k P(B) + \sum_{j=k}^{\infty} \alpha_k (P(B_k)/P(D_k)) \cdots (P(B_{j-1})/P(D_{j-1}))P(B_j)\right\} \leqq 6\alpha_k$$

and $V^{-1} \leqq \alpha_k^{-1}$ on $B$. Thus $A_1(V) \leqq 6$. But on $B_k$

$$E[V^{1+\delta}|F_{i(k)}] \geqq \alpha_k^{1+\delta}(P(B_k)/P(D_k))^{\delta}/2$$

and $E[V|F_{i(k)}] \leqq 4\alpha_k$. Thus $H_\delta(V) = \infty$ for any $\delta > 0$.

## References

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