ON ASYMPTOTICALLY ABSOLUTE CONVERGENCE

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Let us consider the series

$$\sum_{n=1}^{\infty} a_n \tag{1}$$

of real numbers a_n. We shall say that the series (1) is *asymptotically absolutely convergent* if there exists an increasing sequence of positive integers $\{n_k\}$ such that $k/n_k \to 1$ as $k \to \infty$ and the subseries

$$\sum_{k=1}^{\infty} a_{n_k} \tag{2}$$

converges absolutely.

We shall establish, in this note, a theorem of Tauberian type and some results for trigonometrical series.

1. Tauberian theorem.

THEOREM 1. *Suppose that the series* (1) *is asymptotically absolutely convergent, and one of the following three conditions is satisfied*:

(i) $\{|a_n|\}$ *is a monotone sequence*;

(ii) $|a_{n+1}| < (1 + C/n)|a_n|$ $(n \geqq n_0)$, *where* C *and* n_0 *are positive constants independent of* n;

(iii) *for some* B *which is independent of* $N = 1, 2, \cdots$,

$$\sum_{n=1}^{N-1} n \big| |a_n| - |a_{n+1}| \big| + N|a_N| \leqq B \sum_{n=1}^{N} |a_n|. \tag{3}$$

Then the series (1) *converges absolutely.*

PROOF. If (i) is satisfied, then the absolute convergence of the series of type (2) implies the decreaseness of $|a_n|$; and (i) is included in (ii). On the other hand, (ii) implies the inequality (3). For, Supposing $n_0 = 1$,

$$\begin{aligned}
&\sum_{n=1}^{N-1} n \big| |a_n| - |a_{n+1}| \big| + N|a_N| \\
&\quad \leqq \sum_{n=1}^{N-1} n \frac{C}{n} |a_n| + \sum_{n=1}^{N} \Big(1 + \frac{C}{N-1}\Big)\Big(1 + \frac{C}{N-2}\Big) \cdots \Big(1 + \frac{C}{n}\Big) |a_n| \\
&\quad \leqq C \sum_{n=1}^{N-1} |a_n| + e^C \sum_{n=1}^{N} |a_n| \leqq (C + e^C) \sum_{n=1}^{N} |a_n|.
\end{aligned}$$

Hence it is sufficient to prove the absolute convergence of (1) under the condition (iii). Suppose that (2) converges absolutely and $k/n_k \to 1$ as $k \to \infty$. Let $\varepsilon_n = 1$ if $n = n_k$, $k = 1, 2, \cdots$, and $\varepsilon_k = 0$ otherwise. Then, as we see easily,

(4) $$\sum_{n=1}^{\infty} \varepsilon_n |a_n| < \infty \quad \text{and} \quad s_n \equiv \frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \to 1 \quad \text{as} \quad n \to \infty.$$

Now, by the Abel transformation

$$\sum_{n=1}^{N} \varepsilon_n |a_n| = \sum_{n=1}^{N-1} n(|a_n| - |a_{n+1}|) s_n - N s_N |a_N|,$$

and if the series (1) does not converge absolutely, we see by (3) that the Toeplitz condition is satisfied for the transform of $\{s_n\}$:

$$\Big(\sum_{n=}^{N} \varepsilon_n |a_n|\Big) \Big/ \Big(\sum_{n=1}^{N} |a_n|\Big);$$

hence from the second relation of (4) we must have $\sum_{n=1}^{\infty} \varepsilon_n |a_n| = \infty$, which contradicts our assumption.

2. Asymptotically absolute convergence of trigonometrical series.

THEOREM 2. *If one of the series*

$$\sum_{n=1}^{\infty} a_n \sin nx, \qquad \sum_{n=1}^{\infty} a_n \cos nx$$

converges absolutely at a point incommensurable with π, *then the series* (1) *is asymptotically absolutely convergent.*

PROOF. Let us consider only the sine series (the consine case may be treated similarly), and suppose that

(5) $$\sum_{n=1}^{\infty} |a_n \sin n\pi x_0| \equiv M < \infty \qquad (x_0 \text{ irrational}).$$

Let $\{\delta_m\}$ be a positive decreasing null sequence. For every integer i, let

(6) $$n_1^{(i)},\ n_2^{(i)},\ n_3^{(i)},\ \cdots\cdot$$

be the n's for which $|\sin n\pi x_0| > \delta_i$, then by the uniform distribution of $\{nx_0\}$ we have

(7) $$\lim_{k\to\infty} k/n_k^{(i)} = 1 - \frac{\arc \sin \delta_i}{\pi}.$$

We put $\varepsilon_i = (\arc \sin \delta_i)/\pi$, then $\varepsilon_i \to 0$ as $i \to \infty$. By (5) and the definition of (6) $(i = 1)$ there exists an integer N_1 such that

$$\sum_{j=N_1}^{\infty} |a_{n_j^{(1)}}| < M/1^2,$$

and by (7) there is an integer $M_1 > N_1$ such that

$$\frac{(n_{N_1}^{(1)} - 1) + (N - N_1 + 1)}{n_N^{(1)}} > 1 - 2\varepsilon_1 \qquad \text{for all } N > M_1.$$

Next, by the similar reason there exist two positive integers N_2 and $M_2 > N_2$ such that

$$\sum_{j=N_2}^{\infty} |a_{n_j^{(2)}}| < M/2^2,$$

$$\frac{(n_{N_1}^{(1)}-1)+(L_1-N_1)+(N-N_2+1)}{n_N^{(2)}}>1-2\varepsilon_2 \quad \text{for all } N>M_2,$$

where L_1 is the maximum of N for which $n_{N_2}^{(2)}>n_N^{(1)}$ (hence $L_1 \geqq M_1$).

Proceeding in this way we obtain an increasing sequence of integers

(8) $\quad 1,2,\cdots\cdot, n_{N_1}^{(1)}-1;\ n_{N_1}^{(1)},\ n_{N_1+1}^{(1)},\ \cdots,\ n_{M_1}^{(1)},\ \cdots,\ n_{L_1}^{(1)};$
$n_{N_2}^{(2)},\cdots,\ n_{M_2}^{(2)},\ \cdots,\ n_{L_2}^{(2)};\ n_{N_3}^{(3)},\ \cdots\cdots\cdots\cdots,$

which we denote newly by $\{m_i\}$. For any integer i the following relations are fulfilled:

(9) $$\sum_{j=N_i}^{\infty}|a_{n_j^{(i)}}|<M/i^2,$$

(10) $$\frac{(n_{N_1}^{(1)}-1)+(L_1-N_1)+\cdots+(L_{i-1}-N_{i-1})+(N-N_i+1)}{n_N^{(i)}}>1-2\varepsilon_i$$

for all $N>M_i$.

For any integer k, if $m_k>n_{N_i}^{(i)}$, we see by (9) that

$$\sum_{j=k}^{\infty}|a_{m_j}| \leqq \frac{M}{i^2}+\frac{M}{(i+1)^2}+\cdots \leqq \frac{M}{i-1};$$

hence we have

(11) $$\sum_{j=1}^{\infty}|a_{m_j}|<\infty.$$

On the other hand, for any k, if $n_{M_i}^{(i)} \leqq m_k \leqq n_{L_i}^{(i)}$, then by (10) we have

$$k/m_k>1-2\varepsilon_i;$$

and if $n_{N_i}^{(i)} \leqq m_k \leqq n_{M_i}^{(i)}$, then putting $m_k=n_K^{(i)}$ we have

(12) $$\frac{k}{m_k}=\frac{(n_{N_1}^{(1)}-1)+(L_1-N_1)+\cdots+(L_{i-1}-N_{i-1})+(K-N_i-1)}{n_K^{(i)}}$$
$$\geqq \frac{(n_{N_1}^{(1)}-1)+(L_1-N_1)+\cdots+(L_{i-1}-N_{i-1})+(H-L_{i-1}+1)}{n_{H+1}^{(i-1)}}$$

where H is the integer such that $n_H^{(i-1)} \leqq n_K^{(i)}<n_{H+1}^{(i-1)}$.

By (10) the last-hand side of (12) is

$$=\frac{(n_{N_1}^{(1)}-1)+\cdots+(L_{i-2}-N_{i-2})+(H+1-N_{i-1})}{n_{H+1}^{(i-1)}}>1-2\varepsilon_{i-1}-\frac{1}{n_{H+1}^{(i-1)}},$$

from which we see immediately that

(13) $$\lim_{k\to\infty} k/m_k=1.$$

From (11) and (13) the theorem is proved.

COROLLARY 1. *If the series $\Sigma\rho_n\cos(nx+\alpha_n)$ $(\rho_n \geqq 0)$ converges absolutely at two points x_0, x_1 and if x_0-x_1 is incommensurable with π, then the series $\Sigma\rho_n$ is asymptotically absolutely convergent.*

From the assumption and Salem's theorem [2] we have

$$\sum \rho_n|\sin n(x_0-x_1)|<\infty;$$

and by Theorem 2 we get the required.

COROLLARY 2. *If the series* $\Sigma a_n \cos nx$ *or* $\Sigma a_n \sin nx$ *converges absolutely at a point incommensurable with* π, *and if* $a_n = O(1)$ *as* $n \to \infty$, *then we have* $(|a_1| + |a_2| + \cdots + |a_n|)/n \to 0$ *as* $n \to \infty$.

PROOF. By theorem 2, there is a sequence $\{m_k\}$ such that $\Sigma |a_{m_k}| < \infty$, and $m_k/k \to 1$ as $k \to \infty$. Let $\{n_k\}$ be its complementary sequence, then clearly $k/n_k \to 0$ as $k \to \infty$. Hence

$$\frac{1}{N}\sum_{k=1}^{N}|a_k| = \frac{1}{N}\left\{\sum_{m_k \leqq N}|a_{m_k}| + \sum_{n_k \leqq N}|a_{n_k}|\right\}$$

$$\leqq \frac{1}{N}O(1) + \frac{1}{N}O(1)\{\text{number of } n_k \text{ not greater than } N\}.$$

Let $n_j \leqq N < n_{j+1}$, then the last hand side is $\leqq O(1/N) + O(1)j/n_j = o(1)$.

COROLLARY 3. *If the series* $\Sigma a_n \cos nx$ *or* $\Sigma a_n \sin nx$ *is a Fourier series of a function of bounded variation, and if its derived series converges absolutely at a point incommensurable with* π, *then the function is continuous everywhere.*

PROOF. From the assumption, we have $a_n = O(1/n)$ or $na = O(1)$. Hence consider the derived series $\Sigma na_n \sin nx$ or $\Sigma na_n \cos nx$, and apply Corollary 2. We have $(|a_1| + 2|a_2| + \cdots + n|a_n|)/n \to 0$ as $n \to \infty$, and Wiener's theorem ([3], p. 221) yields the conclusion.

COROLLARY 4. *If* $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\rho_n \cos(nx + \alpha_n)$ *is a Fourier series of a function of bounded variation, and if its derived series converges absolutely at two points* x_0 *and* x_1 *where* $x_0 - x_1$ *is incommensurable with* π, *then the function is continuous everywhere.*

PROOF. An easy combination of the corollaries 1 and 3.

REMARKS (i). In Theorem 2, $\sin nx$ or $\cos nx$ may be replaced by any function $f(nx)$, where $f(x) \not\equiv 0$ is of period π and integrable in the Riemann sense. In fact, the set $(x; |f(x)| > \delta)$ being Jordan measurable for any $\delta > 0$, the sequence $\{n_i\}$ of n's for which $|f(nx_0)| > \delta$, has the property: i/n_i tends to the measure of the set as $i \to \infty$, in virtue of the uniform distribution of $\{nx_0\}$. And clearly the Jordan measure of the sets $(x; |f(x)| > \delta)$ tends to π as $\delta \to 0$. Hence the same argument as Theorem 2 leads us to the conclusion.

Again, to assert the above remark, it is enough to suppose that there exists a sequence $\{\delta_i\}$ such that $\delta_i \downarrow 0$ as $i \to \infty$, and the sets $(x; |f(x)| > \delta_i)$ are Jordan measurable. For an example we shall construct a function with this property but not integrable in the Riemann sense. Let E_1 be a non dense perfect set of Jordan measure $\frac{1}{3}$ in $(0,1)$; and in each of the contiguous intervals of E_1 we construct a non-dense perfect set of relative Jordan measure $\frac{1}{3^2}\Big/\left(1 - \frac{1}{3}\right)$, let the sum of them be E_2; and consider the contiguous intervals of E_2, and so on; we get the sequence of Jordan measurable set E_1, E_2, $\cdots$, of measure $\frac{1}{3}$, $\left(\frac{1}{3}\right)^2$, $\cdots$ respectively. Let $f(x) = 1/j$ if $x = E_j$ $(j = 1, 2, \cdots)$ and $f(x) = 0$ elsewhere. Then $f(x)$ is not Riemann integrable, for the set $E_1 \cup E_2 \cup \cdots$ is everywhere dense in $(0, 1)$ and of Lebesgue measure $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots = \frac{1}{2} < 1$; and for every $j > 0$ the set $E_j = (x; |f(x)| < 1/j)$ is Jordan measurable.

(ii) By the above remark and Theorem 1 we get easily a theorem of Szàsz ([1] Theorem 6):

If $f(x) \not\equiv 0$ is a Riemann integrable function of period 1, *if one of the conditions of Theorem* 1 *holds, and if* $\sum |a_n f(nx)| < \infty$ *for some irrational* x, *then* $\sum |a_n| < \infty$.

References

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