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SATURATION OF BOUNDED LINEAR OPERATORS

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1. Introduction. An asymptotic relation of the type appearing in the theorem of Voronovskaja (see, [5; p. 22]) plays an important role in an investigation of saturation problems in the theory of approximation of linear operators. This relation can be formulated in a more general form on Banach spaces as follows:

Let B be a Banach space and let M be a dense linear subspace of B. Let $\{L_n\}$ be a sequence of bounded linear operators of B into itself. We say that $\{L_n\}$ satisfies a Voronovskaja condition of type $(\phi(n); L)$ on M if there exists a sequence $\{\phi(n)\}$ of positive real numbers tending to infinity and a linear operator L of M into B such that for every f in M

$$(1) \qquad \lim_{n\to\infty} \phi(n)(L_n(f) - f) = L(f)$$

(in the sence of strong convergence).

The purpose of this paper lies in considering the saturation problem of the sequence $\{L_n\}$ of contractive linear operators of B into itself satisfying the condition (1) under the additional hypothesis that for each g in M there exists a finite dimensional linear subspace of M which contains g and is invariant under L and L_n for all n.

Our results will be concerned with semigroups of operators, and recover the result of R. Schnabl [7] on the saturation of generalized Bernstein operators.

Throughout this paper B will be a fixed (real or complex) Banach space endowed with norm $\|\cdot\|$, and we will denote by the same symbol $\|\cdot\|$ the norm of a bounded linear operator.

2. Semigroups of operators and relative completions of linear subspaces. In this section, we present the necessary results concerning semigroups of operators and relative completions of linear subspaces, which will be needed in our development.

Let $\{S(t); t \geqq 0\}$ be a family of bounded linear operators of B into itself. $\{S(t); t \geqq 0\}$ is said to be a semigroup (of operators) on B if $S(t+u) = S(t)S(u)$ for each $t, u \geqq 0$ and $S(0) = I$, where I is the identity operator. The semigroup $\{S(t); t \geqq 0\}$ on B is said to be of class $(C_0)_s$

if $\lim_{t\to 0+}||S(t)(f)-f||=0$ for every f in B; it is said to be of class $(C_0)_u$ if $\lim_{t\to 0+}||S(t)-I||=0$. The infinitesimal generator of the semigroup $\{S(t); t \geqq 0\}$ is the operator G in B defined by

$$G(f)=\lim_{t\to 0+}(1/t)(S(t)(f)-f),$$

with domain $D(G)$ consisting of all f in B for which this limit exists in the sense of strong convergence.

The basic properties of semigroups are the following (see, for instance, [1; Propositions 1.1.4 and 1.1.6]):

PROPOSITION A. *Let $\{S(t); t \geqq 0\}$ be a semigroup on B with its infinitesimal generator G. Then the following statements hold:*

(a) *If $\{S(t); t \geqq 0\}$ is of class $(C_0)_s$, then $D(G)$ is a dense linear subspace of B and G is a closed linear operator on $D(G)$.*

(b) *If $D(G)=B$, then $\{S(t); t \geqq 0\}$ is of class $(C_0)_u$ and $S(t)=\exp(tG)$ for all $t \geqq 0$.*

A linear subspace M of B is said to be a (normalized) Banach subspace if it is a Banach space with norm $||\cdot||_M$ such that $||f|| \leqq ||f||_M$ for every f in M. Note that if L is a closed linear operator with domain $D(L)$ and range in B, then $D(L)$ becomes a Banach subspace of B under the norm $||\cdot||_{D(L)}=||\cdot||+||L(\cdot)||$.

If M is a Banach subspace of B, then the relative completion of M in B, denoted by $\tilde{M}$, is defined as the set consisting of all f in B for which there exists a sequence $\{f_n\}$ of elements in M and a constant $C>0$ such that $||f_n||_M \leqq C$ for all n and $\lim_{n\to\infty}||f_n-f||=0$. Evidently, $\tilde{M}$ is a linear subspace of B. For each f in $\tilde{M}$ set

$$(2) \qquad ||f||_{\tilde{M}}=\inf\{C>0; \{f_n\}\subset M, \sup_n ||f_n||_M \leqq C, \lim_{n\to\infty}||f_n-f||=0\},$$

which defines a norm on $\tilde{M}$. Note that $\tilde{M}$ contains M and $||f||_{\tilde{M}} \leqq ||f||_M$ for all f in M.

Important properties of relative completions are the following (see, [2; Propositions 10.4.2 and 10.4.3]):

PROPOSITION B. *Let M be a Banach subspace of B with norm $||\cdot||_M$. Then the following statements hold:*

(a) *$\tilde{M}$ is a Banach subspace of B under the norm $||\cdot||_{\tilde{M}}$ defined by* (2).

(b) *If M is reflexive, then $M=\tilde{M}$ and $||\cdot||_M=||\cdot||_{\tilde{M}}$.*

The following theorem concerning the saturation of semigroups illustrates the usefulness of the concept of relative completions (see, [2; Theorem 13.4.4], cf. [1; Theorem 2.1.2]):

THEOREM A. *Let $\{S(t); t \geqq 0\}$ be a uniformly bounded semigroup of class $(C_0)_s$ on B with its infinitesimal generator G. Then the following statements hold:*

(a) *If f in B satisfies $\|S(t)(f) - f\| = o(t)$ $(t \to 0+)$, then $S(t)(f) = f$ for all $t \geqq 0$.*

(b) *For an element f in B the following are equivalent:*

(i) $\|S(t)(f) - f\| = O(t)$;

(ii) $f \in \widetilde{D(G)}$;

(iii) $f \in D(G)$ *if B is reflexive.*

A generalization of Theorem A can be found in [2; p. 502].

3. **Preliminary results.** In this section, we state and prove certain results needed in the proof of main results of Section 4. We first begin with the following lemma, which will be needed in the proof of Proposition 1.

LEMMA. *Let $\{E(t)\}$ be a family of mappings of B into itself, where a parameter t varies over the real line (or complex plane), such that for each f in B $E(\cdot)(f)$ is strongly continuous. Let t be a real (or complex) number and let $\{\xi_n\}$ be a sequence of nonzero real (or complex) numbers with $\lim_{n\to\infty} |\xi_n| = +\infty$. Then for every sequence $\{k_n\}$ of positive integers satisfying $\lim_{n\to\infty} k_n/\xi_n = t$ and every f in B,*

$$\lim_{n\to\infty} (1/(k_n + 1)) \sum_{j=0}^{k_n} E(j/\xi_n)(f) = \int_0^1 E(tu)(f)du \, .$$

We omit the proof, which can be elementary. With the help of the above lemma and a similar argument used in the proof of Hilfssatz 1 in [7] we have the following:

PROPOSITION 1. *Let A be a real (or complex) Banach algebra with unit e and norm $\|\cdot\|$. Let $\{f_n\}$ be a sequence of elements in A satisfying $\lim_{n\to\infty} \|f_n\| = 0$. Let t be a real (or complex) number and let $\{t_n\}$ be a sequence of nonzero real (or complex) numbers converging to zero. Then for every sequence $\{k_n\}$ of positive integers satisfying $\lim_{n\to\infty} k_n t_n = t$ and every f in A,*

$$\lim_{n\to\infty} \|(e + t_n f + t_n f_n)^{k_n} - \exp(tf)\| = 0$$

and

$$\lim_{n\to\infty} \left\|(1/(k_n + 1)) \sum_{j=0}^{k_n} (e + t_n f + t_n f_n)^j - \int_0^1 \exp(tuf)du \right\| = 0 \, .$$

PROOF. First let us show that

$$\begin{aligned} \|(e + g + h)^k - \exp(kh)\| &\leqq k\|g\|\exp((k-1)(\|g\| + \|h\|)) \\ &\quad + \exp(k\|h\|) - (1 + \|h\|)^k, \end{aligned} \tag{3}$$

for all g, h in A and for $k = 0, 1, 2, \cdots$.

An induction argument reveals that

$$\|(g + h)^j - h^j\| \leqq j\|g\|(\|g\| + \|h\|)^{j-1},$$

for $j = 1, 2, 3, \cdots$. Since

$$\begin{aligned} &(e + g + h)^k - \exp(kh) \\ &= \sum_{j=0}^{k} \binom{k}{j}(g + h)^j - \sum_{j=0}^{\infty}(kh)^j/j! \\ &= \sum_{j=0}^{k} \binom{k}{j}((g + h)^j - h^j) + \sum_{j=0}^{k}\left(\binom{k}{j} - k^j/j!\right)h^j \\ &\quad - \sum_{j=k+1}^{\infty}(kh)^j/j!, \end{aligned}$$

we have

$$\begin{aligned} &\|(e + g + h)^k - \exp(kh)\| \\ &\leqq \sum_{j=0}^{k}\binom{k}{j}\|(g + h)^j - h^j\| \\ &\quad + \sum_{j=0}^{k}\left(k^j/j! - \binom{k}{j}\right)\|h\|^j + \sum_{j=k+1}^{\infty}(k\|h\|)^j/j! \\ &\leqq \sum_{j=1}^{k}\binom{k}{j} j\|g\|(\|g\| + \|h\|)^{j-1} + \exp(k\|h\|) \\ &\quad - (1 + \|h\|)^k \\ &\leqq k\|g\|\sum_{j=0}^{k-1}((k-1)^j/j!)(\|g\| + \|h\|)^j \\ &\quad + \exp(k\|h\|) - (1 + \|h\|)^k, \end{aligned}$$

which yields (3).

Now putting $k = k_n$, $g = t_n f_n$ and $h = t_n f$ in (3), and letting n tend to infinity, we have

$$\lim_{n\to\infty}\|(e + t_n f_n + t_n f)^{k_n} - \exp(k_n t_n f)\| = 0,$$

and so

$$\lim_{n\to\infty}\|(e + t_n f_n + t_n f)^{k_n} - \exp(tf)\| = 0$$

since $\lim_{n\to\infty} k_n t_n = t$.

Next putting $g = t_n f_n$ and $h = t_n f$ in (3), we have

$$\begin{aligned}&\|(e + t_n f_n + t_n f)^k - \exp(kt_n f)\| \\ &\quad \leqq k|t_n|\,\|f_n\| \exp((k-1)|t_n|(\|f_n\| + \|f\|)) \\ &\qquad + \exp(k|t_n|\,\|f\|) - (1 + |t_n|\,\|f\|)^k ,\end{aligned}$$

which yields

$$\begin{aligned}&\left\|(1/(k_n+1))\sum_{j=0}^{k_n}((e + t_n f_n + t_n f)^j - \exp(jt_n f))\right\| \\ &\quad \leqq (|t_n|\,\|f_n\|/(k_n+1)) \\ &\qquad \times \sum_{j=0}^{k_n} j \exp((j-1)|t_n|(\|f_n\| + \|f\|)) \\ &\qquad + (1/(k_n+1))\sum_{j=0}^{k_n}(\exp(j|t_n|\,\|f\|) - (1 + |t_n|\,\|f\|)^j) \\ &\quad \leqq k_n|t_n|\,\|f_n\| \exp((k_n-1)|t_n|(\|f_n\| + \|f\|)) \\ &\qquad + (\exp(k_n|t_n|\,\|f\|) - (1 + |t_n|\,\|f\|)^{k_n}) .\end{aligned}$$

Since $\lim_{n\to\infty} k_n|t_n| = |t|$, $\lim_{n\to\infty}|t_n| = 0$ and $\lim_{n\to\infty}\|f_n\| = 0$, the both terms of the right-hand side of the above inequality converge to zero as n tends to infinity, and therefore we have

$$\lim_{n\to\infty}\left\|(1/(k_n+1))\sum_{j=0}^{k_n}((e + t_n f_n + t_n f)^j - \exp(jt_n f))\right\| = 0 .$$

On the other hand, applying lemma to $B = A$ considered as a Banach space, $E(t)(\cdot) = \exp(t\cdot)$ and $\xi_n = 1/t_n$, we have

$$\lim_{n\to\infty}\left\|(1/(k_n+1))\sum_{j=0}^{k_n}\exp(jt_n f) - \int_0^1 \exp(tuf)du\right\| = 0 ,$$

and consequently

$$\lim_{n\to\infty}\left\|(1/(k_n+1))\sum_{j=0}^{k_n}(e + t_n f_n + t_n f)^j - \int_0^1 \exp(tuf)du\right\| = 0 ,$$

which completes the proof.

We say that a sequence $\{L_n\}$ of bounded linear operators of B into itself satisfies a type $[\phi(n); P]$ on B if there exists a sequence $\{\phi(n)\}$ of positive real numbers tending to infinity and a bounded linear operator P of B into itself such that for all n,

$$L_n P = P \tag{4}$$

and for every sequence $\{k_n\}$ of positive integers with $\lim_{n\to\infty} k_n/\phi(n) = +\infty$,

$$\lim_{n\to\infty} L_n^{k_n} = P , \tag{5}$$

strongly on B, where L_n^k denotes the k-th iteration of L_n.

PROPOSITION 2. *Let* $\{L_n\}$, $||L_n|| \leqq 1$ *be a sequence of bounded linear operators of* B *into itself satisfying a type* $[\phi(n); P]$ *on* B. *Then for an element* f *in* B *the following are equivalent*:

(i) $||L_n(f) - f|| = o(1/\phi(n))$ $(n \to \infty)$;

(ii) $P(f) = f$;

(iii) $L_n(f) = f$ *for all* n.

PROOF. Suppose that $||L_n(f)-f|| = o(1/\phi(n))$ $(n \to \infty)$. Set $\phi(n)||L_n(f) - f|| = a_n$. Then we have for every positive integer k and every n

$$||L_n^k(f) - f|| \leqq ka_n/\phi(n) .$$

We now choose a sequence $\{k_n\}$ of positive integers so that

$$\lim_{n\to\infty} k_n/\phi(n) = +\infty \quad \text{and} \quad \lim_{n\to\infty} a_n k_n/\phi(n) = 0 .$$

Putting $k = k_n$ in the above inequality, and letting n tend to infinity, we have

$$\lim_{n\to\infty} ||L_n^{k_n}(f) - f|| = 0 ,$$

and therefore (5) yields $P(f) = f$. By (4), it is immediate that (ii) implies (iii). It is clear that (iii) implies (i). The proof is complete.

4. **Main results** (**saturation theorems**). Throughout this section let M be a fixed dense linear subspace of B. We will now consider a sequence $\{L_n\}$ of bounded linear operators of B into itself satisfying the following conditions (I) and (II):

(I) $\{L_n\}$ satisfies a Voronovskaja condition of type $(\phi(n); L)$ on M and $||L_n|| \leqq 1$ for all n.

(II) For each g in M there exists a finite dimensional linear subspace of M which contains g and is invariant under L and L_n for all n.

Note that the density of M and the condition (I) imply that $\{L_n\}$ is a strong approximation process on B, i.e., $\lim_{n\to\infty} ||L_n(f) - f|| = 0$ for every f in B. The sequence $\{L_n\}$ satisfying the conditions (I) and (II) induces a semigroup of class $(C_0)_s$ on B. That is, we have the following:

THEOREM 1. *Let* $\{L_n\}$ *be a sequence of bounded linear operators of* B *into itself satisfying the conditions* (I) *and* (II). *Then there exists a unique contractive semigroup* $\{T(t); t \geqq 0\}$ *of class* $(C_0)_s$ *on* B *such that for every* f *in* B *and every sequence* $\{k_n\}$ *of positive integers with* $\lim_{n\to\infty} k_n/\phi(n) = t$,

$$\lim_{n\to\infty} ||L_n^{k_n}(f) - T(t)(f)|| = 0 \tag{6}$$

and

$$(7) \qquad \lim_{n\to\infty}\left\|(1/(k_n+1))\sum_{j=0}^{k_n} L_n^j(f) - \int_0^1 T(tu)(f)du\right\| = 0$$

for all $t \geqq 0$.

PROOF. Let g be an element in M. Then by the condition (II) there exists a finite dimensional linear subspace V of M such that $g \in V$, $L(V) \subset V$ and $L_n(V) \subset V$ for all n. Denote by $[V]$ the Banach algebra of all bounded linear operators of V into itself. Let F_n denote the restriction of $\phi(n)(L_n - I) - L$ to V. Then we have for all h in V $\lim_{n\to\infty} \|F_n(h)\| = 0$, and so $\lim_{n\to\infty}\|F_n\| = 0$ since the dimension of V is finite. Applying Proposition 1 to $A = [V]$, $f = L|V$ the restriction of L to V, $f_n = F_n$ and $t_n = 1/\phi(n)$ we have

$$\lim_{n\to\infty}(I + (1/\phi(n))L|V + (1/\phi(n))F_n)^{k_n} = \exp(tL|V)$$

and

$$\lim_{n\to\infty}(1/(k_n+1))\sum_{j=0}^{k_n}(I + (1/\phi(n))L|V + (1/\phi(n))F_n)^j$$
$$= \int_0^1 \exp(tu(L|V))du$$

in the uniform operator topology on $[V]$. Therefore, we obtain

$$\lim_{n\to\infty}\|L_n^{k_n}(g) - \exp(tL)(g)\| = 0$$

and

$$\lim_{n\to\infty}\left\|(1/(k_n+1))\sum_{j=0}^{k_n} L_n^j(g) - \int_0^1 \exp(tuL)(g)du\right\| = 0\,,$$

from which it follows that $\exp(tL)$ and $\int_0^1 \exp(tuL)du$ are contractive linear operators of M into itself. Since M is dense in B, there exists a bounded linear operator $T(t)$ of B into itself so that $T(t) = \exp(tL)$ on M and $\|T(t)\| = \|\exp(tL)\| \leqq 1$.

Note that $\int_0^1 T(tu)du$ is the norm-preserving extension of $\int_0^1 \exp(tuL)du$. Now it can be verified that the family $\{T(t); t \geqq 0\}$ is a contractive semigroup of class $(C_0)_s$ on B which satisfies the relations (6) and (7). The uniqueness of $\{T(t); t \geqq 0\}$ is clear, and the proof of the theorem is complete.

REMARK. If B is a Banach lattice with order unit, and if $\{L_n\}$ is a sequence of positive linear operators of B into itself satisfying the conditions (I) and (II), then the semigroup $\{T(t); t \geqq 0\}$ of class $(C_0)_s$ on B

as in Theorem 1 can be taken to be positive.

We are now in a position to state the result concerning the saturation of $\{L_n\}$.

THEOREM 2. *Let* $\{L_n\}$ *be a sequence of bounded linear operators of* B *into itself satisfying the conditions* (I) *and* (II), *and let* $\{T(t); t \geqq 0\}$ *be the contractive semigroup of class* $(C_0)_s$ *on* B, *uniquely determined by* $\{L_n\}$ *as in Theorem* 1, *having* G *as its infinitesimal generator. Set* $S[L_n, \phi(n)] = \{f \in B; \|L_n(f) - f\| = O(1/\phi(n))\}$. *Then the following statements hold:*

(a) *If* f *and* g *in* B *satisfy* $\liminf_{n\to\infty} \|\phi(n)(L_n(f) - f) - g\| = 0$, *then* f *belongs to the domain* $D(G)$ *of* G *and* $G(f) = g$. *In case* $g = 0$ *we have* $T(t)(f) = f$ *for all* $t \geqq 0$.

(b) $M \subset S[L_n, \phi(n)] \subset \widetilde{D(G)}$.

(c) *If* L *is closed, then* $\tilde{M} \subset S[L_n, \phi(n)] \subset \widetilde{D(G)}$.

(d) *If* B *is reflexive, then* $M \subset S[L_n, \phi(n)] \subset D(G)$.

(e) *If* $M = B$, *then* L *is bounded,* $T(t) = \exp(tL)$ *for all* $t \geqq 0$ *and* $S[L_n, \phi(n)] = B$.

PROOF. We first show that

$$\begin{aligned} &\|(1/t)(T(t)(f) - f) - g\| \\ &\leqq \liminf_{n\to\infty} \|\phi(n)(L_n(f) - f) - g\| \\ &\quad + \left\|(1/t)\int_0^t T(u)(g)du - g\right\| \end{aligned} \tag{8}$$

for all f, g in B and all $t > 0$.

Let k be a positive integer. Then we have

$$\begin{aligned} &\phi(n)(L_n^{k+1}(f) - f) - \sum_{j=0}^{k} L_n^j(g) \\ &= \sum_{j=0}^{k} L_n^j(\phi(n)(L_n(f) - f) - g)\,, \end{aligned}$$

and so

$$\begin{aligned} &\left\|\phi(n)(L_n^{k+1}(f) - f) - \sum_{j=0}^{k} L_n^j(g)\right\| \\ &\leqq \sum_{j=0}^{k} \|L_n\|^j \|\phi(n)(L_n(f) - f) - g\| \\ &\leqq (k+1)\|\phi(n)(L_n(f) - f) - g\| \end{aligned}$$

since $\|L_n\| \leqq 1$. Consequently,

$$\begin{aligned}&\|\phi(n)/(k+1)(L_n^{k+1}(f)-f)-g\|\\&\quad\leqq\left\|\phi(n)/(k+1)(L_n^{k+1}(f)-f)-(1/(k+1))\sum_{j=0}^{k}L_n^j(g)\right\|\\&\qquad+\left\|(1/(k+1))\sum_{j=0}^{k}L_n^j(g)-g\right\|\\&\quad\leqq\|\phi(n)(L_n(f)-f)-g\|\\&\qquad+\left\|(1/(k+1))\sum_{j=0}^{k}L_n^j(g)-g\right\|.\end{aligned}$$

Now we choose a sequence $\{k_n\}$ of positive integers so that $\lim_{n\to\infty} k_n/\phi(n)=t$. Putting $k=k_n$ in the above inequality, from (6) and (7), we obtain

$$\begin{aligned}&\|(1/t)(T(t)(f)-f)-g\|\\&\quad\leqq\liminf_{n\to\infty}\|\phi(n)(L_n(f)-f)-g\|\\&\qquad+\left\|\int_0^1 T(tu)(g)du-g\right\|,\end{aligned}$$

which yields (8).

(a) Suppose that f, g in B and $\liminf_{n\to\infty}\|\phi(n)(L_n(f)-f)-g\|=0$. Then from (8) we have for all $t>0$

$$\|(1/t)(T(t)(f)-f)-g\|\leqq\left\|(1/t)\int_0^t T(u)(g)du\right\|,$$

which yields $f\in D(G)$ and $G(f)=g$.

(b) Let f be in $S[L_n,\phi(n)]$. Then there exists a constant $C>0$ such that $\|\phi(n)(L_n(f)-f)\|\leqq C$ for all n. Putting $g=0$ in (8), we have $\|T(t)(f)-f\|\leqq Ct$ for all $t\geqq 0$. Hence, by Theorem A, f belongs to $\widetilde{D(G)}$. By the condition (I), the inclusion $M\subset S[L_n,\phi(n)]$ is clear.

(c) Suppose that L is closed. Then we recall that M becomes a Banach subspace of B with the norm $\|\cdot\|_M=\|\cdot\|+\|L(\cdot)\|$. By the condition (I), for each f in M the sequence $\{\|\phi(n)(L_n(f)-f)\|\}$ is bounded. Therefore, by the uniform boundedness principle, there exists a constant $C>0$ such that

$$(9)\qquad \phi(n)\|L_n(g)-g\|\leqq C\|g\|_M$$

for all g in M and all n. Now let f be in $\tilde{M}$. Then there exists a sequence $\{f_k\}$ of elements in M and a constant $C'>0$ such that $\|f_k\|_M\leqq C'$ for all k and $\lim_{k\to\infty}\|f_k-f\|=0$. Replacing g by f_k in (9), and letting k tend to infinity, we have for all n

$$\phi(n)\|L_n(f)-f\|\leqq CC',$$

which implies that f belongs to $S[L_n, \phi(n)]$.

(d) If B is reflexive, then by Theorem A (cf. Proposition B (b)), $D(G) = \widetilde{D(G)}$, and so the above (b) yields the desired result.

(e) Suppose that $M = B$. By the above (a), we have $M \subset D(G)$ and $L = G$ on M. Therefore, $M = D(G) = B$ and $L = G$. Hence, by Proposition A, the semigroup $\{T(t); t \geqq 0\}$ is of class $(C_0)_u$ on B and $T(t) = \exp(tG) = \exp(tL)$ for all $t \geqq 0$. Furthermore, by the above (b), we have $S[L_n, \phi(n)] = B$. The proof of the theorem is complete.

THEOREM 3. *In addition to hypotheses of Theorem* 1, *suppose that* $\{L_n\}$ *satisfies a type* $[\phi(n); P]$ *on* B. *Let* $\{T(t); t \geqq 0\}$ *be the contractive semigroup of class* $(C_0)_s$ *on* B, *uniquely determined by* $\{L_n\}$ *as in Theorem* 1, *and let* G *be its infinitesimal generator. Suppose that* $\lim_{t\to\infty} T(t) = P$ *strongly on* B. *Then for an element* f *in* B *the following are equivalent*:

(i) $\|L_n(f) - f\| = o(1/\phi(n))$ $(n \to \infty)$;

(ii) $L_n(f) = f$ *for all* n;

(iii) $P(f) = f$;

(iv) $\|T(t)(f) - f\| = o(t)$ $(t \to 0+)$;

(v) $T(t)(f) = f$ *for all* $t \geqq 0$, *or, equivalently*, $G(f) = 0$.

PROOF. The equivalence of (i), (ii), and (iii) follows from Proposition 2. Now since $L_n P = P$, denoting the largest integer not exceeding $t\phi(n)$ by j_n, we have $L_n^{j_n} P = P$, and so, letting n tend to infinity, by Theorem 1, $T(t)P = P$ for all $t \geqq 0$. Suppose now that $\|T(t)(f) - f\| = o(t)$ $(t \to 0+)$. Let $\{t_n\}$ be a sequence of positive real numbers converging to zero. Then the sequence $\{T(t_n)\}$ satisfies a type $[1/t_n; P]$. In fact, $T(t_n)P = P$ for all n and by the hypothesis, we have for every sequence $\{k_n\}$ of positive integers with $\lim_{n\to\infty} k_n t_n = +\infty$, $\lim_{n\to\infty} T(t_n)^{k_n} = \lim_{n\to\infty} T(k_n t_n) = P$, strongly on B. Therefore, by Proposition 2, we have $P(f) = f$.

Conversely, if $P(f) = f$, then $T(t)(f) = T(t)(P(f)) = P(f) = f$ for all $t \geqq 0$. The proof of the theorem is complete.

5. An application to generalized Bernstein operators on Bauer simplex. Let X be a compact convex subset of a locally convex Hausdorff vector space over the field of real numbers. Let $C(X)$ denote the Banach algebra of all real-valued continuous functions defined on X with the supremum norm $\|\cdot\|_\infty$. Let $A(X)$ denote the closed linear subspace of $C(X)$ of all real-valued continuous affine functions defined on X. For each positive integer k, we denote by $M_k(X)$ the linear subspace of $C(X)$ spanned by the set

$$\{f_1 f_2 \cdots f_k \in C(X); f_1, f_2, \cdots, f_k \in A(X)\},$$

and define

$$M(X) = \bigcup \{M_k(X); k = 1, 2, \cdots\} ,$$

which is a subalgebra of $C(X)$ separating points of X and containing the unit function 1_X on X. By the theorem of Stone-Weierstrass, $M(X)$ is dense in $C(X)$. We shall now consider $C(X)$ as a Banach lattice with order unit 1_X.

We will need the following proposition, which may be an immediate consequence of Proposition 1 in [6], and omit the proof (see, [6; Remark 2]):

PROPOSITION 3. *Let $\{W_t; t > 0\}$ be a family of positive linear operators of $C(X)$ into itself, and let W be a positive linear operator of $C(X)$ into itself satisfying $W(h) = h$ for every h in $A(X)$. Suppose that $W_t W = W$ for all $t > 0$ and $\lim_{t\to\infty} \|W_t(h^2) - W(h^2)\|_\infty = 0$ for every h in $A(X)$. Then $\lim_{t\to\infty} \|W_t(f) - W(f)\|_\infty = 0$ for every f in $C(X)$.*

Suppose now that X is a Bauer simplex. Let $\{B_n; n \geqq 1\}$ be the sequence of generalized Bernstein operators on $C(X)$ with respect to the lower triangular stochastic matrix (p_{nj}) whose entries are chosen as

$$p_{nj} = 1/n , \qquad (1 \leqq n, 1 \leqq j \leqq n) ,$$

and

$$p_{nj} = 0 , \qquad (1 \leqq n < j) \text{ ([3], [4])} .$$

Hence, each B_n $(n = 1, 2, \cdots)$ can be expressed by

$$B_n(f)(x) = \int_X \cdots \int_X f\Big((1/n) \sum_{j=1}^n x_j\Big) d\mu_x(x_1) \cdots d\mu_x(x_n) ,$$

where μ_x is an $A(X)$-representing measure for x supported by the extremal points of X. Note B_n is a positive linear operator and the operator norm of B_n is equal to one and B_1 is a positive projection of $C(X)$ onto $A(X)$. The author [6] showed that $\{B_n; n \geqq 1\}$ satisfies a type $[n; B_1]$ on $C(X)$. Now one verifies exactly by a modification of the proof of Hilfssatz 2 in [7] that there exists a linear operator L of $M(X)$ into itself such that

$$L(f_1 f_2 \cdots f_k) = \sum_{1 \leqq i < j \leqq k} (B_1(f_i f_j) - f_i f_j) \prod_{\substack{r=1 \\ r \neq i,j}}^k f_r ,$$

$$f_1, f_2, \cdots, f_k \in A(X), k = 1, 2, \cdots ,$$

and

$$\lim_{n\to\infty} \|n(B_n(f) - f) - L(f)\|_\infty = 0$$

for all f in $M(X)$, and for each g in $M(X)$ there exists a finite dimensional linear subspace of $M(X)$ which contains g and is invariant under

L and B_n for all n. Therefore, $\{B_n; n \geqq 1\}$ satisfies the conditions (I) and (II). Now let $\{U(t); t \geqq 0\}$ denote the positive contractive semigroup of class $(C_0)_s$ on $C(X)$, uniquely determined by $\{B_n; n \geqq 1\}$ in the sense of Theorem 1. Then we have the following theorem, which should be compared with the result of the author [6] and of R. Schnabl [7].

THEOREM 4. *The following statements hold*:

(a) $M(X) \subset S[B_n, n] \subset \widetilde{D(G)}$, *where* G *is the infinitesimal generator of* $\{U(t); t \geqq 0\}$.

(b) *For an element* f *in* $C(X)$ *the following are equivalent*:

(i) $\|B_n(f) - f\|_\infty = o(1/n)$ $(n \to \infty)$;

(ii) $B_n(f) = f$ *for all* $n \geqq 1$;

(iii) $f \in A(X)$;

(iv) $\|U(t)(f) - f\|_\infty = o(t)$ $(t \to 0+)$;

(v) $U(t)(f) = f$ *for all* $t \geqq 0$.

PROOF. Let $\{k_n\}$ be a sequence of positive integers with $\lim_{n\to\infty} k_n/n = t$, $t \geqq 0$. Then we have for all $n \geqq 1$ and all h in $A(X)$,

$$B_n^{k_n} B_1 = B_1$$

and

$$B_n^{k_n}(h^2) = B_1(h^2) + (1 - 1/n)^{k_n}(h^2 - B_1(h^2))$$

since $B_n B_1 = B_1$ and $B_n(h^2) = h^2 + (1/n)(B_1(h^2) - h^2)$. Therefore, letting n tend to infinity, from the relation (6), we have for all $t \geqq 0$ and all h in $A(X)$,

$$U(t)B_1 = B_1$$

and

$$U(t)(h^2) = B_1(h^2) + \exp(-t)(h^2 - B_1(h^2)) .$$

Hence, by Proposition 3, we have for all f in $C(X)$,

$$\lim_{t\to\infty} \|U(t)(f) - B_1(f)\|_\infty = 0 ,$$

and so the desired results follow from Theorems 2 and 3. The proof of the theorem is complete.

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