ON SUBPROJECTIVE SPACES III

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(Received July 14, 1951)

§ 1. Subprojective space admitting a parallel vector field.

A Riemannian space V_n, which has Christoffel symbols of the form

$$\left\{ {\lambda \atop \mu\nu} \right\} = \varphi_\mu \delta^\lambda_\nu + \varphi_\nu \delta^\lambda_\mu + \varphi_{\mu\nu} x^\lambda \qquad (\varphi_{\mu\nu} = \varphi_{\nu\mu}), \tag{1.1}$$

for a suitable coordinate system, was called the subprojective space by B. Kagan [3], [4].

P. Rachevsky [5] proved that φ_μ is a gradient vector and by a transformation of coordinates

$$'x^\lambda = e^\varphi x^\lambda \qquad \left(\varphi_\mu = \frac{\partial \varphi}{\partial x^\mu}\right), \tag{1.2}$$

the Christoffel symbols take the form

$$\left\{ {\lambda \atop \mu\nu} \right\} = u_{\mu\nu} x^\lambda. \tag{1.3}$$

Furthermore, making use of (1.3), he introduced three conditions for the subprojective space, that is to say,

$$\begin{aligned} &(A) \quad R^\lambda_{\cdot\mu\nu\omega} = T^\lambda_{\cdot\omega} g_{\mu\nu} - T^\lambda_{\cdot\nu} g_{\mu\omega} + T_{\mu\nu}\delta^\lambda_\omega - T_{\mu\omega}\delta^\lambda_\nu, \\ &(A') \quad T_{\mu\nu;\omega} - T_{\mu\omega;\nu} = 0, \\ &(B) \quad T_{\lambda\mu} = \rho g_{\lambda\mu} + \rho_\lambda \sigma_\mu, \end{aligned} \tag{1.4}$$

where

$$T_{\lambda\mu} = \frac{1}{n-2}\left(R_{\lambda\mu} - \frac{R}{2(n-1)} g_{\lambda\mu}\right),$$

$$\rho_\mu = \frac{\partial \rho}{\partial x^\mu}, \qquad \sigma_\mu = \frac{\partial \sigma}{\partial x^\mu}, \qquad \sigma = \sigma(\rho).$$

Now since the covariant derivatives of the vector x^λ with respect to (1.3) are

$$x^\lambda_{;\mu} = \delta^\lambda_\mu + u_{\nu\mu} x^\nu x^\lambda,$$

the vector x^λ is a concircular or concurrent vector field [1], [10]. This result is also obtained from (1.4), since we have by virtue of (A') and (B)

$$\sigma_{\lambda;\mu} = g_{\lambda\mu} + \kappa\sigma_\lambda\sigma_\mu.$$

Therefore the case when the subprojective space admits a parallel vector field did not be treated by P. Rachevsky, H. Shapiro [6] and other authors [7], [8].

On the other hand, by covariant differentiation with respect to (1.1), we have

$$x^\lambda_{;\mu} = (1 + \varphi_\nu x^\nu)\delta^\lambda_\mu + (\varphi_\mu + \varphi_{\mu\nu} x^\nu) x^\lambda. \tag{1.5}$$

Therefore if the vector x^λ is a parallel vector field, we must have

$$1+\varphi_\nu x^\nu=0. \tag{1.6}$$

However, by the transformation of coordinates (1.2), we have

$$\frac{\partial \bar{x}^\lambda}{\partial x^\alpha}=e^{\varphi}(\varphi_\alpha x^\lambda+\delta_\alpha^\lambda),$$

from which follows

$$\frac{\partial \bar{x}^\lambda}{\partial x^\alpha}x^\alpha=(1+\varphi_\nu x^\nu)\bar{x}^\lambda.$$

Consequently components of the vector x^α may be transformed to $(1+\varphi_\nu x^\nu)\bar{x}^\lambda$ by (1.2). Moreover, the determinant of the transformation (1.2) becomes

$$\left|\frac{\partial \bar{x}^\lambda}{\partial x^\alpha}\right|=e^{n\varphi}|\varphi_\alpha x^\lambda+\delta_\alpha^\lambda|=e^{n\varphi}(1+\varphi_\nu x^\nu).$$

From these results, we find that, if (1.1) may be reducible to (1.3) by the transformation (1.2), we must have $1+\varphi_\nu x^\nu \neq 0$ and consequently by virtue of (1.6), if the vector x^λ is a parallel vector field, (1.1) can not be transformable to (1.3).

In this paper, we shall seek conditions for the subprojective space admitting a parallel vector field and relations which distinguish from it the subprojective space admitting a concircular or concurrent vector field.

§2. Rachevsky's condition (A), (A').

Let us assume that ξ^λ is a parallel vector field satisfying

$$\xi^\lambda{}_{;\mu}=\beta_\mu\xi^\lambda \tag{2.1}$$

and, for a suitable coordinate system, Christoffel symbols of the second kind take the form

$$\left\{ {\lambda \atop \mu\nu} \right\}=\varphi_\mu\delta_\nu^\lambda+\varphi_\nu\delta_\mu^\lambda+\varphi_{\mu\nu}\xi^\lambda, \tag{2.2}$$

where φ_μ and $\varphi_{\mu\nu}$ are certain covariant vector and symmetric tensor respectively. Then we have readily the curvature tensor

$$R^\lambda_{\cdot\mu\nu\omega}=2u_{\mu[\nu}\delta^\lambda_{\omega]}+2U_{\mu\nu\omega}\xi^\lambda+2\varphi_{[\nu;\omega]}\delta^\lambda_\mu, \tag{2.3}$$

where

$$u_{\mu\nu}=-(\varphi_{\mu;\nu}+\varphi_\mu\varphi_\nu+\xi^\sigma\varphi_\sigma\varphi_{\mu\nu}), \tag{2.4}$$

$$2U_{\mu\nu\omega}=\varphi_{\mu\nu;\omega}-\varphi_{\mu\omega;\nu}-\xi^\sigma(\varphi_{\mu\nu}\varphi_{\sigma\omega}-\varphi_{\mu\omega}\varphi_{\sigma\nu})+\varphi_{\mu\nu}\beta_\omega-\varphi_{\mu\omega}\beta_\nu. \tag{2.5}$$

From (2.3) we can obtain the following equations by the same method in the previous paper [1]. Namely

$$\varphi_{[\mu;\nu]}=0,\qquad \beta_{[\mu;\nu]}=0,\qquad u_{\mu\nu}=2\rho g_{\mu\nu}+u\xi_\mu\xi_\nu, \tag{2.6}$$

$$U_{\mu\nu\omega}=ug_{\mu[\nu}\xi_{\omega]}, \tag{2.7}$$

$$R^\lambda_{\cdot\mu\nu\omega}=2u_{\mu[\nu}\delta^\lambda_{\omega]}+2U_{\mu\nu\omega}\xi^\lambda, \tag{2.8}$$

(2.9) $$T_{\lambda\mu} = \rho g_{\lambda\mu} + u\xi_\lambda\xi_\mu.$$

Thus we have Rachevsky's condition

(2.10) (A) $$R^{\lambda}_{\cdot\mu\nu\omega} = T^{\lambda}_{\cdot\omega} g_{\mu\nu} - T^{\lambda}_{\cdot\nu} g_{\mu\omega} + \delta^{\lambda}_{\omega} T_{\mu\nu} - \delta^{\lambda}_{\nu} T_{\mu\omega}.$$

Moreover, from (2.6) and (2.7) we have

(2.11) $$\begin{aligned} - u_{\mu\nu;\omega} + u_{\mu\omega;\nu} &= - \varphi_\sigma R^{\sigma}_{\cdot\mu\nu\omega} + 2\xi^\sigma \varphi_\sigma U_{\mu\nu\omega} \\ &+ (u_{\mu\nu}\varphi_\omega - u_{\mu\omega}\varphi_\nu) - \xi^\sigma(\varphi_{\mu\nu} u_{\sigma\omega} - \varphi_{\mu\omega} u_{\sigma\nu}). \end{aligned}$$

Substituting (2.8), we have

(2.12) $$- u_{\mu\nu;\omega} + u_{\mu\omega;\nu} = - \xi^\sigma(\varphi_{\mu\nu} u_{\sigma\omega} - \varphi_{\mu\omega} u_{\sigma\nu}).$$

In consequence of (2.6), we obtain

(2.13) $$\begin{aligned} 2(\rho_\omega g_{\mu\nu} - \rho_\nu g_{\mu\omega}) + \xi_\mu\{(u_\omega + 2u\beta_\omega)\xi_\nu - (u_\nu + 2u\beta_\nu)\xi_\omega\} \\ = (2\rho + u\xi^\sigma\xi_\sigma)(\xi_\omega\varphi_{\mu\nu} - \xi_\nu\varphi_{\mu\omega}). \end{aligned}$$

On the other hand, substituting (2.1) and (2.8) in the Ricci identities

$$\xi_{\mu;\nu\omega} - \xi_{\mu;\omega\nu} = - \xi_\sigma R^{\sigma}_{\cdot\mu\nu\omega},$$

we have

$$(2\rho + u\xi^\sigma\xi_\sigma)(\xi_\omega g_{\mu\nu} - \xi_\nu g_{\mu\omega}) = 0 ,$$

from which follows

(2.14) $$2\rho + u\xi^\sigma\xi_\sigma = 0.$$

Substituting (2.14) in (2.13), we have

(2.15) $$2(\rho_\omega g_{\mu\nu} - \rho_\nu g_{\mu\omega}) + \xi_\mu\{(u_\omega + 2u\beta_\omega)\xi_\nu - (u_\nu + 2u\beta_\nu)\xi_\omega\} = 0.$$

Multiplying any vector η^μ orthogonal to ξ^μ and contracting for μ, we have

$$\rho_\omega\eta_\nu - \rho_\nu\eta_\omega = 0,$$

from which we find that

(2.16) $\rho_\omega = 0$, that is, $\rho = c =$ const.,

and (2.14) becomes

(2.17) $$2c + u\xi^\sigma\xi_\sigma = 0,$$

Consequently from (2.15) we have

(2.18) $$u_\mu + 2u\beta_\mu = q\xi_\mu,$$

where q is a scalar.

Thus from (2.9) we have

$$\begin{aligned} T_{\lambda\mu;\nu} &= u_\nu\xi_\lambda\xi_\mu + u\xi_{\lambda;\nu}\xi_\mu + u\xi_\lambda\xi_{\mu;\nu} \\ &= (u_\nu + 2u\beta_\nu)\xi_\lambda\xi_\mu. \end{aligned}$$

Because of (2.17), we obtain

(2.19) $$T_{\lambda\mu;\nu} = q\xi_\lambda\xi_\mu\xi_\nu,$$

from which follows Rachevsky's condition

(2.20) (A′) $$T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0.$$

§ 3. Rachevsky's condition (B′).

Differentiating (2.17) covariantly, we have

$$u_\lambda \xi^\sigma \xi_\sigma + 2u \xi^\sigma_{;\lambda} \xi_\sigma = 0.$$

Substituting (2.1), we have

$$(u_\lambda + 2u\beta_\lambda)\xi^\sigma \xi_\sigma = 0.$$

If we assume $\xi^\sigma \xi_\sigma \neq 0$, we obtain

(3.1) $$u_\lambda + 2u\beta_\lambda = 0,$$

from which follows

(3.2) $$ue^{2\beta} = k = \text{const.} \qquad \left(\beta_\lambda = \frac{\partial \beta}{\partial x^\lambda}\right).$$

Now if we put

(3.3) $$\eta_\lambda = e^{-\beta}\xi_\lambda,$$

we can easily find that

$$\eta_{\lambda;\mu} = 0$$

and by virtue of (2.9) and (3.2)

(3.4) $$T_{\lambda\mu} = cg_{\lambda\mu} + k\eta_\lambda \eta_\mu.$$

Moreover, from (2.18), (2.19) and (3.1), we have

$$T_{\lambda\mu;\nu} = 0.$$

Thus we obtain the next three conditions:

$$\begin{array}{lll} & (A) & R^{\lambda}_{\cdot\mu\nu\omega} = T^{\lambda}_{\cdot\omega} g_{\mu\nu} - T^{\lambda}_{\cdot\nu} g_{\mu\omega} + T_{\mu\nu}\delta^\lambda_\omega - T_{\mu\omega}\delta^\lambda_\nu , \\ (3.5) & (A') & T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0 \quad (\text{or } T_{\lambda\mu;\nu} = 0), \\ & (B') & T_{\lambda\mu} = cg_{\lambda\mu} + k\eta_\lambda\eta_\mu \end{array}$$

where η_λ is a gradient vector and $\eta^\lambda \eta_\lambda = \text{const.} \neq 0$.

Coversely, let us assume that (A') and (B') hold. Then

$$T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = k(\eta_{\lambda;\nu}\eta_\mu - \eta_{\lambda;\mu}\eta_\nu) = 0,$$

from which we have

$$\eta_{\lambda;\nu}\eta^\mu \eta_\mu = \eta^\mu \eta_{\lambda;\mu}\eta_\nu = \eta^\mu \eta_{\mu;\lambda}\eta_\nu .$$

However, since $\eta^\mu \eta_\mu = \text{const.}$, we have $\eta^\mu \eta_{\mu;\lambda} = 0$ and consequently $\eta_{\lambda;\nu} = 0$, which follows that η_λ is a parallel vector field.

Especially, when $\xi^\sigma \xi_\sigma = 0$, from (2.14) we have

$$\rho = 0.$$

Therefore $T_{\lambda\mu} = u\xi_\lambda \xi_\mu$, from which we have, substituting (3.3),

$$T_{\lambda\mu} = ue^{2\beta}\eta_\lambda \eta_\mu.$$

However, since from (2.18) we find that $ue^{2\beta}$ is a function of η, we can obtain the equation of the form

(3.6) $$T_{\lambda\mu} = v(\eta)\eta_\lambda\eta_\mu.$$

Thus we have [8]

$$\begin{array}{lll} & (A) & R^{\lambda}_{\cdot\mu\nu\omega} = T^{\lambda}_{\cdot\omega} g_{\mu\nu} - T^{\lambda}_{\cdot\nu} g_{\mu\omega} + T_{\mu\nu}\delta^\lambda_\omega - T_{\mu\omega}\delta^\lambda_\nu , \\ (3.7) & (A') & T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0, \\ & (B'') & T_{\lambda\mu} = v(\eta)\eta_\lambda\eta_\mu, \end{array}$$

where $\eta_\lambda = \dfrac{\partial \eta}{\partial x^\lambda}$ and $\eta^\lambda \eta_\lambda = 0$.

Conversely, if (A') and (B'') are satisfied, we have

$$T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = v(\eta_{\lambda;\nu}\eta_\mu - \eta_{\lambda;\mu}\eta_\nu) = 0.$$

Multiplying by ζ^μ, where $\zeta^\mu \eta_\mu \neq 0$, and contracting for μ, we have the equation of form

$$\eta_{\lambda;\nu} = \kappa \eta_\lambda \eta_\nu, \tag{3.8}$$

which follows that η_λ is a parallel vector field. Furthermore, when (A) holds, we have

$$\begin{aligned} \eta_{\lambda;\mu\nu} - \eta_{\lambda;\nu\mu} &= \eta_\lambda(\kappa_\nu \eta_\mu - \kappa_\mu \eta_\nu) \\ &= -\eta_\sigma R^{\sigma}_{\cdot\lambda\mu\nu} \\ &= 0. \end{aligned}$$

Thus we find that κ is a function of η and consequently (3.8) is transformable to the form

$$\eta_{\lambda;\nu} = 0.$$

Finally we shall introduce some relations. From (2.6) and (2.17) we can readily obtain

$$\xi^\mu u_{\mu\nu} = (2c + u\xi^\sigma \xi_\sigma)\xi_\nu = 0 \tag{3.9}$$

and consequently, by virtue of $u_{\mu\nu} = T_{\mu\nu} + cg_{\mu\nu}$,

$$\xi^\mu T_{\mu\nu} = -c\xi_\nu\ ,$$
$$u_{\mu\nu;\omega} = T_{\mu\nu;\omega}\ ,$$

that is,

$$u_{\mu\nu;\omega} - u_{\mu\omega;\nu} = 0. \tag{3.10}$$

§ 4. Transformation to the form $\left\{ {\lambda \atop \mu\nu} \right\} = \varphi_\mu \delta^\lambda_\nu + \varphi_\nu \delta^\lambda_\mu + \varphi_{\mu\nu} x^\lambda$.

Let us assume that (3.5) holds and consequently $\eta_{\lambda;\mu} = 0$. From (3.3) we have (2.1) and

$$T_{\lambda\mu} = cg_{\lambda\mu} + u\xi_\lambda \xi_\mu, \tag{4.1}$$

which follows (3.1).

Moreover, we have, by virtue of Ricci identities

$$\xi_{\mu;\nu\omega} - \xi_{\mu;\omega\nu} = -\xi_\sigma R^{\sigma}_{\cdot\mu\nu\omega}\ ,$$
$$\xi_\sigma T^{\sigma}_{\cdot\omega} g_{\mu\nu} - \xi_\sigma T^{\sigma}_{\cdot\nu} g_{\mu\omega} + T_{\mu\nu}\xi_\omega - T_{\mu\omega}\xi_\nu = 0.$$

Substituting (4.1), we obtain

$$(2c + u\xi^\sigma \xi_\sigma)(\xi_\omega g_{\mu\nu} - \xi_\nu g_{\mu\omega}) = 0,$$

from which we have

$$2c + u\xi^\sigma \xi_\sigma = 0. \tag{4.2}$$

We consider now differential equations

$$Z_{\lambda;\mu} = -Z_\lambda \varphi_\mu - Z_\mu \varphi_\lambda - Z_\sigma \xi^\sigma \varphi_{\lambda\mu}\ , \tag{4.3}$$

where φ_μ is a gradient vector and $\varphi_{\lambda\mu}$ a symmetric tensor. We shall first calculate the integrability conditions

$$Z_{\lambda;\mu\nu} - Z_{\lambda;\nu\mu} = - Z_\sigma R^\sigma_{\cdot\lambda\mu\nu}. \tag{4.4}$$

Substituting (4.3) in the left-hand member, we have

$$Z_{\lambda;\mu\nu} - Z_{\lambda;\nu\mu} = - Z_\sigma(u_{\lambda\mu}\delta^\sigma_\nu - u_{\lambda\nu}\delta^\sigma_\mu + 2U_{\lambda\mu\nu}\xi^\sigma), \tag{4.5}$$

where $u_{\lambda\mu}$ and $U_{\lambda\mu\nu}$ are defined by (2.4) and (2.5) respectively. On the other hand, from (3.5) (A) and (4.1) we have

$$\begin{aligned} - Z_\sigma R^\sigma_{\cdot\lambda\mu\nu} &= - Z_\sigma(T^\sigma_{\cdot\nu} g_{\lambda\mu} - T^\sigma_{\cdot\mu} g_{\lambda\nu} + T_{\lambda\mu}\delta^\sigma_\nu - T_{\lambda\nu}\delta^\sigma_\mu) \\ &= - Z_\sigma\{(2cg_{\lambda\mu} + u\xi_\lambda\xi_\mu)\delta^\sigma_\nu - (2c + u\xi_\lambda\xi_\nu)\delta^\sigma_\mu \\ &\quad + u(g_{\lambda\mu}\xi_\nu - g_{\lambda\nu}\xi_\mu)\xi^\sigma\}. \end{aligned} \tag{4.6}$$

Let us assume that φ_μ is an arbitrary gradient vector satisfying $\xi^\sigma\varphi_\sigma \neq 0$. Then we can define a symmetric tensor $\varphi_{\mu\nu}$ by the equation

$$u_{\mu\nu} = 2cg_{\mu\nu} + u\xi_\mu\xi_\nu. \tag{4.7}$$

From (3.5) (A′) we have (3.10). Furthermore, from (4.2) we have (3.9) and from (3.5) (A)

$$\varphi_\sigma R^\sigma_{\cdot\mu\nu\omega} = u_{\mu\nu}\varphi_\omega - u_{\mu\omega}\varphi_\nu + u(g_{\mu\nu}\xi_\omega - g_{\mu\omega}\xi_\nu)\varphi_\sigma\xi^\sigma.$$

Substituting these results in (2.11), we obtain

$$2U_{\mu\nu\omega} = u(g_{\mu\nu}\xi_\omega - g_{\mu\omega}\xi_\nu). \tag{4.8}$$

Therefore substituting (4.7) and (4.8) in (4.5) and comparing with (4.6), we find that (4.4) is satisfied identically and consequently (4.3) is completely integrable.

If we represent n linearly independent solutions of (4.3) by Z^α_λ ($\alpha = 1, 2, \ldots . n$), then we have

$$Z^\alpha_\lambda = \frac{\partial \bar{x}^\alpha}{\partial x^\lambda},$$

where x^α are independent functions of x^λ, that is,

$$x^\alpha = x^\alpha(x^\lambda).$$

We consider now the above equations as a transformation of coordinates. Then

$$\begin{aligned} \overline{\left\{ {\alpha \atop \beta\gamma} \right\}} &= \frac{\partial x^\mu}{\partial x^\beta}\frac{\partial x^\nu}{\partial \bar{x}^\gamma}\left(\left\{ {\lambda \atop \mu\nu} \right\}\frac{\partial \bar{x}^\alpha}{\partial x^\lambda} - \frac{\partial^2 x^\alpha}{\partial x^\mu \partial x^\nu}\right) \\ &= -\frac{\partial x^\mu}{\partial x^\beta}\frac{\partial x^\nu}{\partial x^\gamma} Z^\alpha_{\mu;\nu}. \end{aligned}$$

Substituting (4.3)

$$\overline{\left\{ {\alpha \atop \beta\gamma} \right\}} = \frac{\partial x^\mu}{\partial x^\beta}\frac{\partial x^\nu}{\partial x^\gamma}(Z^\alpha_\mu\varphi_\nu + Z^\alpha_\nu\varphi_\mu + Z^\alpha_\sigma\xi^\sigma\varphi_{\mu\nu}),$$

that is,

$$\overline{\left\{ {\alpha \atop \beta\gamma} \right\}} = \bar{\varphi}_\beta\delta^\alpha_\gamma + \bar{\varphi}_\gamma\delta^\alpha + \bar{\varphi}_{\beta\gamma}\xi^\alpha, \tag{4.9}$$

where ξ^α, $\overline{\varphi}_\beta$ and $\overline{\varphi}_{\beta\gamma}$ are respectively components of the vectors ξ^α, φ_β and the tensor $\varphi_{\beta\gamma}$ in the x' s.

Thus, when (3.5) holds, the Christoffel symbols may by expressible in the form (2.2) for a suitable coordinate system.

Now, if $\xi^\sigma\varphi_\sigma \neq 0$, we may assume that

(4.10) $$\xi^\sigma\varphi_\sigma = -1,$$

replacing φ_σ (or ξ^σ) by $-\dfrac{\varphi_\sigma}{\sqrt{\xi^\mu\varphi_\mu}}\left(\text{or } -\dfrac{\xi^\sigma}{\sqrt{\xi^\mu\varphi_\mu}}\right)$. Then differentiating with respect to x^μ, we have

$$\xi^\sigma_{;\mu}\varphi_\sigma + \xi^\sigma\varphi_{\sigma;\mu} = 0,$$

that is to say,

$$\beta_\mu\xi^\sigma\varphi_\sigma + \xi^\sigma\varphi_{\sigma;\mu} = 0.$$

Substituting (4.10), we have

(4.11) $$\beta_\mu = \xi^\sigma\varphi_{\sigma;\mu}.$$

Moreover

$$\xi^\sigma u_{\sigma\mu} = -(\xi^\sigma\varphi_{\sigma;\mu} + \xi^\sigma\varphi_\sigma\varphi_\mu + \xi^\omega\varphi_\omega\xi^\sigma\varphi_{\sigma\mu}) = 0,$$

from which we have, by virtue of (4.10),

$$\xi^\sigma\varphi_{\sigma;\mu} - \varphi_\mu - \xi^\sigma\varphi_{\sigma\mu} = 0.$$

Substituting (4.11), we obtain

$$\beta_\mu - \varphi_\mu - \xi^\sigma\varphi_{\sigma\mu} = 0.$$

Therefore

$$\begin{aligned}(\xi^\sigma Z_\sigma)_{;\mu} &= \xi^\sigma_{;\mu}Z_\sigma + \xi^\sigma Z_{\sigma;\mu} \\ &= \beta_\mu\xi^\sigma Z_\sigma - \xi^\sigma Z_\sigma\varphi_\mu - \xi^\sigma\varphi_\sigma Z_\mu - \xi^\sigma\varphi_{\sigma\mu}\xi^\omega Z_\omega \\ &= (\beta_\mu - \varphi_\mu - \xi^\omega\varphi_{\omega\mu})\xi^\sigma Z_\sigma + Z_\mu \\ &= Z_\mu.\end{aligned}$$

Thus we find

$$\xi^\sigma Z^\alpha_\sigma = x^\alpha$$

and consequently (4.9) becomes

$$\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\} = \overline{\varphi}_\beta\delta^\alpha_\gamma + \overline{\varphi}_\gamma\delta^\alpha_\beta + \overline{\varphi}_{\beta\gamma}x^\alpha,$$

from which follows the

THEOREM 4.1. *A Riemannian space whose Christoffel symbols take the form*

$$\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\} = \varphi_\mu\delta^\lambda_\nu + \varphi_\nu\delta^\lambda_\mu + \varphi_{\mu\nu}\xi^\lambda,$$

where ξ^λ is a parallel vector field, for a suitable coordinate system, is a subprojective space in the sense of Kagan.

Especially, when $\xi^\sigma\xi_\sigma = 0$, we can prove the theorem by the analogous method. Thus we have the

THEOREM 4.2. *A subprojective space which admits a parallel vector field may be characterized by the conditions* (3.5) *or* (3.7).

Furthermore comparing with the results in the previous paper [1], we have the

THEOREM 4.3. *In order that a Riemannian space is a subprojective one, it is necessary and sufficient that the next relations hold*:

$$(A)\qquad R^{\lambda}_{\cdot\mu\nu\omega} = T^{\lambda}_{\cdot\omega} g_{\mu\nu} - T^{\lambda}_{\cdot\nu} g_{\mu\omega} + T_{\mu\nu}\delta^{\lambda}_{\omega} - T_{\mu\omega}\delta^{\lambda}_{\nu},$$

$$(A')\qquad T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0,$$

$$(B)\qquad T_{\lambda\mu} = \rho(\sigma) g_{\lambda\mu} + \kappa(\sigma)\sigma_{\lambda}\sigma_{\mu},$$

where

$$T_{\lambda\mu} = \frac{1}{n-2}\left(R_{\lambda\mu} - \frac{R}{2(n-1)} g_{\lambda\mu}\right), \qquad \sigma_{\lambda} = \frac{\partial\sigma}{\partial x^{\lambda}}.$$

In this case, if $\rho = \text{const.}$, we can prove that σ_{λ} is a parallel vector field and consequently the space is a subprojective space admitting a parallel vector field.

§ 5. Some theorems on a subprojective space admitting a parallel vector field.

The fundamental quadratic differential form of a Riemannian space which admits a parallel vector field ξ^{λ} satisfying $\xi^{\sigma}\xi_{\sigma} \neq 0$, may be written in the form [9], [10]

$$(5.1)\qquad ds^2 = g_{jk}(x^i) dx^j dx^k + (dx^n)^2, \qquad (i, j, k = 1, 2, \ldots, n-1)$$

for a suitable coordinate system. From it we can readily obtain the following equations [1]

$$(5.2)\qquad R_{ij} = R_{ij}, \quad R_{in} = R_{nn} = 0,$$

$$(5.3)\qquad R = R,$$

$$(5.4)\qquad \begin{cases} T_{ij} = \dfrac{1}{n-2}\left(R_{ij} - \dfrac{R}{2(n-1)} g_{ij}\right), \\ T_{nn} = -\dfrac{R}{2(n-1)(n-2)}, \\ T_{in} = 0, \end{cases}$$

$$(5.5)\qquad \begin{cases} T_{nn;i} - T_{ni;n} = -\dfrac{1}{2(n-1)(n-2)} \dfrac{\partial \overline{R}}{\partial x^i}, \\ T_{ij;k} - T_{ik;j} = \dfrac{1}{n-2}\left(R_{ij|k} - \dfrac{1}{2(n-1)} \dfrac{\partial \overline{R}}{\partial x^k} g_{ij}\right) \\ \qquad\qquad - \dfrac{1}{n-2}\left(R_{ik|j} - \dfrac{1}{2(n-1)} \dfrac{\partial \overline{R}}{\partial x^j} g_{ik}\right), \\ T_{ij;n} - T_{in;j} = T_{ni;j} - T_{nj;i} = 0, \end{cases}$$

where $\overline{R}$ and $\overline{R}_{ij}$ are respectively Riemann curvature and Ricci tensor of the hypersurfaces $x^n = \text{const.}$ and $R_{ij|k}$ is a covariant derivative of R_{ij} with respect to g_{ij}.

From (5.2) we have the

THEOREM 5.1. *In a Riemannian space V_n admitting a parallel vector field, there exists a family of ∞^1 totally geodesic hypersurfaces and the vector field are defined as the normals to these hypersurfaces. In this case, in order that tangential directions to these hypersurfaces are Ricci principal directions, it is necessary and sufficient that these hypersurfaces are all Einsein spaces* $(n>3)$.

Calculating $C^{\lambda}_{\cdot\mu\nu\omega}$ and making use of (5.5), we have [1] the

THEOREM 5.2. *In a Riemannian space V_n admitting a parallel vector field, in order that the hypersurfaces in the above theorm are of constant curvatur, it is necessary and sufficient that V_n is a conformally flat space.*

When $n>3$, if the above-mentioned hypersurfaces $x^n=$ const. are Einstein spaces, from (5.5) we have

$$T_{\lambda\mu;\nu}-T_{\lambda\nu;\mu}=0.$$

Thus we have the

THEOREM 5.3. *In a Riemannian space V_n admitting a parallel vector field, if the above-mentioned totally geodesic hypersurfaces are all Einstein spaces, then we have*

$$T_{\lambda\mu;\nu}-T_{\lambda\nu;\mu}=0 \qquad (n>3).$$

When the hypersurfaces $x^n=$ const. are Einstein spaces, we have, from (5.2) and (5.3),

$$R_{ij}=\frac{R}{n-1}g_{ij}.$$

Consequently (5.4) becomes

$$\text{(5.6)}\qquad \begin{cases} T_{ij}=\dfrac{R}{2(n-1)(n-2)}g_{ij}, \\ T_{nn}=-\dfrac{R}{2(n-1)(n-2)}, \\ T_{in}=0. \end{cases}$$

Now let us assume that η_λ is a parallel vector field, where $\eta_\lambda=\dfrac{\partial\eta}{\partial x^\lambda}$, and the totally geodesic hypersurfaces $\eta=$ const. are Einstein spaces. Then, since tangential directions to these hypersurfaces are Ricci directions by virtue of the Theorem 5.1, the tensor $T_{\lambda\mu}$ may be written in the form

$$\text{(5.7)}\qquad T_{\lambda\mu}=\rho g_{\lambda\mu}+\kappa\eta_\lambda\eta_\mu.$$

Comparing with (5.6), we have, because η_λ corresponds to δ^n_λ,

$$\text{(5.8)}\qquad \rho=\frac{R}{2(n-1)(n-2)},$$

$$\text{(5.9)}\qquad \kappa=-\frac{R}{(n-1)(n-2)}.$$

However, when $n > 3$, from (5.3) we find $R =$ const.. Thus we obtain

$$\rho = c = \text{const.} \neq 0,$$

which follows

$$R = R = 2(n-1)(n-2)c.$$

Moreover, from (5.7) we have

$$T_{\lambda\mu;\nu} = \kappa_\nu \eta_\lambda \eta_\mu + \kappa \eta_{\lambda;\nu} \eta_\mu + \kappa \eta_\lambda \eta_{\mu;\nu} .$$

Therefore, by virtue of the Theorem 5.3, we have

$$\begin{aligned} T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} &= \eta_\lambda(\kappa_\nu \eta_\mu - \kappa_\mu \eta_\nu) + \kappa(\eta_{\lambda;\nu}\eta_\mu - \eta_{\lambda;\mu}\eta_\nu) \\ &= 0. \end{aligned}$$

Since η_λ is a parallel vector field, we find that

$$\kappa_\nu \eta_\mu - \kappa_\mu \eta_\nu = 0,$$

that is, κ is a function of η.

Thus, when $n > 3$, if η_λ is a parallel vector field and (5.7) holds, then we have

$$\rho = c = \text{const.}, \quad \kappa = \kappa(\eta),$$

(5.10) $$R = R = 2(n-1)(n-2)c,$$

(5.11) $$T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0,$$

that is to say, we get the

THEOREM 5.4. *If η_λ is a parallel vector field, where $\eta_\lambda = \dfrac{\partial \eta}{\partial x^\lambda}$, and tangential directions to the hypersurfaces $\eta =$ const. are Ricci directions, then we have*

$$T_{\lambda\mu} = c g_{\lambda\mu} + \kappa(\eta)\eta_\lambda\eta_\mu ,$$
$$R = R = 2(n-1)(n-2)c,$$

where c is a constant and R is the Riemann curvature of the hypersurfaces $\eta =$ const. ($n > 3$).

Moreover, if $\eta^\lambda \eta_\lambda = 1$, from (5.8) and (5.9) we have

(5.12) $$T_{\lambda\mu} = c(g_{\lambda\mu} - 2\eta_\lambda\eta_\mu).$$

When $n = 3$, if η_λ is a parallel vector field and

$$T_{\lambda\mu} = c g_{\lambda\mu} + \kappa \eta_\lambda \eta_\mu,$$

from (5.8) and (5,9) we obtain (5.10) and (5.12). Cosequently the hypersurfaces $\eta =$ const. are of constant curvature and from (5.5) we have (5.11), namely V_3 is conformally flat. Therefore κ is a function of η and V_3 is subprojective.

Furthermore, we can find the next fundamental theorems [1].

THEOREM 5.5. *A conformally flat space which admits a parallel vector field is a subprojective space of Kagan.*

THEOREM 5.6. *A Riemannian space which contains a family of ∞^1 totally* *geodesic hypersurfaces, whose Riemann curvatures are all constant and orthogonal trajectories are geodesics, is a subprojective space of Kagan.*

Finally, we can easily find that the fundamental quadratic differential form of the subprojective space admitting a parallel vector field takes the form [2]

$$ds^2 = \frac{(dx^1)^2 + (dx^2)^2 + \dots + (dx^{n-1})^2}{\pm K\left\{\frac{1}{4}\sum_{i=1}^{n-1}(x^i)^2 \pm 1\right\}^2} + (dx^n)^2,$$

where $K = \frac{R}{(n-1)(n-2)} = \text{const.} \neq 0$ and '$\pm$' takes '+' or '$-$' according as R is positive or negative.

§ 6. Subprojective space admitting a concircular vector field.

Let us assume that a tensor $T_{\lambda\mu}$ of a space admitting a concircular vector field ξ_λ takes the form

$$T_{\lambda\mu} = \rho g_{\lambda\mu} + u\xi_\lambda\xi_\mu \tag{6.1}$$

and

$$\xi^\lambda_{;\mu} = \alpha\delta^\lambda_\mu + \beta_\mu\xi^\lambda, \tag{6.2}$$

$$\alpha\beta_\mu - \alpha_\mu = p\xi_\mu\,. \tag{6.3}$$

From these equations, we shall introduce some relations which hold in the subprojective space.

From (6.2) and (6.3) we have

$$\begin{aligned}\xi_{\lambda;\mu\nu} - \xi_{\lambda;\nu\mu} &= (\alpha\beta_\mu - \alpha_\mu)g_{\lambda\nu} - (\alpha\beta_\nu - \alpha_\nu)g_{\lambda\mu}\\ &= -p(\xi_\nu g_{\lambda\mu} - \xi_\mu g_{\lambda\nu}).\end{aligned}$$

Making use of Ricci identities, we have

$$\xi_\sigma R^\sigma_{\cdot\lambda\mu\nu} = p(\xi_\nu g_{\lambda\mu} - \xi_\mu g_{\lambda\nu}),$$

from which follows

$$\xi_\sigma R^\sigma_{\cdot\nu} = (n-1)p\xi_\nu\,. \tag{6.4}$$

Therefore from (6.1) we have

$$\begin{aligned}\xi_\sigma T^\sigma_{\cdot\mu} &= \frac{1}{n-2}\left\{(n-1)p - \frac{R}{2(n-1)}\right\}\xi_\mu\\ &= (\rho + u\xi^\sigma\xi_\sigma)\xi_\mu\,.\end{aligned}$$

Hence we have

$$-\frac{R}{2(n-1)(n-2)} + \frac{n-1}{n-2}p = \rho + u\xi^\sigma\xi_\sigma\,. \tag{6.5}$$

On the other hand, calculating $g^{\lambda\mu}T_{\lambda\mu}$, we have

$$\frac{R}{2(n-1)} = n\rho + u\xi^\sigma\xi_\sigma\,.$$

Eliminating $u\xi^\sigma\xi_\sigma$ from these two equations, we obtain

$$\rho = \frac{R}{2(n-1)(n-2)} - \frac{p}{n-2}\,. \tag{6.6}$$

Eliminating R from (6.5) and (6.6), we have

(6.7) $$2\rho + u\xi^\sigma\xi_\sigma = p.$$

From (6.7) we have

$$u = \frac{p-2\rho}{\xi^\sigma\xi_\sigma} = \frac{-2\rho}{\xi^\sigma\xi_\sigma} + \frac{p}{\xi^\sigma\xi_\sigma}$$

and consequently (6.1) becomes

(6.8) $$T_{\lambda\mu} = \rho(g_{\lambda\mu} - 2\eta_\lambda\eta_\mu) + p\eta_\lambda\eta_\mu,$$

where $\eta_\lambda = \xi_\lambda / \sqrt{\xi^\sigma\xi_\sigma}$, that is, $\eta_\lambda\eta_\lambda = 1$.

If we put $\kappa = \sqrt{\xi^\sigma\xi_\sigma}$, we have

$$\eta_\lambda = \frac{\xi_\lambda}{\kappa}$$

and by virtue of (6.2)

$$\eta_{\mu;\nu} = \frac{\alpha}{\kappa} g_{\mu\nu} + \left(\beta_\nu - \frac{\kappa_\nu}{\kappa}\right)\eta_\mu.$$

However

$$\kappa_\nu = \frac{\xi^\sigma_{;\nu}\xi_\sigma}{\sqrt{\xi^\sigma\xi_\sigma}} = \frac{1}{\kappa}(\alpha\delta^\sigma_\nu + \beta_\nu\xi^\sigma)\xi_\sigma = \alpha\eta_\nu + \kappa\beta_\nu,$$

that is,

(6.9) $$\frac{\kappa_\nu}{\kappa} = \frac{\alpha}{\kappa}\eta_\nu + \beta_\nu.$$

Thus we have

$$\eta_{\mu;\nu} = \frac{\alpha}{\kappa}(g_{\mu\nu} - \eta_\mu\eta_\nu),$$

where, if $\eta_\mu = \frac{\partial\eta}{\partial x^\mu}$, $\frac{\alpha}{\kappa}$ is a function of η, because η_μ is a concircular vector field.

Furthermore if we assume that $\xi_\lambda = \theta\sigma_\lambda$ and

$$\sigma_{\lambda;\mu} = \frac{\alpha}{\theta} g_{\lambda\mu} + \gamma\sigma_\lambda\sigma_\mu \quad \left(\sigma_\lambda = \frac{\partial\sigma}{\partial x^\lambda}\right),$$

then (6.1) becomes

$$T_{\lambda\mu} = \rho g_{\lambda\mu} + u\theta^2\sigma_\lambda\sigma_\mu$$

and we can prove [2] that, when $n > 3$, $\rho_\mu = u\theta^2\frac{\alpha}{\theta}\sigma_\mu$, that is,

(6.10) $$\rho_\mu = \alpha u\xi_\mu,$$

(6.11) $$T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0.$$

Hence we have

$$\rho_\mu = \frac{\alpha(p-2\rho)}{\kappa}\eta_\mu,$$

that is,

$$\frac{\rho_\mu}{p-2\rho} = \frac{\alpha}{\kappa}\eta_\mu.$$ [1]

1) We assume that $\xi^\sigma\xi_\sigma \neq 0$ and consequently $p - 2\rho \neq 0$. If $\xi^\sigma\xi_\sigma = 0$, $(\xi^\sigma\xi_\sigma)_{;\mu} = 2\xi^\sigma\xi_{\sigma;\mu} = 2\alpha\xi_\mu = 0$ which follows $\alpha = 0$. Hence ξ^λ is a parallel vector field.

Therefore ρ and $p-2\rho$ are a function of η and consequently p is also a function of η.

Especially when $n=3$, if V_3 is a subprojective space, it is evident that (6.10) and (6.11) hold and consequently we have the same results.

From these results we find that Rachevsky's conditions for the subprojective space may be written as

$$(6.12)\qquad \begin{array}{ll} \text{(A)} & R^{\lambda}_{\cdot\mu\nu\omega} = T^{\lambda}_{\cdot\omega} g_{\mu\nu} - T^{\lambda}_{\cdot\nu} g_{\mu\omega} + T_{\mu\nu}\delta^{\lambda}_{\omega} - T_{\mu\omega}\delta^{\lambda}_{\nu}, \\ \text{(A}'\text{)} & T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0, \\ \text{(B)} & T_{\lambda\mu} = \rho(\eta)(g_{\lambda\mu} - 2\eta_\lambda\eta_\mu) + p\eta_\lambda\eta_\mu, \end{array}$$

where $\eta_\lambda = \dfrac{\partial\eta}{\partial x^\lambda}$ and $\eta^\lambda\eta_\lambda = 1$.

Conversely let us assume that (A′) and (B) hold. Then we have from (B)

$$T_{\lambda\mu;\nu} = \rho_\nu(g_{\lambda\mu} - 2\eta_\lambda\eta_\mu) - 2\rho(\eta_{\lambda;\nu}\eta_\mu + \eta_\lambda\eta_{\mu;\nu}) + p_\nu\eta_\lambda\eta_\mu + p(\eta_{\lambda;\nu}\eta_\mu + \eta_\lambda\eta_{\mu;\nu}),$$

from which follows

$$(6.13)\qquad \begin{aligned} T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} &= (\rho_\nu g_{\lambda\mu} - \rho_\mu g_{\lambda\nu}) + (p-2\rho)(\eta_{\lambda;\nu}\eta_\mu - \eta_{\lambda;\mu}\eta_\nu) \\ &\quad + \eta_\lambda(p_\nu\eta_\mu - p_\mu\eta_\nu) \\ &= 0. \end{aligned}$$

Multiplying by η^λ and summing for λ, we have

$$p_\nu\eta_\mu - p_\mu\eta_\nu = 0,$$

because ρ is a function of η and $\eta^\lambda\eta_{\lambda;\nu} = 0$. Thus we find that p is a function of η. Consequently from (6.13) we have

$$(\rho_\nu g_{\lambda\mu} - \rho_\mu g_{\lambda\nu}) + (p-2\rho)(\eta_{\lambda;\nu}\eta_\mu - \eta_{\lambda;\mu}\eta_\nu) = 0.$$

Multiplying by η^μ and summing for μ, we have

$$\rho_\nu\eta_\lambda - \eta^\mu\rho_\mu g_{\lambda\nu} + (p-2\rho)\eta_{\lambda;\nu} = 0,$$

from which we have

$$\eta_{\lambda;\nu} = \frac{\eta^\mu\rho_\mu}{p-2\rho} g_{\lambda\nu} - \frac{\rho_\nu\eta_\lambda}{p-2\rho}.$$

Consequently putting $\dfrac{\rho_\mu}{p-2\rho} = f(\eta)\eta_\mu$, we have

$$\eta_{\lambda;\mu} = f(\eta)(g_{\lambda\nu} - \eta_\lambda\eta_\nu).$$

Thus we find that if (6.12) (A′) and (B) hold, then p is a function of η and η_λ is a concircular vector field.

§ 7. Subprojective space admitting a concurrent vector field.

When ξ^λ is a concurrent vector field, (6.3) becomes [1]

$$(7.1)\qquad \alpha\beta_\mu - \alpha_\mu = 0,$$

that is, $p=0$. Therefore (6.6) and (6.7) reduce to

$$(7.2)\qquad \rho = \frac{R}{2(n-1)(n-2)},$$

$$(7.3)\qquad 2\rho + u\xi^\sigma\xi_\sigma = 0.$$

Consequently if we put

$$\eta_\lambda = \frac{\xi_\lambda}{\kappa}, \qquad \kappa = \sqrt{\xi^\sigma \xi_\sigma},$$

we have

$$T_{\lambda\mu} = \rho(g_{\lambda\mu} - 2\eta_\lambda \eta_\mu),$$
$$\eta_{\mu;\nu} = \frac{\alpha}{\kappa}(g_{\mu\nu} - \eta_\mu \eta_\nu).$$

Now eliminating β_ν from (6.9) and (7.1), we have

$$\frac{\alpha}{\kappa}\eta_\mu - \frac{1}{\kappa}\kappa_\mu + \frac{1}{\alpha}\alpha_\mu = 0,$$

from which follows

$$\eta_\mu = \frac{\alpha\kappa_\mu - \kappa\alpha_\mu}{\alpha^2} = \frac{\partial}{\partial x^\mu}\frac{\kappa}{\alpha}.$$

Therefore we have $\eta = \frac{\kappa}{\alpha} + \text{const.}$, Hence putting $\frac{\kappa}{\alpha} = \eta$, we obtain

(7.4) $$\eta_{\mu;\nu} = \frac{1}{\eta}(g_{\mu\nu} - \eta_\mu \eta_\nu).$$

Furthermore, when $n > 3$, from (6.10) we have

(7.5) $$\rho_\mu = -\frac{2\alpha\rho}{\kappa}\eta_\mu = -\frac{2\rho}{\eta}\eta_\mu,$$

from which follows

$$\frac{\rho_\mu}{2\rho} + \frac{\eta_\mu}{\eta} = 0,$$

that is,

$$\rho\eta^2 = \text{const.} \neq 0.$$

When $n = 3$, for the subprojective space the above equation holds.

Hence we can conclude that a subprojective space admitting a concurrent vector field satisfies the next three conditions

(7.6)
$$\text{(A)} \quad R^\lambda_{\cdot\mu\nu\omega} = T^\lambda_{\cdot\omega} g_{\mu\nu} - T^\lambda_{\cdot\nu} g_{\mu\omega} + T_{\mu\nu}\delta^\lambda_\omega - T_{\mu\omega}\delta^\lambda_\nu,$$
$$\text{(A}'\text{)} \quad T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0,$$
$$\text{(B)} \quad T_{\lambda\mu} = \rho(g_{\lambda\mu} - 2\eta_\lambda \eta_\mu),$$

where $\eta^\lambda \eta_\lambda = 1$, $\eta_\lambda = \frac{\partial \eta}{\partial x^\lambda}$ and $\rho\eta^2 = \text{const.} \neq 0$.

Conversely, if (A′) and (B) hold, we have

$$T_{\lambda\mu;\nu} = \rho_\nu(g_{\lambda\mu} - 2\eta_\lambda\eta_\mu) - 2\rho(\eta_{\lambda;\nu}\eta_\mu + \eta_\lambda\eta_{\mu;\nu}).$$

Consequently we have

$$T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = (\rho_\nu g_{\lambda\mu} - \rho_\mu g_{\lambda\nu}) - 2\rho(\eta_{\lambda;\nu}\eta_\mu - \eta_{\lambda;\mu}\eta_\nu)$$
$$= 0.$$

Since we have $\eta^\lambda \eta_{\lambda;\mu} = 0$ from $\eta^\lambda \eta_\lambda = 1$, multiplying by η^μ and contracting for μ, we obtain

$$\rho_\nu \eta_\lambda - \eta^\mu \rho_\mu g_{\lambda\nu} - 2\rho\eta_{\lambda;\nu} = 0,$$

from which follows

$$\eta_{\lambda;\nu} = -\frac{\eta^\mu \rho_\mu}{2\rho} g_{\lambda\nu} + \frac{\eta_\lambda \rho_\nu}{2\rho}.$$

Substituting (7.5), we have

$$\eta_{\lambda;\nu} = \frac{1}{\eta}(g_{\lambda\nu} - \eta_\lambda \eta_\nu)$$

and, in consequence of

$$\frac{1}{\eta}\frac{-\eta_\lambda}{\eta} - \frac{\partial}{\partial x^\lambda}\frac{1}{\eta} = 0,$$

we find that if (7.6) (A′) and (B) are satisfied, then η_λ is a concurrent vector field.

If η_λ is a parallel vector field and $\eta^\lambda \eta_\lambda = 1$, we can easily obtain $T_{\lambda\mu} = c(g_{\lambda\mu} - 2\eta_\lambda \eta_\mu)$, where $c = \text{const.}$.

Thus from (3.5), (6.12), (7.6) and the above result, we find the

THEOREM. *A subprojective Riemannian space is characterized as follows:*
(I) *The space is conformally flat.*
(II) *If* $\eta^\lambda \eta_\lambda = 1$ *and* $\eta_\lambda = \dfrac{\partial \eta}{\partial x^\lambda}$,
(1) *when* η^λ *is a concircular vector field,*
$T_{\lambda\mu} = \rho(\eta)(g_{\lambda\mu} - 2\eta_\lambda \eta_\mu) + p\eta_\lambda \eta_\mu,$
(2) *when* η_λ *is a concurrent vector field,*
$T_{\lambda\mu} = \rho(g_{\lambda\mu} - 2\eta_\lambda \eta_\mu), \quad \rho\eta^2 = const. \neq 0,$
(3) *when* η^λ *is a parallel vector field,*
$T_{\lambda\mu} = c(g_{\lambda\mu} - 2\eta_\lambda \eta_\mu), \quad c = const. \neq 0,$

where

$$T_{\lambda\mu} = \frac{1}{n-2}\left(R_{\lambda\mu} - \frac{R}{2(n-1)} g_{\lambda\mu}\right).$$

Finally we shall note on the fundamental quadratic differential form of the subprojective space admitting a concurrent vector field. According to the previous paper [2], it takes the next form, for a suitable coordinate system,

$$ds^2 = \frac{(x^n)^2}{k} \frac{(dx^1)^2 + (dx^2)^2 + \dots + (dx^{n-1})^2}{\left\{\frac{1}{4}\sum_{i=1}^{n-1}(x^i)^2 \pm 1\right\}^2} + (dx^n)^2 \quad (k > 0),$$

from which follows

$$K \equiv \frac{R}{(n-1)(n-2)} = \frac{\pm k}{(x^n)^2},$$

where Riemann curvatures $\bar{R}$ of the hypersurfaces $x^n = \text{const.}$ are positive or negative according as the sign '$\pm$' takes '+' or '−'. Consequently we have

$$R^i_{\cdot jkl} = \bar{R}^i_{\cdot jkl} - \frac{1}{(x^n)^2}(g_{jk}\delta^i_l - g_{jl}\delta^i_k)$$

$$= \frac{\pm k - 1}{(x^n)^2}(g_{jk}\delta^i_l - g_{jl}\delta^i_k)$$

and the other components are zero. It follows

$$R = (n-1)(n-2)\frac{\pm k - 1}{(x^n)^2}.$$

Hence when $K = \frac{1}{(x^n)^2}$, that is,

$$ds^2 = (x^n)^2 \frac{(dx^1)^2 + (dx^2)^2 + \dots + (dx^{n-1})^2}{\left\{\frac{1}{4}\sum_{i=1}^{n-1}(x^i)^2 + 1\right\}^2} + (dx^n)^2,$$

the space is flat.

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