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TOEPLITZ OPERATORS FOR UNIFORM ALGEBRAS

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Let $C(\boldsymbol{T})$ be the Banach algebra of complex valued continuous functions on the unit circle $\boldsymbol{T}$ in the complex plane and $P(\boldsymbol{T})$ be the subalgebra of $C(\boldsymbol{T})$ consisting of those functions with continuous extensions to the closed unit disc which are holomorphic in the open unit disc. Let m be the normalized Lebesgue measure on $\boldsymbol{T}$ and $H^2(\boldsymbol{T})$ be the $L^2(m)$-closure of $P(\boldsymbol{T})$. Let P denote the orthogonal projection of $L^2(m)$ onto $H^2(\boldsymbol{T})$. For ϕ in $L^\infty(m)$ the Toeplitz operator T_ϕ on $H^2(\boldsymbol{T})$ is defined by $T_\phi(f) = P(\phi f)$ for f in $H^2(\boldsymbol{T})$. Let $\mathscr{T}(C(\boldsymbol{T}))$ be the C^*-algebra generated by the set $\{T_\phi;\ \phi \in C(\boldsymbol{T})\}$ and $\mathscr{C}(C(\boldsymbol{T}))$ be the commutator ideal of $\mathscr{T}(C(\boldsymbol{T}))$. Then it is known that there exists a $*$-homomorphism ρ from $\mathscr{T}(C(\boldsymbol{T}))$ onto $C(\boldsymbol{T})$ such that the following sequence is exact,

$$(*)\qquad \{0\} \longrightarrow \mathscr{C}(C(\boldsymbol{T})) \xrightarrow{\ i\ } \mathscr{T}(C(\boldsymbol{T})) \xrightarrow{\ \rho\ } C(\boldsymbol{T}) \longrightarrow \{0\}$$

and $\rho(T_\phi) = \phi$, where i is the inclusion map. Further in this case $\mathscr{C}$ coincides with the closed ideal $\mathscr{L}\mathscr{C}(H^2(\boldsymbol{T}))$ consisting of all bounded linear compact operators on $H^2(\boldsymbol{T})$ and it holds

$$(**)\qquad \{0\} \longrightarrow \mathscr{L}\mathscr{C}(H^2(\boldsymbol{T})) \longrightarrow \mathscr{T}(C(\boldsymbol{T})) \xrightarrow{\ \rho\ } C(\boldsymbol{T}) \longrightarrow \{0\}$$

is exact and $\rho(T_\phi) = \phi$ for all $\phi \in C(\boldsymbol{T})$. This fact is generalized to many cases, to multiply connected domains in the complex plane [1] and to strongly pseudo-convex domains in $\boldsymbol{C}^n$ [16], [10], [17] and in Stein spaces [13]. In order to obtain the exact sequence $(**)$ it is important to get the exact sequence $(*)$. On the other hand, from an exact sequence of type $(*)$, itself, one can deduce some consequences (see [7, Section 3] and Corollaries 1.6, 2.2, 2.3 and Proposition 2.8 in this note).

In this note we regard the notion of $\mathscr{T}(C(\boldsymbol{T}))$ as a linear representation τ of $C(X) = C(\boldsymbol{T})$ on a compact Hausdorff space $X = \boldsymbol{T}$ into the C^*-algebra of all bounded linear operators on a Hilbert space $H = H^2(\boldsymbol{T})$ satisfying

(1) $\|\tau\| \leqq 1$ and $\tau(1) = 1$: identity operator,
(2) τ is isometric on the uniform algebra $A = P(\boldsymbol{T})$,
(3) $\tau(\phi\varphi) = \tau(\phi)\tau(\varphi)$ for all $\phi \in C(X)$ and $\varphi \in A$.

In Section 1 we set up this formulation for any uniform algebra and will obtain an exact sequence of type $(*)$ (Theorem 1.4). Relating to it we give an ideal theoretic characterization of joint approximate point spectrum within the category of C^*-algebras. These are closely related to the Bunce's results in [2]. In Section 2 we introduce a notion of Toeplitz operator for uniform algebras and apply our results in Section 1. In Section 3 we treat applications of the results in Section 2 to some concrete cases.

For any topological space X we always denote by $C(X)$ the C^*-algebra of all complex valued continuous functions on X, endowed with supremum norm.

1. Toeplitz operators in an abstract setting. Let X be a compact Hausdorff space and A be a uniform algebra on X, i.e., A is a uniformly closed subalgebra of $C(X)$ which contains the constants and separates points in X. We will denote by $\Gamma(A)$ the Shilov boundary of A and by $Q(A)$ the Choquet boundary (=strong boundary) of A. A bounded linear functional α of A is said to be a state if it satisfies the condition $\alpha(1) = 1 = \|\alpha\|$. The set of all states of A forms a weak* compact convex subset in the dual space of A. We also call an extreme point of this state space a pure state of A. One may employ these definitions for any linear subspace of a C^*-algebra which contains the unit of the algebra. In case of the algebra A, the set of pure states corresponds to the evaluations of the Choquet boundary points (cf. [12, Section 6]). Thus, a pure state of A has a unique pure state extension to $C(X)$, and hence by the theorem of Krein-Milman its state extension is unique, too. We shall often use this observation in our subsequent discussions. We identify the points of X with characters. The Shilov boundary is, as a subset of characters of A, the weak* closure of the set of pure states.

Now let τ be a linear representation of $C(X)$ into the algebra of all bounded linear operators, $\mathscr{L}(H)$, on a Hilbert space H. We assume that τ is contractive, i.e., $\|\tau\| \leqq 1$, and $\tau(1) = 1$, the identity operator in H. This implies that τ is a positive map of $C(X)$ into $\mathscr{L}(H)$.

PROPOSITION 1.1. *Suppose that τ is isometric on A. Then for every function ϕ of $C(X)$ we have the inequality $\|\tau(\phi)\| \geqq \max\{|\phi(t)|;\, t \in \Gamma(A)\}$. In particular, if $X = \Gamma(A)$ the representation τ is an isometry.*

PROOF. Let t be a point in $Q(A)$, then it gives rise to a pure state α of $\tau(A)$. Let $\hat{\alpha}$ be a norm preserving extension of α on the subspace $\tau(C(X))$. One easily sees that the functional $\hat{\alpha}\circ\tau$ is a state extension of the character t of A. From the observation stated before, we have

$$|\phi(t)| = |\hat{\alpha} \circ \tau(\phi)| \leqq \|\tau(\phi)\| .$$

Taking the closure of $Q(A)$, we get the conclusion.

From now on we keep the following conditions for τ:

(a) τ is isometric on A,

(b) $\tau(\varphi\phi) = \tau(\varphi)\tau(\phi)$ for all $\varphi \in C(X)$ and $\phi \in A$.

The assumptions are abstract setting of the family of Toeplitz operators (cf. [7]) and our later result (Theorem 1.4) gives a general structure theorem for the C^*-algebra generated by the family $\{\tau(\phi); \phi \in C(X)\}$. With these conditions $\tau|_A$, the restriction τ in A, becomes an algebraic isomorphism and $\tau(A)$ is a commutative Banach subalgebra of $\mathscr{L}(H)$ containing the identity operator in H. Moreover we can show the following

LEMMA 1.2. *For every function ϕ of A, $\tau(\phi)$ is a subnormal operator and hence a hyponormal operator.*

We recall that an operator T is hyponormal if $TT^* \leqq T^*T$.

PROOF. Since the map τ is a positive map of $C(X)$ into $\mathscr{L}(H)$ with $\tau(1) = 1$, it is completely positive and by the theorem of Stinespring [14] there exists a dilation space K of H and a *-homomorphism π of $C(X)$ into $\mathscr{L}(K)$ such that $\tau(\phi) = P\pi(\phi)P$, where P denotes the projection of K onto the subspace H. Hence, by the assumption (b), we have for every $\phi \in A$

$$P\pi(\phi)^*\pi(\phi)P = P\pi(\phi)^*P\pi(\phi)P .$$

Since $1 - P$ is also a projection, we get

$$[P\pi(\phi)^*(1 - P)][(1 - P)\pi(\phi)P] = 0 ,$$

and hence

$$(1 - P)\pi(\phi)P = 0 ,$$

i.e.,

$$\tau(\phi) = P\pi(\phi)P = \pi(\phi)P .$$

This means $\tau(\phi)$ is subnormal. As is well known, every subnormal operator is hyponormal.

Let $\mathscr{T}(C(X))$ be the C^*-algebra generated by the operators $\{\tau(\phi); \phi \in C(X)\}$. Since A is a uniform algebra on X the algebra $C(X)$ is the closed linear span of the set $\{\varphi\bar{\psi}; \varphi, \psi \in A\}$ by the Stone-Weierstrass theorem, and so one may regard $\mathscr{T}(C(X))$ as the C^*-algebra generated by the set $\{\tau(\varphi); \varphi \in A\}$, the commutative Banach algebra of subnormal operators in H. Let α_t be the pure state of $\tau(A)$ induced by a point t

of $Q(A)$. The following is a key lemma for our main result.

LEMMA 1.3. *The state α_t extends uniquely to a state of $\mathscr{T}(C(X))$ and the extended state is a character of $\mathscr{T}(C(X))$.*

PROOF. Since $\tau(A)$ separates the characters of $\mathscr{T}(C(X))$, it suffices to prove that any state extension $\hat{\alpha}$ of α_t is necessarily a character of $\mathscr{T}(C(X))$. Let L be the left kernel of $\hat{\alpha}$ i.e., $L=\{S\in\mathscr{T}(C(X)); \hat{\alpha}(S^*S)=0\}$. We note first that $\hat{\alpha}(\tau(\phi))=\phi(t)$ for every ϕ of $C(X)$ by the uniqueness of the state extension of the state t on A. Hence, for every function φ of A we have

$$(1)\qquad \hat{\alpha}((\tau(\varphi)-\varphi(t))^*(\tau(\varphi)-\varphi(t))=\hat{\alpha}(\tau(\bar{\varphi}-\overline{\varphi(t)})\tau(\varphi-\varphi(t)))$$
$$=\hat{\alpha}\circ\tau(|\varphi-\varphi(t)|^2)=0\,.$$

Thus, the operator $\tau(\varphi)-\varphi(t)$ belongs to L and as L is a left ideal of $\mathscr{T}(C(X))$ the set $\mathscr{T}(C(X))(\tau(\varphi)-\varphi(t))$ is contained in L. Therefore, $\hat{\alpha}(S(\tau(\varphi)-\varphi(t)))=0$ for every element S of $\mathscr{T}(C(X))$. This implies that

$$\hat{\alpha}(ST)=\hat{\alpha}(S)\hat{\alpha}(T)\quad\text{for all}\quad S\in\mathscr{T}(C(X))\quad\text{and}\quad T\in\tau(A)\,.$$

Next, by Lemma 1.2 $\tau(\varphi)-\varphi(t)$ is hyponormal and so using (1) we get

$$\hat{\alpha}((\tau(\varphi)-\varphi(t))(\tau(\varphi)-\varphi(t))^*)=0\,,$$

and $\tau(\varphi-\varphi(t))^*$ belongs to L. Hence the same argument as above shows that

$$\hat{\alpha}(ST^*)=\hat{\alpha}(S)\hat{\alpha}(T^*)=\hat{\alpha}(S)\overline{\hat{\alpha}(T)}\,,$$

and

$$\hat{\alpha}(TS)=\hat{\alpha}(T)\hat{\alpha}(S)\quad\text{for all}\quad S\in\mathscr{T}(C(X))\quad\text{and}\quad T\in\tau(A)\,.$$

As $\mathscr{T}(C(X))$ is the C^*-algebra generated by $\tau(A)$ one may easily conclude that $\hat{\alpha}$ is a character of $\mathscr{T}(C(X))$. This completes the proof.

We denote by Δ the character space of $\mathscr{T}(C(X))$ with the weak* topology. Let $\mathscr{C}(C(X))$ be the commutator ideal of $\mathscr{T}(C(X))$. We define a map of Δ into X as follows. Take a character β of $\mathscr{T}(C(X))$, and consider the state $\alpha=(\beta|\tau(C(X)))\circ\tau$ of $C(X)$ where $\beta|\tau(C(X))$ means the restriction of β to $\tau(C(X))$. We have

$$\alpha(\varphi\phi)=\alpha(\phi\varphi)=\beta(\tau(\phi\varphi))=\beta(\tau(\phi)\tau(\varphi))$$
$$=\beta(\tau(\phi))\beta(\tau(\varphi))=\alpha(\phi)\alpha(\varphi)=\alpha(\varphi)\alpha(\phi)$$

for all $\phi\in C(X)$ and $\varphi\in A$. Therefore, since $C(X)$ is the closed linear span of the set $\{\varphi\bar{\psi};\ \varphi,\psi\in A\}$, α is a character of $C(X)$ and there is a point t_β of X such that $\alpha(\phi)=\phi(t_\beta)$ for every $\phi\in C(X)$. The map, $\beta\to t_\beta$, is clearly a one-to-one continuous mapping of Δ into X. We denote by $\Gamma(\tau)$ the image of this map. The set $\Gamma(\tau)$ is compact and it is homeomorphic to Δ. With this preparation we state our main result in the following.

THEOREM 1.4. *$\Gamma(\tau)$ is a closed boundary for A. Moreover, there is a $*$-homomorphism ρ of $\mathscr{T}(C(X))$ to $C(\Gamma(\tau))$ such that the short sequence*

$$\{0\} \longrightarrow \mathscr{C}(C(X)) \xrightarrow{i} \mathscr{T}(C(X)) \xrightarrow{\rho} C(\Gamma(\tau)) \longrightarrow \{0\}$$

is exact and $\rho(\tau(\phi)) = \phi|\Gamma(\tau)$ for all $\phi \in C(X)$, where i is the inclusion map.

PROOF. By Lemma 1.3, $\Gamma(\tau)$ is a compact subset of X which contains $Q(A)$, hence it is a boundary for A. On the other hand, the homeomorphism between $\Gamma(\tau)$ and Δ induces a $*$-isomorphism between the algebras $C(\Gamma(\tau))$ and $C(\Delta)$, but the latter algebra may be regarded as the quotient C^*-algebra $\mathscr{T}(C(X))/\mathscr{C}(C(X))$ because the ideal $\mathscr{C}(C(X))$ is the intersection of the kernels of all characters of $\mathscr{T}(C(X))$. Thus, this defines naturally a $*$-homomorphism ρ of $\mathscr{T}(C(X))$ onto $C(\Gamma(\tau))$ and by definition $\rho(\tau(\phi)) = \phi|\Gamma(\tau)$ for every $\phi \in C(X)$. This completes the proof.

Next, let J be the kernel of τ, then J is a closed self-adjoint subspace of $C(X)$. Besides, from the assumption for τ, the function $\phi\varphi$ belongs to J for all $\phi \in J$ and $\varphi \in A$. Hence, $\phi\bar{\varphi} = \overline{(\bar{\phi}\varphi)} \in J$. It follows that J is an ideal of $C(X)$, and there exists a closed subset $S(\tau)$ of X such that

$$J = \{\phi \in C(X);\ \phi|_{S(\tau)} = 0\} .$$

One easily verifies that $S(\tau)$ is the smallest closed set for which τ annihilates the set $\{\phi \in C(X);\ \phi|_{S(\tau)} = 0\}$. We call $S(\tau)$ the support of τ. Then the following corollary of the above theorem is a sharpened estimation of the norm of $\tau(\phi)$ in the present situation.

COROLLARY 1.5. *For every function ϕ of $C(X)$, we have the estimation;*

$$\max\{|\phi(t)|;\ t \in S(\tau)\} \geqq \|\tau(\phi)\| \geqq \|\tau(\phi)\|_{sp} \geqq \max\{|\phi(t)|;\ t \in \Gamma(\tau)\} .$$

In particular, if $\tau(\phi)$ is quasi-nilpotent ϕ vanishes on $\Gamma(\tau)$. Here $\|\ \|_{sp}$ denotes the spectral norm.

PROOF. The last estimation is a consequence of the theorem. For the first inequality it is enough to mention that the maximum is equal to the quotient norm of ϕ in $C(X)/J$.

The above inequalities show that $S(\tau)$ contains $\Gamma(\tau)$ but the support

$S(\tau)$ may or may not coincide with $\Gamma(\tau)$. One can say the same thing about $\Gamma(\tau)$ and $\Gamma(A)$ as well.

COROLLARY 1.6. *Suppose that the weak closure of* $\mathscr{C}(C(X))$ *coincides with the weak closure of* $\mathscr{T}(C(X))$. *Then, if* $\sum_{i=1}^{n}\prod_{j=1}^{m}\tau(\phi_{i,j})$ *is compact, the function* $\sum_{i=1}^{n}\prod_{j=1}^{m}\phi_{ij}$ *vanishes on* $\Gamma(\tau)$.

The assumption is particularly satisfied when $\mathscr{T}(C(X))$ *is irreducible and* $\mathscr{C}(C(X))\neq\{0\}$.

PROOF. Let β be a character of $\mathscr{T}(C(X))$ and $\hat{\beta}$ be its pure state extension to $\mathscr{L}(H)$. Let $\mathscr{L}\mathscr{C}(H)$ be the ideal of $\mathscr{L}(H)$ consisting of all compact operators on H. Since the dual space of $\mathscr{L}(H)$ is the l_1-sum of the dual of $\mathscr{L}\mathscr{C}(H)$, i.e., the space of σ-weakly continuous linear functionals on $\mathscr{L}(H)$ and the polar of $\mathscr{L}\mathscr{C}(H)$ (Dixmier [4; Theorem 3]), $\hat{\beta}$ is either σ-weakly continuous or vanishing on $\mathscr{L}\mathscr{C}(H)$. From the assumption, however, $\hat{\beta}$ can not be σ-weakly continuous, hence it vanishes on $\mathscr{L}\mathscr{C}(H)$. It follows that the intersection $\mathscr{L}\mathscr{C}(H)\cap\mathscr{T}(C(X))$ is necessarily contained in $\mathscr{C}(C(X))$, so that the function $\sum_i\prod_j\phi_{ij}=\rho(\sum_i\prod_j\tau(\phi_{ij}))$ vanishes on $\Gamma(\tau)$.

In general, the ideal $\mathscr{C}(C(X))$ could be zero so that $\mathscr{T}(C(X))$ would become a commutative C^*-algebra. But this happens only in trivial cases because if it is the case then τ turns out to be a $*$-homomorphism of $C(X)$ to $\mathscr{T}(C(X))=\tau(C(X))$ and there are no nontrivial dilations for τ.

Let $(T_1, T_2, \cdots, T_n)$ be an n-tuple of commuting bounded operators on H. Define the joint approximate point spectrum $\sigma_\pi(T_1, T_2, \cdots, T_n)$ to be the set of all complex n-tuples $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ such that the set

$$\mathscr{L}(H)(T_1-\lambda_1)+\mathscr{L}(H)(T_2-\lambda_2)+\cdots+\mathscr{L}(H)(T_n-\lambda_n)$$

forms a proper left ideal of $\mathscr{L}(H)$. In [2], Bunce has shown that the joint spectrum $\sigma_\pi(T_1, T_2, \cdots, T_n)$ for those operators T_i's in $\tau(A)$ consists of the n-tuples $\{(\alpha(T_1), \alpha(T_2), \cdots, \alpha(T_n)); \alpha\in\Delta\}$. In connection with this we shall show an improved version of the essential part of Bunce's arguments in [2]. Namely, we have

PROPOSITION 1.7. *Let* $(T_1, T_2, \cdots, T_n)$ *be an n-tuple of commuting operators on H and let B be a* C^**-subalgebra of* $\mathscr{L}(H)$ *with unit which contains* $\{T_1, T_2, \cdots, T_n\}$. *Then, the following statements are equivalent for a complex n-tuple* $(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

(1) $(\lambda_1, \lambda_2, \cdots, \lambda_n)\in\sigma_\pi(T_1, T_2, \cdots, T_n)$

(2) $B(T_1-\lambda_1)+B(T_2-\lambda_2)+\cdots+B(T_n-\lambda_n)$ *is a proper left ideal of B,*

(3) *there exists a state* α *of* B *such that* $\alpha(T_i) = \lambda_i$ *and* $\alpha(ST_i) = \alpha(S)\alpha(T_i)$ *for all* $S \in B$ *and* i,

(4) *there exists a state* β *of* $\mathscr{L}(H)$ *such that* $\beta(T_i) = \lambda_i$ *and* $\beta(ST_i) = \beta(S)\beta(T_i)$ *for all* $S \in B$ *and* i.

PROOF. (1) $\Rightarrow$ (2). Suppose that

$$B(T_1 - \lambda_1) + B(T_2 - \lambda_2) + \cdots + B(T_n - \lambda_n) = B .$$

Then, there exists an n-tuple $(R_1, R_2, \cdots, R_n)$ of operators in B such that

$$R_1(T_1 - \lambda_1) + R_2(T_2 - \lambda_2) + \cdots + R_n(T_n - \lambda_n) = 1 .$$

But this means that

$$\mathscr{L}(H)(T_1 - \lambda_1) + \mathscr{L}(H)(T_2 - \lambda_2) + \cdots + \mathscr{L}(H)(T_n - \lambda_n) = \mathscr{L}(H) ,$$

whence $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ does not belong to $\sigma_\pi(T_1, T_2, \cdots, T_n)$.

(2) $\Rightarrow$ (3). From the assumption one sees that the closure of the set, $B(T_1 - \lambda_1) + B(T_2 - \lambda_2) + \cdots + B(T_n - \lambda_n)$, is a proper left ideal of B, too. Hence, there exists a state of B which vanishes on this left ideal (by [5, Theorem 2.9.5] the state can even be a pure state). Thus, the assertion (3) follows.

(3) $\Rightarrow$ (4). Let $\hat{\alpha}$ be a state extension of α to $\mathscr{L}(H)$ and let $\hat{L}$ and L be left kernels of $\hat{\alpha}$ and α, respectively. By the assumption, $T_i - \lambda_i \in L \subset \hat{L}$ for all i, so that the set $\mathscr{L}(H)(T_i - \lambda_i)$ is contained in $\hat{L}$ for all i. Hence, $\hat{\alpha}$ vanishes on these sets. The assertion (4) $\Rightarrow$ (1) is immediate, for β vanishes on the set

$$\mathscr{L}(H)(T_1 - \lambda_1) + \mathscr{L}(H)(T_2 - \lambda_2) + \cdots + \mathscr{L}(H)(T_n - \lambda_n) .$$

This completes the proof.

The proposition says that at least within the category of C^*-algebras we get a nice ideal theoretic characterization of the joint spectrum that does not depend on the choice of the algebra containing $\{T_1, T_2, \cdots, T_n\}$. It is also to be noticed that in case $B = C^*(T_1, T_2, \cdots, T_n)$ the required state α in the assertion (3) turns out to be a character whenever we have an implication, $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \sigma_\pi(T_1, T_2, \cdots, T_n) \Rightarrow (\bar{\lambda}_1, \bar{\lambda}_2, \cdots, \bar{\lambda}_n) \in \sigma_\pi(T_1^*, T_2^*, \cdots, T_n^*)$. The hyponormality of operators in Bunce's theorem is one of the typical conditions to yield the above implication.

2. Toeplitz operators for uniform algebras. As in Section 1 let X be a compact Hausdorff space and A be a uniform algebra on X, and $\Gamma(A)$ be the Shilov boundary of A. Now let μ be a finite nonnegative regular Borel measure on X. Then we define $H^2(\mu)$ as the $L^2(\mu)$ closure of A and for every $\phi \in L^\infty(\mu)$ we denote by M_ϕ the multiplication operator on $L^2(\mu)$, defined by $M_\phi f = \phi f$ for $f \in L^2(\mu)$ and by P the orthogonal projection of $L^2(\mu)$ onto $H^2(\mu)$. We define Toeplitz operator T_ϕ on $H^2(\mu)$ with $L^\infty(\mu)$ symbol ϕ by

$$T_\phi f = PM_\phi f \qquad \text{for} \quad f \in H^2(\mu) .$$

Let $H^\infty(\mu)$ be the weak* closure of A in $L^\infty(\mu)$. Then it is easily seen that for each $\phi \in L^\infty(\mu)$ and $\varphi \in H^\infty(\mu)$

$$T_\varphi f = \varphi f \quad \text{and} \quad T_\phi T_\varphi f = T_{\phi\varphi} f \quad \text{for} \quad f \in H^2(\mu) .$$

Hence T_φ is a subnormal operator on $H^2(\mu)$ if $\varphi \in H^\infty(\mu)$. It is also clear that the mapping $\phi \longrightarrow T_\phi$ is contractive for $\phi \in L^\infty(\mu) \cup C(X)$. Further we have

$$\| T_\varphi \| = \| \varphi \|_{L^\infty(\mu)} \quad \text{for} \quad \varphi \in H^\infty(\mu) .$$

Indeed, if $\varphi \in H^\infty(\mu)$, we have

$$\left(\int |\varphi|^j d\mu\right)^{1/j} = \left(\int | T_\varphi^j 1 | d\mu\right)^{1/j} \leqq \| T_\varphi^j 1 \|_2^{1/j} \| 1 \|_2^{1/j} \leqq \| T_\varphi \| \, \| 1 \|_2^{2/j}$$
$$(j = 1, 2, \cdots) .$$

Letting $j \longrightarrow \infty$ we have $\|\varphi\|_{L^\infty(\mu)} \leqq \| T_\varphi \|$. Since T_φ is a contraction we have $\|\varphi\|_{L^\infty(\mu)} = \| T_\varphi \|$. Now let τ be a linear mapping from $C(X)$ into $\mathscr{L}(H^2(\mu))$ defined by

$$\tau(\phi) = T_\phi \quad \text{for} \quad \phi \in C(X) .$$

Then τ is a linear representation of $C(X)$ into $\mathscr{L}(H^2(\mu))$ satisfying all the conditions for τ in Section 1, whenever supp $\mu \supset \Gamma(A)$. Hence applying Theorem 1.4 we have

THEOREM 2.1. *Suppose* support $\mu = \Gamma(A)$. *Then there exists a *-homomorphism ρ from $\mathscr{T}(C(X))$ onto $C(\Gamma(A))$ such that the short sequence*

$$\{0\} \longrightarrow \mathscr{C}(C(X)) \xrightarrow{\ i\ } \mathscr{T}(C(X)) \xrightarrow{\ \rho\ } C(\Gamma(A)) \longrightarrow \{0\}$$

is exact and $\rho(T_\phi) = \phi|_{\Gamma(A)}$ for all $\phi \in C(X)$, where i is the inclusion map.

PROOF. In this case we have

$$\Gamma(A) = \text{supp}\, \mu \supset S(\tau) \supset \Gamma(\tau) \supset \Gamma(A) ,$$

and hence $\Gamma(A) = \Gamma(\tau)$. Using Theorem 1.4 we get the desired conclusion.

REMARK. From a uniform algebra A we can always construct a model satisfying the assumptions in Theorem 2.1. Let Φ be a nonzero multiplicative linear functional on A and μ be a representing measure for Φ, i.e., $\int f d\mu = \Phi(f)$ for all $f \in A$, concentrated on the strong boundary of A (see Gamelin [9, p. 60] for the definition of strong boundary and p. 60 for the existence of such measures). Then $\Gamma(A|_{\operatorname{supp}\mu}) = \operatorname{supp}\mu$. Let A' be the uniform closure of $A|_{\operatorname{supp}\mu}$. Then A' is a uniform algebra on $\operatorname{supp}\mu$ and A' and μ satisfy the assumptions in Theorem 2.1.

As consequences of Theorem 2.1 we have

COROLLARY 2.2. *Let A, μ be as in Theorem 2.1. Then if $\phi \in C(X)$,*

$$\| T_\phi \| = \|T_\phi\|_{sp} = \max\{|\phi(x)|;\ x \in \Gamma(A)\} .$$

In particular, if $\phi \in C(X)$, T_ϕ is quasi-nilpotent if and only if $\phi|_{\Gamma(A)} = 0$.

COROLLARY 2.3. *Let A, μ be as in Theorem 2.1. Suppose $H^2(\mu) \neq L^2(\mu)$ and every real valued function in $H^2(\mu)$ is a constant function. Then if $\{\phi_{ij}, i=1,2,\cdots,n, j=1,2,\cdots,m\}$ are functions in $C(X)$ and $\sum_{i=1}^n \prod_{j=1}^m T_{\phi_{ij}}$ is compact, the function $\sum_{i=1}^n \prod_{j=1}^m \phi_{ij} = 0$ on $\Gamma(A)$. In particular, if $\phi \in C(X)$, then T_ϕ is compact if and only if $\phi = 0$ on $\Gamma(A)$.*

PROOF. By corollary 1.6 it suffices to show that $\mathscr{T}(C(X))$ is irreducible and $\mathscr{C}(C(X)) \neq \{0\}$. Assume $\mathscr{T}(C(X))$ is not irreducible. Then there exists an orthogonal projection Q of $H^2(\mu)$ such that $Q \neq 0, 1$ and $QT_\phi = T_\phi Q$ for all ϕ in A. Set $g = Q1 \in H^2(\mu)$. Then for any $\varphi \in A$ we have $Q\varphi = Q(T_\varphi 1) = T_\varphi(Q1) = \varphi g$. Hence for any $\varphi, \psi \in A$ we have

$$(g\varphi, \psi) = (Q\varphi, \psi) = (Q^2\varphi, \psi) = (Q\varphi, Q\psi) = (g\varphi, g\psi) = (|g|^2\varphi, \psi) ,$$

and hence

$$\int (g - |g|^2)\varphi\bar{\psi}d\mu = 0 .$$

Since A separates points in X, the set $\{\varphi\bar{\psi};\ \varphi, \psi \in A\}$ is linearly dense in $C(X)$ by the Stone-Weierstrass theorem. Hence we get

$$g = |g|^2 \quad \mu - \text{a.e.} .$$

By the assumption g must be constant, and hence either $g = 0$ or $g = 1$. This contradicts $Q \neq 0, 1$. Hence $\mathscr{T}(C(X))$ is irreducible. Thus it is weakly dense in $\mathscr{L}(H^2(\mu))$. Now assume $\mathscr{C}(C(X)) = \{0\}$. Then $\mathscr{T}(C(X))$ is commutative and hence $\mathscr{L}(H^2(\mu))$ is also commutative in this case. This implies $\dim H^2(\mu) = 1$ and so $H^2(\mu) = L^2(\mu) = \boldsymbol{C}$, a contradiction. This completes the proof.

If we apply Theorem 2.1 to the Banach algebra $H^\infty(\mu)$, we have

THEOREM 2.4. *Let A be a uniform algebra on a compact Hausdorff space X and μ a finite nonnegative regular Borel measure on X. Suppose*

*$\Gamma(H^\infty(\mu))$ is homeomorphic with $\mathscr{M}(L^\infty(\mu))$: the space of all non-zero multiplicative linear functionals on $L^\infty(\mu)$. Then, there exists a *-homomorphism σ from $\mathscr{T}(L^\infty(\mu))$ onto $L^\infty(\mu)$ such that the short sequence*

$$\{0\} \longrightarrow \mathscr{C}(L^\infty(\mu)) \xrightarrow{i} \mathscr{T}(L^\infty(\mu)) \xrightarrow{\sigma} L^\infty(\mu) \longrightarrow \{0\}$$

is exact and $\sigma(T_\phi) = \phi$ for all $\phi \in L^\infty(\mu)$, where i is the inclusion map and $\mathscr{T}(L^\infty(\mu))$ is the C^*-algebra generated by the set $\{T_\phi; \phi \in L^\infty(\mu)\}$ and $\mathscr{C}(L^\infty(\mu))$ is the commutator ideal of $\mathscr{T}(L^\infty(\mu))$.

PROOF. Let $Y = \mathscr{M}(L^\infty(\mu))$. Then, as is well known, $L^\infty(\mu) \cong C(Y)$ and μ can be seen as a measure on Y with support $\mu = Y$. Further $H^\infty(\mu)$ can be seen as a closed subalgebra of $C(Y)$. The assumption $\Gamma(H^\infty(\mu)) = Y$ implies that $H^\infty(\mu)$ is a uniform algebra on Y. Hence applying Theorem 2.1 and using the isomorphism $L^\infty(\mu) \cong C(Y)$ we get the desired conclusion.

We shall state some conditions to satisfy the assumptions in Theorem 2.4 as a lemma.

LEMMA 2.5. *The following statement* (1) *implies* (2). (2), (3), *and* (4) *are equivalent each other.*

(1) *The set $\{f = \sum_{\text{finite}} |g_j|; g_j \in H^\infty(\mu)\}$ is $L^\infty(\mu)$-norm dense in the set of positive $L^\infty(\mu)$ functions.*

(2) *$H^\infty(\mu)$ separates elements in $\mathscr{M}(L^\infty(\mu))$ and $\Gamma(H^\infty(\mu)) = \mathscr{M}(L^\infty(\mu))$.*

(3) *$\Gamma(H^\infty(\mu)) = \mathscr{M}(L^\infty(\mu))$.*

(4) *$\Gamma(H^\infty(\mu)) \supset \mathscr{M}(L^\infty(\mu))|_{H^\infty(\mu)}$ and the set $\{\varphi\bar{\psi}; \varphi, \psi \in H^\infty(\mu)\}$ is linearly dense in $L^\infty(\mu)$.*

PROOF. (1) $\Rightarrow$ (2). Let $Y = \mathscr{M}(L^\infty(\mu))$ and η be the Gelfand transform from $L^\infty(\mu)$ onto $C(Y)$. Then η is an isometrical isomorphism. Note also $\alpha(\bar{f}) = \overline{\alpha(f)}$ and $\alpha(g) \geqq 0$ for all $\alpha \in Y$, $f \in L^\infty(\mu)$ and $g \in L^\infty(\mu)$ with $g \geqq 0$, since α is a state. Now let $\alpha, \beta \in Y$ and $\alpha(f) = \beta(f)$ for all $f \in H^\infty(\mu)$. Then for any $g \in H^\infty(\mu)$ we have $(\alpha(|g|))^2 = \alpha(|g|^2) = \alpha(g\bar{g}) = \alpha(g)\alpha(\bar{g}) = \alpha(g)\overline{\alpha(g)} = \beta(g)\overline{\beta(g)} = (\beta(|g|))^2$. Since $\alpha(|g|) \geqq 0$, $\beta(|g|) \geqq 0$, we get $\alpha(|g|) = \beta(|g|)$. Hence by the assumption and the linearity and continuity of α, β we have $\alpha(f) = \beta(f)$ for all $f \in L^\infty(\mu)$ with $f \geqq 0$, and hence for all $f \in L^\infty(\mu)$. This shows that $H^\infty(\mu)$ separates elements in Y. Thus the mapping $\xi: \alpha \in Y \rightarrow \alpha|_{H^\infty(\mu)}$ is a homeomorphism from Y into $\mathscr{M}(H^\infty(\mu))$. Now suppose V is a neighborhood of a point v in Y. By Urysohn's lemma there is an $h \in L^\infty(\mu)$ such that $0 \leqq \eta(h) \leqq 1$, $\eta(h)(v) = 1$, and $\eta(h)(y) = 0$ for $y \in Y \setminus V$. By the assumption there are a finite number of $g_1, \cdots, g_k \in H^\infty(\mu)$ such that $\|h - \sum_{j=1}^k |g_j|\|_\infty < 1/4$. Hence we get

$$\sum_{j=1}^{k} \eta(|g_j|)(v) > 3/4 ,$$

$$\sum_{j=1}^{k} \eta(|g_j|)(y) < 1/4 \quad \text{for} \quad y \in Y \backslash V .$$

Here we can take $\eta(|g_j|)(v) = \eta(g_j)(v) \geqq 0$ $(j = 1, \cdots, k)$, multiplying each g_j by a constant of modulus 1, if necessary. Hence we have

$$\eta\Big(\sum_{j=1}^{k} g_j\Big)(v) > 3/4 ,$$

$$\Big|\eta\Big(\sum_{j=1}^{k} g_j\Big)(y)\Big| < 1/4 \quad \text{for} \quad y \in Y \backslash V .$$

Thus $\xi(v)$ is a point in $\Gamma(H^\infty(\mu))$ and hence we have $\xi(Y) = \Gamma(H^\infty(\mu))$. (2) $\Rightarrow$ (4). Since $H^\infty(\mu)$ separates points in $\Gamma(H^\infty(\mu)) = Y$ and the set $\{\varphi\bar{\psi}; \varphi, \psi \in H^\infty(\mu)\}$ is conjugate closed, that set is linearly dense in $C(Y) = L^\infty(\mu)$ by the Stone-Weierstrass theorem. (4) $\Rightarrow$ (3). As in the proof of the step (1) $\rightarrow$ (2) the mapping $\xi: \alpha \in Y \rightarrow \alpha|_{H^\infty(\mu)}$ is one-to-one. Hence $\Gamma(H^\infty(\mu)) = \xi(Y)$, since $\Gamma(H^\infty(\mu)) \supset \xi(Y)$. (3) $\Rightarrow$ (2). Clear, since $H^\infty(\mu)$ separates points in $\Gamma(H^\infty(\mu))$.

Also in this case we can formulate results similar to Corollaries 2.2 and 2.3.

COROLLARY 2.6. *Suppose* $\Gamma(H^\infty(\mu)) = \mathscr{M}(L^\infty(\mu))$. *Then* $\|T_\phi\|_{\mathrm{sp}} = \|T_\phi\|_{\mathrm{op}} = \|\phi\|_\infty$ *for all* $\phi \in L^\infty(\mu)$. *In particular, if* $\phi \in L^\infty(\mu)$, T_ϕ *is quasi-nilpotent if and only if* $\phi = 0$.

COROLLARY 2.7. *Suppose* $\Gamma(H^\infty(\mu)) = \mathscr{M}(L^\infty(\mu))$ *and* $H^2(\mu) \neq L^2(\mu)$ *and every real valued function in* $H^2(\mu)$ *is constant. Then, if* $\phi \in L^\infty(\mu)$, T_ϕ *is compact if and only if* $\phi = 0$.

Now as to joint approximate point spectrum for n-tuple of functions in A or $H^\infty(\mu)$ we have the following.

PROPOSITION 2.8. *Let* A, μ *satisfy the assumptions in Theorem* 2.1 *(resp. Theorem* 2.4*). Then for* $\phi_1, \phi_2, \cdots, \phi_n \in A$ *(resp.* $H^\infty(\mu)$*) we have*

$$\sigma_\pi(T_{\phi_1}, T_{\phi_2}, \cdots, T_{\phi_n}) = \{(\phi_1(x), \phi_2(x), \cdots, \phi_n(x)); x \in \operatorname{supp} \mu (\textit{resp. } \mathscr{M}(L^\infty(\mu)))\} .$$

PROOF. Immediate from Theorem 2.1 (resp. Theorem 2.4) and Proposition 1.7.

REMARK. One can prove the above proposition by a result of Żelazko ([18] p. 240 in the proof of theorem), and then can prove Theorem 2.1 and 2.4 using it and the Bunce's theorem. Similarly one can prove Theorem 1.4 combining a result of Żelazko and the Bunce's theorem.

Finally in this section we mention essential spectra of Toeplitz operators.

PROPOSITION 2.9. *Let* A, μ *satisfy the assumptions in Corollary* 2.3 (*resp. Corollary* 2.7). *Suppose further* $\mathscr{C}(C(X)) \cap \mathscr{L}\mathscr{C}(H^2(\mu)) \neq \{0\}$. *Then, if* $\phi \in C(X)$(*resp.* $L^\infty(\mu)$), *the spectrum* $\sigma(M_\phi)$ *of* M_ϕ *is contained in the essential spectrum* $\sigma_e(T_\phi)$ *of* T_ϕ.

PROOF. In these cases $\mathscr{T}(C(X))$ and $\mathscr{T}(L^\infty(\mu))$ are irreducible and $\mathscr{C}(C(X))$ and $\mathscr{C}(L^\infty(\mu))$ are non-trivial respectively. Hence the assumption $\mathscr{C}(C(X)) \cap \mathscr{L}\mathscr{C}(H^2(\mu)) \neq \{0\}$ implies $\mathscr{L}\mathscr{C}(H^2(\mu)) \subset \mathscr{C}(C(X)) \subset \mathscr{C}(L^\infty(\mu))$. Thus from Theorem 2.1 (resp. Theorem 2.4) it follows that $\sigma(M_\phi)$ is contained in the essential spectrum (spectrum modulo compact operators) of T_ϕ.

3. **Applications.** a) Let A be a hypo-Dirichlet algebra on a compact Hausdorff space X, Φ be a non-zero multiplicative linear functional, and μ be the unique logmodular measure for Φ. Then it is known that $\Gamma(H^\infty(\mu)) = \mathscr{M}(L^\infty(\mu))$. Hence one can apply Theorem 2.4.

b) Let D be a bounded domain in the complex plane whose boundary $X = \partial D$ consists of n non-intersecting analytic Jordan curves and let a be a point in D. Let A be the uniform algebra on X consisting of continuous functions on $\bar{D}$ which are holomorphic in D and let μ be the harmonic measure on X with respect to a and D. Then A is a hypo-Dirichlet algebra and supp $\mu = X$. Hence by Theorem 2.1 we have $C(X) \cong \mathscr{T}(C(X))/\mathscr{C}(C(X))$. Thus after proving $\mathscr{C}(C(X)) = \mathscr{L}\mathscr{C}(H^2(\mu))$ as in Abrahamse [1, p. 275] one can give a somewhat different proof of the theorem of Abrahamse.

c) Let Ω be a relatively compact strongly pseudo-convex domain with smooth boundary X in a Stein manifold M, and A be the uniform closure on X of functions holomorphic in a neighborhood of $\bar{\Omega}$. Then A is a uniform algebra on X and $\Gamma(A) = X$. Let μ be the canonical measure on X induced by a hermitian metric on M. Then supp $\mu = X$. Further by a theorem of Folland-Kohn ([8], p. 102, Theorem 5.4.12) one can show $\mathscr{C}(C(X)) \subset \mathscr{L}\mathscr{C}(H^2(\mu))$. Clearly $H^2(\mu) \neq L^2(\mu)$ and every real valued $H^2(\mu)$ function is a constant. Hence $\mathscr{T}(C(X))$ is irreducible and $\mathscr{C}(C(X)) \neq \{0\}$, and hence $\mathscr{C}(C(X)) = \mathscr{L}\mathscr{C}(H^2(\mu))$. Now applying Theorem 2.1 we have $\mathscr{T}(C(X))/\mathscr{L}\mathscr{C}(H^2(\mu)) \cong C(X)$. This is also true for some Stein spaces. Similar results are gained if we replace μ by the volume form on Ω. These are generalizations of the theorem of Venugopalkrishna and Janas. Details will appear elsewhere.

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