ON THE MAXIMAL HILBERT ALGEBRAS

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H. Nakano [4][1] has extended the results of W. Ambrose concerning his "proper H-algebras" (see [1]), by introducing the notion of "Hilbert algebras". In his paper, he showed, among others, that, to each Hilbert algebra, there exists a distinguished extension of it (*maximal extension*) which cannot be extended properly in any way ([4; Theorem 2.2]). After he told to the author this result, W. Ambrose's second paper [2] concering "H-systems" has appeared. Considering the inner relations of these notions, the author was able to show that every Hilbert algebra can be extended uniquely to a maximal one, and the considerations of maximal Hilbert algebras and H-systems are the same thing, *i. e.* the "bounded algebra" of an H-system is no other than our maximal Hilbert algebra. The structure of this algebra was also determined completely in some extent (*i. e.*, except that we have to introduce the separability assumption at a certain point) by the use of the F. J. Murray and J. von Neumann's theory on rings of operators.

In this paper we shall concern with the existence and unicity of a given Hilbert algebra and also some principal properties of the maximal Hilbert algebras deduced from it. As to the structure, we shall only give the results, as the proof is considerably long though the method is not so new. The fundamentals for the proof will be mentioned. The notions and notations in [4; §1] will be used freely.

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1. The maximal extension of a Hilbert algebra.

Let a Hilbert algebra $\mathfrak{A}$ in $\mathfrak{H}$ be given.

DEFINITION 1. A Hilbert algebra $\widetilde{\mathfrak{A}}$ in $\mathfrak{H}$ is called the *extension* of $\mathfrak{A}$ if $\mathfrak{A}$ is contained in $\widetilde{\mathfrak{A}}$, and the multiplication and the adjoint operation in $\mathfrak{A}$ are preserved in $\widetilde{\mathfrak{A}}$. If there exists no proper extension of $\mathfrak{A}$ we call the Hilbert algebra $\mathfrak{A}$ *maximal.* If $\widetilde{\mathfrak{A}}$ is a maximal Hilbert algebra and at the same time the extension of $\mathfrak{A}$ we call it the *maximal extension* of $\mathfrak{A}$.

We shall state the existence of the maximal extension of the given Hilbert algebra $\mathfrak{A}$ and deduce some fundamental properties of it.

For this purpose, let us consider first the nature of the extension $\widetilde{\mathfrak{A}}$ of an $\mathfrak{A}$. As $\mathfrak{A}$ is contained in $\widetilde{\mathfrak{A}}$, there correspond to an arbitrary $a \in \mathfrak{A}$ two

1) Numbers in brackets denote the number of literature at the end of the paper.

sorts of bounded linear operators as operators of right multiplication: the one is the operator S_a which assigns for every $x \in \mathfrak{A}$ the element xa and the other is the operator $\widetilde{S}_a$ which assigns for every $\widetilde{x} \in \widetilde{\mathfrak{A}}$ the element $\widetilde{xa}$. But as we assumed that the operations between element of $\mathfrak{A}$ are preserved in $\widetilde{\mathfrak{A}}$, the product xa, when considered in $\mathfrak{A}$, is the same as when considered in $\widetilde{\mathfrak{A}}$, that is, $S_a x = \widetilde{S}_a x$ for all $x \in \mathfrak{A}$. Now S_a and $\widetilde{S}_a$ are both bounded linear operators and $\mathfrak{A}$ is dense in $\mathfrak{H}$, S_a and $\widetilde{S}_a$ have the same extension over the whole $\mathfrak{H}$, since they coincide on a dense subset : $\widetilde{S}_a = S_a$ (on $\mathfrak{A}$). We can consider also for every $\widetilde{a} \in \widetilde{\mathfrak{A}}$ the bounded linear operator $\widetilde{T}_{\widetilde{a}} : \widetilde{T}_{\widetilde{a}} \widetilde{x} = \widetilde{a}\,\widetilde{x} (\widetilde{x} \in \widetilde{\mathfrak{A}})$, and by the above argument $\widetilde{T}_{\widetilde{a}} x = \widetilde{S}_x \widetilde{a} = S_x \widetilde{a}$ ($x \in \mathfrak{A}$). This shows that the linear operator which assigns for every $x \in \mathfrak{A}$ the element $S_x f$, where f is an arbitrary fixed element of $\mathfrak{H}$, is bounded for $f \in \widetilde{\mathfrak{A}}$ and just the same as $\widetilde{T}_f$. We shall proceed now keeping this fact as a key to the further considerations of the maximal extension of $\mathfrak{A}$.

DEFINITION 2. Let f be an arbitrary fixed element in $\mathfrak{H}$. We denote by T_f^0 the operator which makes the element $S_x f$ correspond to each $x \in \mathfrak{A}$. That is

$$T_f^0 : \quad T_f^0 x = S_x f \qquad (x \in \mathfrak{A}),$$

and in the same way we define the operator S_f^0 (with domain $\mathfrak{A}$)

$$S_f^0 : \quad S_f^0 x = T_x f \qquad (x \in \mathfrak{A}).$$

LEMMA 1: (i) *T_f^0, S_f^0 $(f \in H)$ are all linear operators with domain $\mathfrak{A}$.*

(ii) *If $T_f^0 x = T_g^0 x$ (or $S_f^0 x = S_g^0 x$) holds for every $x \in \mathfrak{A}$, then $f = g$.*

(iii) $T^0_{\alpha f + \beta g} = \alpha T_f^0 + \beta T_g^0$, $S^0_{\alpha f+\beta g} = \alpha S_f^0 + \beta S_g^0$ *(for arbitrary complex numbers α, β).*

We define next the adjoint element f^* of an $f \in \mathfrak{H}$. Let us consider an arbitrary sequence $\{x_\nu\}_{\nu=1,2,\cdots}$ which satisfies

$$(1) \qquad x_\nu \in \mathfrak{A} \quad (\nu = 1, 2, \cdots), \qquad \lim_{\nu\to\infty} x_\nu = f.$$

It follows from this that $\lim_{\nu,\mu\to\infty} \|x_\nu - x_\mu\| = 0$ and as we have $\|x_\nu - x_\mu\| = \|x_\nu^* - x_\mu^*\|$ by [3; theorem 1.7], we see that $\lim_{\nu,\mu\to\infty}\|x_\nu^* - x_\mu^*\| = 0$ too. Thus $\lim_{\nu\to\infty} x_\nu^*$ exists. Consider now another sequence $\{y_\nu\}_{\nu=1,2,\cdots}$, which also satisfies (1), then we have $\lim_{\nu\to\infty}(x_\nu - y_\nu) = 0$ and by the same argument as above we have $\lim_{\nu\to\infty}(x_\nu^* - y_\nu^*) = 0$, and *a fortiori* $\lim_{\nu\to\infty} x_\nu^* = \lim_{\nu\to\infty} y_\nu^*$. Therefore independently from the choice of the sequence $\{x_\nu\}_{\nu=1,2,\cdots}$ which satisfies (1), we have a definite element $\lim x_\nu^*$ depending only on f. This element we denote as f^* and call the *adjoint element of f*. Then we have

LEMMA 2: (i) *$f \in \mathfrak{A}$ implies $f^* \in \mathfrak{A}$, and this coincides with that which is defined in $\mathfrak{A}$ in advance.*

(ii) $f^{**} = f$.

(iii) $(\alpha f + \beta g)^* = \bar{\alpha} f^* + \bar{\beta} g^*$ *(for arbitrary complex numbers α, β).*

(iv) $\quad ||f|| = ||f^*||, \qquad (f,g) = (g^*,f^*).$

LEMMA 3: *Let $a \in \mathfrak{A}$, $f \in \mathfrak{H}$, then*

(i) $$(T_f^0 a)^* = (S_a f)^* = T_{a^*} f^* = S_{f^*}^0 a,$$
$$(S_f^0 a)^* = (T_a f)^* = S_{a^*} f^* = T_{f^*}^0 a^*,$$

(ii) $$T_{f^*}^0 \subset T_f^{0*}, \qquad S_{f^*}^0 \subset S_f^{0*}.$$ [2]

PROOF: (i) We have only to prove $(S_a f)^* = T_{a^*} f^*$. We construct a sequence $\{x_\nu\}_{\nu=1,2,\cdots}$ for f as in (1). Then $S_a x_\nu \in \mathfrak{A}(\nu = 1,2,\cdots)$, and $\lim_{\nu\to\infty} S_a x_\nu = S_a f$, thus

$$(S_a f)^* = \lim_{\nu\to\infty} (S_a x_\nu)^* = \lim_{\nu\to\infty} (x_\nu a)^* = \lim_{\nu\to\infty} a^* x_\nu^*$$
$$= \lim_{\nu\to\infty} T_{a^*} x_{\nu^*} = T_{a^*} f^*.$$

(ii) For arbitrary $x, y \in \mathfrak{A}$, we have

$$(T_{f^*}^0 x, y) = (S_x f^*, y) = (y^*, (S_x f^*)^*) = (y^*, T_x^* f) = (T_x y^*, f)$$
$$= (S_{y^*} x, f) = (x, S_y f) = (x, T_f^0 y).$$

Thus $T_{f^*}^0 \subset T_f^{0*}$.

LEMMA 4: *If one of the linear operators T_f^0, $T_{f^*}^0$, S_f^*, $S_{f^*}^0$, is bounded (on $\mathfrak{A}$), all others are bounded too.*

PROOF: It suffices to show that, if T_f^0 is bounded, then $T_{f^*}^0$, S_f^0 are also bounded.

Since $T_{f^*}^0 \subset T_f^{0*}$ by the preceding lemma and T_f^0 is bounded by the assumption, T_f^{0*}, hence $T_{f^*}^0$ is bounded.

To prove the case of S_f^0, take an arbitrary $x \in \mathfrak{A}$. Then, using Lemma 3,

$$||S_f^0 x|| = ||(T_{f^*}^0 x^*)^*|| = ||T_{f^*}^0 x^*|| \leqq |||T_{f^*}^0|||\,||x^*|| = |||T_{f^*}^0|||\,||x||$$ [3]

thus S_f^0 is bounded.

Put now

(2) $$\mathfrak{A}^0 = \{f; f \in \mathfrak{H},\ T_f^0 \text{ is bounded}\}.$$

Then

LEMMA 5: (i) *$\mathfrak{A}^0$ is a linear manifold in $\mathfrak{H}$ and contains $\mathfrak{A}$, therefore $\mathfrak{A}^0$ is dense in $\mathfrak{H}$ too.*

(ii) *If h belongs to $\mathfrak{A}^0$, then h^* also belongs to $\mathfrak{A}^0$.*

Now if h belongs to $\mathfrak{A}^0$, T_h^0 and S_h^0 are bounded linear operators. Thus these operators can be extended uniquely so as to have $\mathfrak{H}$ as their domains. We shall write them indifferently by

(3) $$T_h, \ S_h \qquad (h \in \mathfrak{A}^0)$$

resp., as there would occur no misunderstanding with the case of $h \in \mathfrak{A}$, in which the newly-defined T_h or S_h is identical with which is defined from the beginning (see the introductory considerations of this section). Fur-

2) For two operators A, B in $\mathfrak{H}$, the ralation $A \subset B$ means that B is a extension of A.

3) A being a bounded linear operator, $|||A|||$ denotes its bound, i.e. the least real number such that $||Af|| \leqq \alpha ||f||$ for arbitrary f in the domain of A.

thermore, we get from Lemmas 3 and 4 that

(4) $$T_{h^*} = T_h^*, \qquad S_{h^*} = S_h^* \qquad (h \in \mathfrak{A}^0).$$

LEMMA 6: *Let $h,\ k \in \mathfrak{A}^0$, then*

(i) $$T_h k, \qquad S_h k \in \mathfrak{A}^0,$$

(ii) $$T_{T_h k} = T_h T_k, \qquad S_{S_h k} = S_h S_k,$$

(iii) $$T_h k = S_k h.$$

PROOF: (i), (ii). Take a sequence $\{x_\nu\}_{\nu=1,2,\cdots}$ which satisfies (1) for k, then, since T_h is bounded, we have

$$T_h k = \lim_{\nu\to\infty} T_h x_\nu = \lim_{\nu\to\infty} S_{x_\nu} h,$$

and so, for arbitrary $x \in \mathfrak{A}$,

$$S_x T_h k = \lim_{\nu\to\infty} S_x S_{x_\nu} h = \lim_{\nu\to\infty} S_{x_\nu x} h = \lim_{\nu\to\infty} T_h(x_\nu x)$$
$$= T_h \lim_{\nu\to\infty} x_\nu x = T_h \lim_{\nu\to\infty} S_x x_\nu = T_h S_x k = T_h T_k x,$$

thus

$$\| S_x T_h k \| = \| T_h T_k x \| \leqq |||T_h|||\ |||T_k|||\, \| x \| \qquad (x \in \mathfrak{A}).$$

This shows that $T^0_{T_h k}$ is bounded, or, what is the same thing by the definition of $\mathfrak{A}^0$, $T_{T_h k} \in \mathfrak{A}^0$, and, at the same time, $T_{T_h k} = T_h T_k$.

(iii) for $x \in \mathfrak{A}$, we have

$$T_{T_h k} x = T_h T_k x = T_h S_x k = S_{S_x k} h = S_x S_k h = T_{S_k h} x,$$

by the use of the above obtained facts (i), (ii). Thus $T^0_{T_h k} = T^0_{S_k h}$, and, by Lemma 1 (ii), this is no other than $T_h k = S_k h$.

LEMMA 7: *$\mathfrak{A}^0$ may be considered as a Hilbert algebra defining the product hk ($h,\ k \in \mathfrak{A}^0$) as*

$$hk: \qquad hk = T_h k = S_k h \ (\in \mathfrak{A}^0)$$

and the adjoint element of h ($\in \mathfrak{A}^0$) as the element h^* which also belongs to $\mathfrak{A}^0$ by Lemma 5 (ii).

PROOF: We assert this by verifying the conditions of the definition (cf. [4; §1]).

(1) We have already noticed that $\mathfrak{A}^0$ is a dense linear manifold in $\mathfrak{H}$ (Lemma 5 (i)).

(2) To see that $\mathfrak{A}^0$ is an algebra over the complex number field, it only needs to show that the associativity of the above defined multiplication holds, others are obvious. But this also can be settled easily from

$$(hk)\, l = (T_h k)\, l = T_{T_h k} l = T_h T_k l = T_h(T_k l) = T_h(kl) = h(kl).$$

(3) By the formula (4), the required properties as the adjoint element are clearly satisfied.

(4) Because the product hk of $h,\ k \in \mathfrak{A}^0$ is defined as $T_h k$, the operator associated with h which assigns to every $k \in \mathfrak{A}^0$ the element hk coincides with T_h on this linear manifold $\mathfrak{A}^0$. But as T_h is bounded, the former is also bounded, whence it is clear, that, when we extend it continuously over the whole space $\mathfrak{H}$, it must coincide with T_h as they coincide on a dense subset $\mathfrak{A}^0$.

(5) It suffices to show that if, for an $f \in \mathfrak{H}$, $T_x f = 0$ for every $x \in \mathfrak{A}^0$, then $f = 0$. But this is clear, as $\mathfrak{A}$ is contained in $\mathfrak{A}^0$ and by the remark after (3), it follows that $T_x f = 0$ is valid for every $x \in \mathfrak{A}$, and thus from our first assumption we must have $f = 0$.

LEMMA 8: *$\mathfrak{A}^0$ is an extension of every Hilbert algebra $\widetilde{\mathfrak{A}}$ which is the extension of $\mathfrak{A}$. Thus, in particular, $\mathfrak{A}^0$ is an extension of $\mathfrak{A}$, which has no proper extension,* i.e. *a maximal Hilbert algebra.*

PROOF: Let $\widetilde{\mathfrak{A}}$ be an extension of $\mathfrak{A}$. Regarding to the linear operator of left multiplication $\widetilde{T}_{\tilde{a}}$ with domain $\widetilde{\mathfrak{A}}$ which assigns to every $\tilde{x} \in \widetilde{\mathfrak{A}}$ the element $\tilde{a}\tilde{x}$, we have, as considered at the beginning of this section,

$$\widetilde{T}_{\tilde{a}} x = S_x \tilde{a} = T^0_{\tilde{a}} x \qquad (\text{for arbitrary } x \in \mathfrak{A}).$$

Therefore $\widetilde{T}_{\tilde{a}} = T^0_{\tilde{a}}$ as linear operators defined on $\mathfrak{A}$. But by the assumption $\widetilde{T}_{\tilde{a}}$ was a bounded linear operator, $T^0_{\tilde{a}}$ must be bounded on $\mathfrak{A}$, therefore, by the definition of $\mathfrak{A}^0$ (cf. (2)), we have $\tilde{a} \in \mathfrak{A}^0$, which asserts thst $\widetilde{\mathfrak{A}} \subset \mathfrak{A}^0$ as sets. To prove that $\widetilde{\mathfrak{A}}$ is a subalgebra of $\mathfrak{A}^0$, it needs to show that the linear operation, the multiplication and the adjoint operation in $\widetilde{\mathfrak{A}}$ are preserved in $\mathfrak{A}^0$. As to the linear operation, this is clear. And once we have extended continuously the operator $\widetilde{T}_{\tilde{a}}$ over the whole space (for an arbitrary $\tilde{a} \in \widetilde{\mathfrak{A}}$), we have $T_{\tilde{a}} = \widetilde{T}^0_{\tilde{a}}$ (see (3)) as $\widetilde{T}_{\tilde{a}} = T^0_{\tilde{a}}$ (on $\mathfrak{A}$). Thus

$$\tilde{a}\tilde{b} \text{ (defined in } \widetilde{\mathfrak{A}}) = \widetilde{T}_{\tilde{a}}\tilde{b} = T^0_{\tilde{a}}\tilde{b} = \tilde{a}\tilde{b} \text{ (defined in } \mathfrak{A}^0).$$

Next, setting the adjoint element of $\tilde{a} \in \widetilde{\mathfrak{A}}$ considered in $\widetilde{\mathfrak{A}}$ as $a^{\sim *\sim}$, we have

$$T_{\tilde{a}^{*\sim}} = \widetilde{T}_{\tilde{a}^{*\sim}} = \widetilde{T}_{\tilde{a}}^{\,*} = T_{\tilde{a}}^{\,*} = T_{\tilde{a}^*}$$

and *a fortiori* $a^{\sim *\sim} = \tilde{a}^*$ by Lemma 1.

From what we have shown, it follows that $\mathfrak{A}^0$ is an extension of $\widetilde{\mathfrak{A}}$. If we take as $\widetilde{\mathfrak{A}}$ in particular $\mathfrak{A}$ itself, we are able to conclude that $\mathfrak{A}^0$ is an extension of $\mathfrak{A}$.

Last of all we have to show that $\mathfrak{A}^0$ is a maximal Hilbert algebra. If there exists any extension of $\mathfrak{A}^0$, then it is one of the extensions of $\mathfrak{A}$. But what we have shown above says that such an extension must have $\mathfrak{A}^0$ as its extension, thus from the inclusion relation as sets, we know that it coincides with $\mathfrak{A}^0$. Therefore $\mathfrak{A}^0$ has no proper extension, which means $\mathfrak{A}^0$ is maximal as Hilbert algebra in $\mathfrak{H}$. Thus our proof is completed.

Now we can state the following

THEOREM 1: *Corresponding to each Hilbert algebra $\mathfrak{A}$ in $\mathfrak{H}$, a Hilbert algebra $\mathfrak{A}^0$ in $\mathfrak{H}$ is determined in a unique way such that*

(i) *$\mathfrak{A}^0$ is a maximal Hilbert algebra.*

(ii) *Each extension of $\mathfrak{A}$ has $\mathfrak{A}^0$ as its own extension, and* a fortiori *is the extension of $\mathfrak{A}$.*

2. Maximal Hilbert algebras.

Most essential property of a maximal Hilbert algebra is expressed in the following

THEOREM 2. *A necessary and sufficient condition that a Hilbert algebra $\mathfrak{A}$ in $\mathfrak{H}$ should be maximal is that, if for an $f \in \mathfrak{H}$ the linear operator T_f^0 or S_f^0 defined in Definition 2 is bounded, then this f necessarily belongs to $\mathfrak{A}$.*

PROOF: *Necessity.* If $\mathfrak{A}$ is maximal, then the algebra $\mathfrak{A}^0$ in Theorem 1 must coincide with $\mathfrak{A}$. As $\mathfrak{A}^0$ is defined by (2) in §1, we see that the above mentioned condition is necessary.

Sufficiency. If $\mathfrak{A}$ were not maximal, then the algebra $\mathfrak{A}^0$ is certainly a proper extension of $\mathfrak{A}$: $\mathfrak{A}^0 \supset \mathfrak{A}$, $\mathfrak{A}^0 \neq \mathfrak{A}$. Thus there exists an element f not belonging to $\mathfrak{A}$ while T_f^0 or S_f^0 being bounded. This shows that our condition is sufficient.

COROLLARY: *A maximal Hilbert algebra is closed.* (Cf. [4; §2]).

PROOF: Let $\mathfrak{A}$ be maximal. We take from $\mathfrak{A}$ a sequence $\{a_\nu\}_{\nu=1,2,\cdots}$ satisfying

$$a_\nu \in \mathfrak{A} \ (\nu = 1, 2, \cdots), \qquad \lim_{\nu\to\infty} a_\nu = f \in \mathfrak{H}, \qquad |||\, T_{a_\nu} |||\leqq \gamma \ (\nu = 1, 2, \cdots).$$

Then, for an arbitrary $x \in \mathfrak{A}$,

$$\| T_f^0 x \| = \| S_x f \| = \| \lim_{\nu\to\infty} S_x a_\nu \| = \lim_{\nu\to\infty} \| S_x a_\nu \| = \lim_{\nu\to\infty} \| T_{a_\nu} x \| \leqq \gamma \| x \|,$$

and this shows that T_f^0 is bounded, and by the preceding theorem $f \in \mathfrak{A}$. This shows that $\mathfrak{A}$ is a closed Hilbert algebra.

We next consider the relation between our Hilbert algebras and W. Ambrose's H-systems. (See [2]. His definition will be freely used in the sequel). As he remarked, the most essential part of his definition is (5): $L_{x^*} = L_x^* = l_x^*$, $R_{x^*} = R_x^* = r_x^*$. By this the bounded algebra $\mathfrak{A}$ defined in (3) turns to be a maximal Hilbert algebra.

That $\mathfrak{A}$ is a Hilbert algebra is easily verified by examining each article of the definition. Especially, as $a \in \mathfrak{A}$ implies $a^* \in \mathfrak{A}$, each operator L_a has, along with its adjoint operator, the whole space $\mathfrak{H}$ as its domain, and so is bounded ([2; Lemma 2.2]). This permits us to accord his notations with ours. Thus, if $a \in \mathfrak{A}$, $L_a = T_a$, $R_a = S_a$, and if $f \in \mathfrak{H}$, $l_f = T_f^0$, $r_f = S_f^0$. And the adjoint element f^* introduced by his assumption (5) is, as he assumed $\|f\| = \|f^*\|$ in advance, only needed to be defined for the elements of $\mathfrak{A}$. Under these circumstances, we show that, if T_f^0 is bounded, $f \in \mathfrak{A}$. From (5), we obtain

$$L_{f^*} = T_f^{0*}, \quad L_f = T_{f^*}^{0\,*}, \quad R_f = S_{f^*}^{0\,*} \qquad \text{in turn.}$$

Now as $T_{f^*}^0$, $S_{f^*}^0$ are bounded as we have seen already (Lemma 4 in §1), L_f and R_f must have the whole space $\mathfrak{H}$ as their domains. Thus, for an arbitrary $x \in \mathfrak{H}$, the products fx and xf are both well defined, and by the definition of $\mathfrak{A}$, we must have $f \in \mathfrak{A}$. Therefore $\mathfrak{A}$ is a maximal Hilbert algebra.

We shall state this result as a theorem.

THEOREM 3: *The bounded algebra $\mathfrak{A}$ discussed in the argument of* W. Ambrose's H-*system is a maximal Hilbert algebra. Thus our arguments on Hilbert algebras run essentially in the same line as his.*

3. Structure of a maximal Hilbert algebra.

In what follows we always assume that $\mathfrak{A}$ denotes a maximal Hilbert algebra in $\mathfrak{H}$.

The operators T_a, S_a associated with each $a \in \mathfrak{A}$ are bounded linear operators in $\mathfrak{H}$, and we shall denote the sets of all these operators as $\mathbf{T}$, $\mathbf{S}$:

$$\mathbf{T} = \{T_a ; a \in \mathfrak{A}\}. \quad \mathbf{S} = \{S_a ; a \in \mathfrak{A}\}.$$

As, for all a, $b \in \mathfrak{A}$, T_a and S_a are mutually commutable along with their adjoint operators (which have the same forms), we see, by expressing the commutator algebra[4] of a set of operators $\mathbf{N}$ as $\mathbf{N}'$,

$$\mathbf{T}' \supset \mathbf{S}, \qquad \mathbf{S}' \supset \mathbf{T}.$$

Moreover, as we know, the smallest algebra of operators which comprises the identity operator and each $\mathbf{T}$ and $\mathbf{S}$, that will be designated by $\mathbf{R}(\mathbf{T},1)$ and $\mathbf{R}(\mathbf{S},1)$ resp., is $\mathbf{T}''$ and $\mathbf{S}''$[5]:

$$\mathbf{R}(\mathbf{T},1) = \mathbf{T}'', \qquad \mathbf{R}(\mathbf{S},1) = \mathbf{S}''.$$

But as we can show below, $\mathbf{T}'' = \mathbf{S}'$, $\mathbf{S}'' = \mathbf{T}'$ in reality, thus the above algebras are commutator algebras of one another.

THEOREM 4: *If $A \in \mathbf{T}'$ or $\mathbf{S}'$, then $A\mathfrak{A} = \{Aa ; a \in \mathfrak{A}\} \subset \mathfrak{A}$.*

PROOF: Let $A \in \mathbf{S}'$ and $a \in \mathfrak{A}$. Then, as $AS_x = S_x A$ for any $x \in \mathfrak{A}$, we see that

$$\| T^0_{Aa} x \| = \| S_x Aa \| = \| AS_x a \| \leqq ||| A ||| \, \| S_x a \| = ||| A ||| \, \| T_a x \| \leqq ||| A ||| \; ||| T_a ||| \, \| x \|$$

namely T^0_{Aa} is bounded on $\mathfrak{A}$. This proves that

THEOREM 5: $\quad \mathbf{S}' = \mathbf{T}'' = \mathbf{R}(\mathbf{T},1), \qquad \mathbf{T}' = \mathbf{S}'' = \mathbf{R}(\mathbf{S}, 1).$

PROOF: For us, the proof of $\mathbf{S}' \subset \mathbf{T}''$ suffices.

Take arbitrarily $A \in \mathbf{S}'$ and $B \in \mathbf{T}'$ then, for every x, $y \in \mathfrak{A}$,

$$\begin{aligned}(AB)(xy) &= A(B(xy)) = A(x(By)) = (Ax)(By) = B((Ax)y) \\ &= B(A(xy)) = (BA)(xy),\end{aligned}$$

and *a fortiori* $AB = BA$ on $\mathfrak{A}\mathfrak{A}$. But as $\mathfrak{A}\mathfrak{A}$ was complete in $\mathfrak{H}$ (Cf [3; Theorem 1.6]), this shows $AB = BA$. From this $\mathbf{S}' \subset \mathbf{T}''$ is clear.

DEFINITION 3: We put $\mathbf{S}'$ as $\mathbf{M}$. Then $\mathbf{M}$ is an algebra of operators, whose commutator algebra $\mathbf{M}'$ is equal to $\mathbf{T}'$. We call these $\mathbf{M}$, $\mathbf{M}'$ the *algebras of left and right multiplications* resp.

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4) *i.e.* the set of all those bounded linear operators with domain $\mathfrak{H}$ which commute with A and A^* for arbitrarily taken $A \in \mathbf{N}$. Cf. [5; II, Definition 3].

Here and in what follows, an algebra of operators means that it is not only the algebra in the algebraic mean but also it is closed with respect to the adjoint operation (of operators) and closed in the weak topology of operators, cf. [5; II].

5) Cf. [5; II, Satz 7].

$\mathbf{Z} = \mathbf{M} \cap \mathbf{M}'$ is a commutative algebra of operators, which we call the *centre* of $\mathfrak{A}$.

Thus we are led naturally to the idea of making use the theory of rings of operators as developed by F. J. Murray and J. von Neumann. Without exposing in its detail, we state here our final resuls on the structure of a Hilbert algebra.

For an arbitrary projection operator P belonging to $\mathbf{Z}$, $P\mathfrak{A}$ constitutes a maximal Hilbert algebra in $P\mathfrak{H}$, and their algebra of left (right) multiplication is $\mathbf{M}_{(P)}$ ($\mathbf{M}'_{(P)}$) obtained from $\mathbf{M}(\mathbf{M}')$ by contracting each operator of it to $P\mathfrak{H}$. (Cf. [5; § 11.3]). Thus the centre of $P\mathfrak{H}$ is $\mathbf{Z}_{(P)}$. If we name a projection operatior $P \neq 0$ in $\mathbf{Z}$, which has the property that the only projection operator $Q : Q \in \mathbf{Z},\ P \geqslant Q \neq 0$ is P, *minimal*, then, for such P, the centre of $P\mathfrak{A}$ consists of constant multiples of the identity operator $1_{(P)}$ in $P\mathfrak{A}$: $\mathbf{Z}_{(P)} = \{\alpha . 1_P\}$, and the algebras of left and right multiplication $\mathbf{M}_{(P)}$, $\mathbf{M}'_{(P)}$, form a couple of factors (Cf. [3; Chap. III]). Be the whole of these minimal projection operators $\{P_\lambda\}_{\lambda\in\Lambda}$, and let $P_0 = 1 - \bigcup_{\lambda\in\Lambda} P_\lambda$, then

(i) P_0, $P_\lambda(\lambda \in \varLambda)$ are orthogonal to each other (and thus, for any a, b chosen from distinct $P_0\mathfrak{A}$, $P_\lambda\mathfrak{A}$ $(\lambda \in \varLambda)$, $ab = 0$).

(ii) The centre of $P_0\mathfrak{A}$ does contain no minimal projection operator.

(iii) Each $P_\lambda\mathfrak{A}$ $(\lambda \in \varLambda)$ has the above simplicity condition.

The *ideal* in $\mathfrak{A}$ defined in [4; § 5] being no other than $P\mathfrak{A}$ for some projection operator P in $\mathbf{Z}$, we can classify our maximal Hilbert algebras in two completely different types:

Simple case: The centre of $\mathfrak{A}$ consists of only the constant multiples of the identity operator 1: $\mathbf{Z} = \{\alpha \cdot 1\}$.

Purely non-simple case: $\mathfrak{A}$ does contain no simple ideal, or, what is the same, centre of $\mathfrak{A}$ does not contain minimal projection operators.

And

THEOREM 5: *A maximal Hilbert algebra can be decomposed into the direct sum (in the algebraical sense and in the sense as Hilbert spaces) of simple Hilbert algebras and a purely non-simple Hilbert algebra.*

THEOREM 6: *Simple Hilbert algebras can be classified into two different types according to the nature of the norms of units* (Cf. [4; § 4]) *in it*:

(I) These values have a minimum. *This case is the very one which is characterized in* [1] *and also in* [4; § 5].

(II) The infimum of these values is equal to 0. *In this case the algebra of left (right) multiplication is a factor of* case (II) (*cf.* [3; Theorem VIII]), *and also here the most general case is constructed as* TOTAL MATRIX ALGEBRA *with elements in some simple Hilbert algebra containing an identity element. The study of algebras containing an identity element is the same with that of the factors of case* (II)

THEOREM 7: *A purely non-simple Hilbert algebra is obtained from a family of Hilbert algebras $\mathfrak{A}_t$ defined on a continuous measure space $\mathfrak{T}$* (i. e.

which contains no points of positive measure) *by an* INTEGRATION PROCESS (*or, according to* J. von Neuman's *terminology, as a* generalized direct sum, cf. [6]).

By the result of [6], we also see that:

THEOREM 8: *When* $\mathfrak{H}$ *is separable, we can take the measure space* $\mathfrak{T}$ *as the interval* $[0,1]$ *with ordinary Lebesgue measure, and in that case,* $\mathfrak{A}_t$ *can be made into simple Hilbert algebras.*

Of course, the meanings of the "total matrix algebra" in Theorem 6, and the "integration process" in Theorem 7 must be explained in detail. But I want to promise it in another place. [6)]

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