ON THE THEORY OF SERIES WITH FUNCTION TERMS

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Introduction. In the theory of series we find many theorems concerning the divergence nature of the series, for example the classical theorem of Abel-Dini, the theorem of Agnew on the rearrangement of terms [2] and the following:

THEOREM A. (J. D. HILL [3]) *If a series $\sum a_i$ is conditionally convergent, then for all $x \in (0, 1)$ except for a set of the first category, we have*

$$\liminf_{n \rightarrow \infty} \sum_{i=1}^n \beta_i a_i = -\infty, \quad \limsup_{n \rightarrow \infty} \sum_{i=1}^n \beta_i a_i = +\infty,$$

where β_i are the coefficients of dyadic expansion of x .

THEOREM B. (R. P. AGNEW [1]). *If a series $\sum a_i$ is not absolutely convergent, then there exists an increasing sequence of positive integers $\{n_i\}$ such that $n_i - n_{i-1} \rightarrow \infty$ as $i \rightarrow \infty$ and the series*

$$\sum_{i=1}^{\infty} a_{n_i}$$

diverges.

In this paper we shall aim somewhat further, and generalize some of these theorems to the series of functions discussing simultaneously the nature of the category as in Theorem A. We shall prove in the first two §§ some theorems on functionals and in §3 we shall treat the series of functions as application of §§1 and 2. The main pattern of the proof of Theorem 1 and 2 is due to S. Saks [4].

1. Let $\{S_n(t)\}$ be a sequence of measurable functions defined on a measurable set $I \subset (-\infty, \infty)$. The sequence $\{S_n(t)\}$ is called *bounded in measure* on I ¹⁾, if, for every $\mu > 0$ there exists a positive number $N = N(\mu)$ such that

$$\text{meas } E \left[t \in I; \sup_n |S_n(t)| > N \right] < \mu.$$

The sequence $\{S_n(t)\}$ is called to *have the property B*(ε) on I ¹⁾, if there is a positive number N such that

$$\text{meas } E \left[t \in I; \sup_n |S_n(t)| > N \right] < \varepsilon.$$

In this and the next §§ we assume that: (α) $\text{meas } I < \infty$; (β) E is a metric space of the second category which consists of sequences of real numbers, and if $x_0 = \{\beta_i\} \in E$ and $\{\gamma_i\} \in E$ then for any integer j , $x_j =$

1) This definition is different from Saks' [4].

$\{\beta_1, \dots, \beta_j, \gamma_{j+1}, \gamma_{j+2}, \dots\} \in E$ and $x_j \rightarrow x_0$ as $j \rightarrow \infty$. For $x = \{\beta_i\} \in E$ and $y = \{\gamma_i\} \in E$ if $\beta_i = \gamma_i$ for every $i > j$, then x and y are called *j-equal*.

Let $\{T_k\}$ ($k = 1, 2, \dots$) be a sequence of measurable sets in I such that the totality of the characteristic functions of T_k forms an everywhere dense set in the space of the measurable characteristic functions, namely, if $J \subset I$ is a measurable set, then for any $\delta > 0$ there exists an integer $k = k(\delta)$ such that

$$\text{meas}(J - JT_k) + \text{meas}(T_k - T_k J) < \delta.$$

THEOREM 1. Let $S_n(x, t)$ ($n = 1, 2, \dots$) be real valued functions defined for all $x \in E$ and $t \in I$, measurable in t and further satisfy the following properties:

- (a) $S_n(x, t)$ is continuous in x for any fixed n and t ;
- (b) If x and x' in E are *j-equal*, then for any $\delta > 0$ there correspond a set $J \subset I$ and positive number K depending on j and δ such that

$$|S_n(x, t) - S_n(x', t)| < K \quad (n = 1, 2, \dots; t \in J)$$

and

$$\text{meas } J > \text{meas } I - \delta.$$

Under these assumptions there exists a measurable set $A \subset I$ such that

- (i) $\{S_n(x, t)\}$ is bounded in measure on A for all $x \in E$;
- and (ii) for all $x \in E$ except perhaps for a set of the first category, $\{S_n(x, t)\}$ is not bounded in measure on any set in $I - A$ of positive measure.

PROOF. Let α_0 be the least upper bound of $\alpha > 0$ for which there exists a set $H(\alpha) \subset E$ of the second category such that $\{S_n(x, t)\}$ is bounded in measure on a set of measure $> \alpha$ for all $x \in H(\alpha)$. If there is no such α_0 , then the set $A = 0$ will be the required; hence we suppose that $\alpha_0 > 0$.

For any positive integer q , we shall show first that there exist a set $H_q \subset E$ of the second category and a set $P_q \subset I$ of measure $> \alpha_0 - 1/q^2$ such that $\{S_n(x, t)\}$ has the property $B(1/q^2)$ on P_q for all $x \in H_q$, that is, for a suitable constant $M_q > 0$,

$$(1.1) \quad \text{meas } E_t[t \in P_q; \sup_n |S_n(x, t)| \geq M_q] < \frac{1}{q^2} \quad (x \in H_q).$$

To $x \in H(\alpha_0 - 1/q^2)$ there correspond a set T_k ($k = k(x)$) and a positive integer $m = m(x)$ such that

$$(1.2) \quad \text{meas } E_t[t \in T_k; \sup_n |S_n(x, t)| \geq m] < 1/q^2, \quad \text{meas } T_k > \alpha_0 - 1/q^2.$$

Let $H_{q, k, m}$ denote the set of all $x \in H(\alpha_0 - 1/q^2)$ for which (1.2) holds for fixed k and m . Since $H(\alpha_0 - 1/q^2) = \sum_{k, m=1}^{\infty} H_{q, k, m}$ and which is of the second category there are a $k = k_q$ and an $m = m_q$ such that H_{q, k_q, m_q} is of the second category. Denoting $H_q = H_{q, k_q, m_q}$, $M_q = m_q$ and $P_q = T_{k_q}$ we see readily that (1.1) is proved.

Now, for any fixed positive integer m we have

$$(1.3) \quad \mathbb{E}_t [t \in P_q; \sup_n |S_n(x, t)| \geq M_q] \supset \mathbb{E}_t [t \in P_q; \max_{n < m} |S_n(x, t)| \geq M_q].$$

If for a fixed t the inequality

$$(1.4) \quad \max_{n < m} |S_n(x, t)| \geq M_q$$

holds for every $x \in H_q$, then (1.4) remains true for every $x \in H_q$ in virtue of the property (a). Hence by (1.1) and (1.3) we have

$$(1.5) \quad \text{meas } \mathbb{E}_t [t \in P_q; \max_{n < m} |S_n(x, t)| \geq M_q] < \frac{1}{q^2} \quad (x \in \bar{H}_q).$$

Making $m \rightarrow \infty$ in (1.5) we deduce that

$$(1.6) \quad \text{meas } \mathbb{E}_t [t \in P_q; \sup_n |S_n(x, t)| \geq M_q] \leq \frac{1}{q^2} \quad (x \in \bar{H}_q).$$

Since the set $\bar{H}_q$ is closed and of the second category, it contains a sphere C_q of center $x_0 = \{\gamma_i\}$ say. By the property of the distance in E , we may find a positive integer j such that the coincidence of the first j terms of x_0 and $x \in E$ implies $x \in C_q$. From the property (b) of $S_n(x, t)$ by putting $\delta = 1/q^2$ we see that if x' and $x'' \in E$ are j -equal, then there exist a set $J_q \subset I$ and a positive number $K_q > 0$ such that

$$(1.7) \quad |S_n(x'; t) - S_n(x'', t)| < K_q \quad (n = 1, 2, \dots; t \in J_q),$$

$$(1.8) \quad \text{meas } J_q > \text{meas } I - 1/q^2.$$

We shall prove next that

$$(1.9) \quad \text{meas } \mathbb{E}_t [t \in P_q; \sup_n |S_n(x, t)| \geq K_q + M_q] \leq \frac{2}{q^2} \quad (x \in E).$$

Let t be contained in the set of the left-hand side of (1.9) for an $x_1 = \{\beta_i\} \in E$. Since x_0 and $x_2 = \{\gamma_1, \dots, \gamma_j, \beta_{j+1}, \dots\}$ have the same first j terms we have $x_2 \in C_q \subset \bar{H}_q$. And if further $t \in P_q J_q$ we have by (1.7)

$$\begin{aligned} \sup |S_n(x_2, t)| &\geq \sup_n |S_n(x_1, t)| - \sup_n |S_n(x_1, t) - S_n(x_2, t)| \\ &\geq (K_q + M_q) - K_q = M_q \quad (n = 1, 2, \dots). \end{aligned}$$

Hence we deduce that

$$\begin{aligned} &\mathbb{E}_t [t \in P_q; \sup_n |S_n(x_1, t)| \geq K_q + M_q] \\ &\subset \mathbb{E}_t [t \in P_q J_q; \sup_n |S_n(x_1, t)| \geq K_q + M_q] + (P_q - J_q) \\ &\subset \mathbb{E}_t [t \in P_q; \sup_n |S_n(x_2, t)| \geq M_q] + (P_q - J_q). \end{aligned}$$

Each set of the last hand side being of measure $< 1/q^2$ by (1.5) and (1.8), we get (1.9).

Now let us put

$$(1.10) \quad A = \limsup_q P_q = \prod_{k=1}^{\infty} \sum_{q=k}^{\infty} P_q.$$

To any $\eta > 0$ there correspond two integers k_0 and s_0 such that

$$(1.11) \quad \sum_{q=k_0}^{\infty} \frac{2}{q^2} < \frac{\eta}{2}, \quad \text{meas} \left(\sum_{q=k_0}^{\infty} P_q - \sum_{q=k_0}^{s_0} P_q \right) < \frac{\eta}{2}.$$

By (1.9) there is a constant $L > 0$ such that

(1.12) $\text{meas } E_t [t \in P_q; \sup_n |S_n(x, t)| \geq L] < \frac{2}{q^2} \quad (x \in E),$
for $q = k_0, k_0 + 1, \dots, s_0$. Finally from (1.11) and (1.12) we have for a fixed $x \in E$,

$$\begin{aligned} & \text{meas } E_t [t \in A; \sup_n |S_n(x, t)| \geq L] \\ & \leq \text{meas } E_t \left[t \in \sum_{q=k_0}^{\infty} P_q; \sup_n |S_n(x, t)| \geq L \right] \\ & \leq \text{meas } E_t \left[t \in \sum_{q=k_0}^{s_0} P_q; \sup_n |S_n(x, t)| \geq L \right] + \text{meas} \left(\sum_{q=k_0}^{\infty} P_q - \sum_{q=k_0}^{s_0} P_q \right) \\ & \leq \sum_{q=k_0}^{s_0} \text{meas } E_t [t \in P_q; \sup_n |S_n(x, t)| \geq L] + \frac{\eta}{2} \\ & \leq \sum_{q=k_0}^{s_0} \frac{2}{q^2} + \frac{\eta}{2} < \eta, \end{aligned}$$

which proves (i).

To prove (ii), suppose, if possible, that there exists a set $H \subset E$ of the second category such that $\{S_n(x, t)\}$ is bounded in measure on a set $Q(x) \subset I - A$ of positive measure. Then from (i) the sequence $\{S_n(x, t)\}$ is bounded in measure on $A + Q(x)$ for all $x \in H$. Since $\text{meas}[A + Q(x)] > \text{meas } A \geq \alpha_0$ we have

$$H = \sum_{n=1}^{\infty} H_n, \quad H_n = E_x \left[x \in H; \text{meas}[A + Q(x)] \geq \alpha_0 + \frac{1}{n} \right].$$

And all the sets H_n being of the first category by the definition of α_0 , these equalities contain a contradiction, q. e. d.

2. The x -set E and t -set I will be assumed the same as in the previous §.

THEOREM 2. *In the hypothesis of Theorem 1 let us replace the property (b) by the following*

(c) *if $x \in E$ and $x' \in E$ are j -equal for some $j > 0$, then for almost all $t \in I$ the sequences $\{S_n(x, t)\}$ and $\{S_n(x', t)\}$ have the same oscillation.*

Then there exists a measurable set $B \subset I$ such that

(i) *$\{S_n(x, t)\}$ is convergent almost everywhere on B for all $x \in E$; and (ii) for all $x \in E$ except for a set of the first category, $\{S_n(x, t)\}$ is divergent almost everywhere on $I - B$.*

PROOF. For $x \in E$ let $\Gamma(x) \subset I$ denote the set of $t \in I$ for which $\{S_n(x, t)\}$ converges. Let δ_0 be the least upper bound of $\delta > 0$ such that the set $H(\delta) = E_x [x \in E; \text{meas } \Gamma(x) > \delta]$ is of the second category. We may suppose that $\delta_0 > 0$.

For any positive integer q and k , denoting by $H_{q,k}$ the set of all $x \in H(\delta_0 - 1/q^2)$ for which $\text{meas } T_k > \delta_0 - 1/q^2$ and $\text{meas}(T_k - T_k \Gamma(x)) < 1/q^2$, we have $H(\delta_0 - 1/q^2) = \sum_{k=1}^{\infty} H_{q,k}$. Then for some $k = k_q$ the set H_{q,k_q}

From (2.3) and (2.8) we have

$$(2.9) \quad \text{meas } E \left[t \in P_q; \sup_{m,n > N_q} |S_n(x_0, t) - S_m(x_0, t)| > 3\eta \right] < 3\eta + \frac{1}{q^2} \quad (x_0 \in \overline{G_q}),$$

where $N_t = N_q(\eta)$ and $G_q = G_q(\eta)$.

The set $\overline{G_q}$ being closed and of the second category, it contains a sphere C_q ; and then for any $x' \in E$ there is an $x^* \in C_q$ which is j -equal to x' for some j . By the property (c) of $S_n(x, t)$ and (2.9) we deduce that

$$\begin{aligned} & \text{meas } E \left[t \in P_q; \limsup_{m,n \rightarrow \infty} |S_n(x', t) - S_m(x', t)| > 3\eta \right] \\ &= \text{meas } E \left[t \in P_q; \limsup_{m,n \rightarrow \infty} |S_n(x^*, t) - S_m(x^*, t)| > 3\eta \right] < 3\eta + \frac{1}{q^2}. \end{aligned}$$

This relation means, η being arbitrary, that for every $x \in E$ the sequence $\{S_n(x, t)\}$ converges for all $t \in P_q$ except for a set of measure $\leq 1/q^2$; that is, (2.1) holds for all $x \in E$.

$B = \limsup_q \text{Let } P_q$, then as we see immediately, the sequence $\{S_n(x, t)\}$ converges almost everywhere on B for all $x \in E$, which proves (i).

As in the proof of (ii) of Theorem 1, we may prove (ii) of Theorem 2 by the inequalities $\text{meas } B \geq \limsup_{q \rightarrow \infty} \text{meas } P_q \geq \delta_0$, q. e. d.

3. For the application of our theorems to the theory of series, we shall define some metric spaces whose elements are sequences of real numbers. They are:

Ω_1 : space of all the sequences of positive numbers which tend to zero and are not increasing from some range on. The distance of the elements $x = \{\beta_i\}$ and $x' = \{\gamma_i\}$ is $\rho(x, x') = \sup_i |\beta_i - \gamma_i|$.

Ω_2 : space of all the sequences of real numbers which tend to zero. The distance of $x = \{\beta_i\}$ and $x' = \{\gamma_i\}$ is $\rho(x, x') = \sup_i |\beta_i - \gamma_i|$.

Ω_3 : space of all the sequences of positive integers $\{n_i\}$ such that $n_1 = 1$ and $\lim_{i \rightarrow \infty} (n_i - n_{i-1}) = +\infty$. The distance of $x = \{n_i\}$ and $x' = \{m_i\}$ is

$$\rho(x, x') = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{||n_{i+1} - n_i| - |m_{i+1} - m_i||}{1 + ||n_{i+1} - n_i| - |m_{i+1} - m_i||}.$$

Ω_4 : space of all the permutations of all the positive integers and a finite number of arbitrary positive integers. The distance of $x = \{n_i\}$ and $x' = \{m_i\}$ is

$$\rho(x, x') = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|n_i - m_i|}{1 + |n_i - m_i|}.$$

Ω_5 : space of all the sequences which consist of infinite number of 0's and infinite number of 1's. For any $x \in \Omega_5$, there corresponds a dyadic irrational in $(0, 1)$ whose coefficients of dyadic expansion is x . The distance is defined as a usual distance for the corresponding dyadic irrationals.

Ω_6 : space of all the sequences which consist of infinite number of +1's and that of -1's. Let $\{r_n(u)\}$ be the Rademacher system $(0 < u < 1)$, then, to any $x \in \Omega_6$ there corresponds a u , $0 < u < 1$ such that $x = \{r_i(u)\}$.

The distance is defined as an ordinary distance in $(0, 1)$ for the corresponding u 's.

The spaces thus defined are all of the second category, in fact Ω_1 , Ω_2 and Ω_3 are complete metric space; for Ω_4 this is proved by R. P. Agnew [2.] and for Ω_5 and Ω_6 this fact is obvious. If $x = \{\gamma_i\} \in \Omega_\lambda$ and $\{\beta_i\} \in \Omega_\lambda$ then for every j , $x_j = \{\gamma_1, \dots, \gamma_j, \beta_{j+1}, \dots\} \in \Omega_\lambda$ and $x_j \rightarrow x$ as $j \rightarrow \infty$ ($\lambda = 1, 2, \dots, 6$).

Let $\sum_{i=1}^{\infty} a_i(t)$ be a series whose terms are finite measurable functions defined in a measurable set I of measure > 0 and $< \infty$ we shall consider the following two cases:

$$(I) \quad \limsup_{n \rightarrow \infty} \left| \sum_{i=1}^n a_i(t) \right| = \infty \quad \text{for almost all } t \in I,$$

$$(II) \quad \text{the series } \sum_{i=1}^{\infty} a_i(t) \text{ diverges almost everywhere in } I.$$

Evidently the case (I) is a special case of (II).

In the case (I) we have the following results.

LEMMA 1. (See [5]). *There exists an $x = \{\lambda_i\} \in \Omega_1$ such that*

$$\limsup_{n \rightarrow \infty} \left| \sum_{i=1}^n \lambda_i a_i(t) \right| = \infty \quad \text{for almost all } t \in I.$$

LEMMA 2. (See [5]). *There exists an $x = \{n_i\} \in \Omega_3$ such that*

$$\limsup_{m \rightarrow \infty} \left| \sum_{i=1}^m a_{n_i}(t) \right| = \infty \quad \text{for almost all } t \in I.$$

LEMMA 3. *There exists a dyadic irrational $u = 0, \beta_1 \beta_2 \dots$ (dyadic expansion) such that*

$$\limsup_{n \rightarrow \infty} \left| \sum_{i=1}^n \beta_i a_i(t) \right| = \infty \quad \text{for almost all } t \in I.$$

LEMMA 4. *There exists a dyadic irrational u such that*

$$\limsup_{n \rightarrow \infty} \left| \sum_{i=1}^n r_i(u) a_i(t) \right| = \infty \quad \text{for almost all } t \in I.$$

The proof of Lemmas 3 and 4 may be done as in the paper [5].

In the case (II) we have the following two lemmas.

LEMMA 5. *There exists a dyadic irrational $u = 0, \beta_1 \beta_2 \dots$ (dyadic expansion) such that $\sum \beta_i a_i(t)$ diverges almost everywhere in I .*

LEMMA 6. *There exists a dyadic irrational u such that $\sum r_i(u) a_i(t)$ diverges almost everywhere in I .*

PROOF OF LEMMA 5. for a positive integer p , let I_p denote the set of all $t \in I$ for which

$$(3.1) \quad \limsup_{m, n \rightarrow \infty} \left| \sum_{i=n}^m a_i(t) \right| \geq \frac{1}{p}.$$

We have evidently

$$(3.2) \quad I = \sum_{p=1}^{\infty} I_p.$$

We can find a positive integer n_1 and a set $I_1^{(1)}$ such that

$$(3.3) \quad I_1^{(1)} = E_t \left[t \in I_1; \sup_{n_1 \leq i \leq j \leq 1} |a_j(t) + \dots + a_i(t)| \geq 1, \right. \\ \left. \text{meas } I_1^{(1)} > \text{meas } I_1 - 1/2; \right.$$

and then we can find an $n_2 > n_1 + 1$ and an $I_2^{(1)}$ such that

$$(3.4) \quad I_2^{(1)} = E_t \left[t \in I_2; \sup_{n_2 \leq i \leq j > n_1+1} |a_j(t) + \dots + a_i(t)| \geq \frac{1}{2} \right], \\ \text{meas } I_2^{(1)} > \text{meas } I_2 - 1/2;$$

next we can find $n_3 > n_2 + 1$ and $I_1^{(2)}$ such that

$$(3.5) \quad I_1^{(2)} = E_t \left[t \in I_1; \sup_{n_3 \leq i \leq j > n_2+1} |a_j(t) + \dots + a_i(t)| \geq 1, \right. \\ \left. \text{meas } I_1^{(2)} > \text{meas } I_1 - 1/2^2. \right.$$

Similarly we shall define $I_p^{(v)}$ ($p, v = 1, 2, \dots$) and n_s ($s = 1, 2, \dots$) considering $I_1; I_2, I_1; I_3, I_2, \dots$ successively. That is, in the $s \equiv \frac{1}{2} [v^2 + (2p-1)v + (p-1)(p-2)]$ -th step we treat the set I_p for every v ; if n_{s-1} was defined we may find a positive integer n_s and a set $I_p^{(v)}$ such that

$$(3.6) \quad I_p^{(v)} = E_t \left[t \in I_p; \sup_{n_s \leq i \leq j > n_{s-1}+1} |a_j(t) + \dots + a_i(t)| \geq \frac{1}{p} \right], \\ \text{meas } I_p^{(v)} > \text{meas } I_p - 1/2^v.$$

The double sequence $\{I_p^{(v)}\}$ and the sequence $\{n_s\}$ were thus defined by induction.

Let now $\beta_i = 0$ if i is of the form $n_s + 1$ and let $\beta_i = 1$ otherwise. Then the number $u = 0, \beta_1 \beta_2 \dots$ is evidently dyadically irrational.

We shall show that the series

$$(3.7) \quad \sum_{i=1}^{\infty} \beta_i a_i(t)$$

diverges almost everywhere in I . Let $I_p^* = \liminf_v I_p^{(v)}$ and $I^* = \sum_{p=1}^{\infty} I_p^*$, then by (3.6) and (3.1) $\text{meas } I_n^* = \text{meas } I_n$ and $\text{meas } I = \text{meas } I^*$.

If $t \in I^*$ we have $x \in I_p^*$ for a certain p , and $x \in I_p^{(v)}$ for all sufficiently large v ; hence by the definition of $I_p^{(v)}$ the series (3.7) has the oscillation $\geq 1/p$, that is (3.7) diverges in I^* , q. e. d.

The proof of Lemma 6 may be done similarly: suppose that $\{n_s\}$ is defined as above. let $\varepsilon_i = -1$ if i is of the form $n_s + 1$ and $\varepsilon_i = +1$ otherwise. Let u be such that $\{r_i(u)\} = \{\varepsilon_i\}$, then u is dyadically irrational. The series $\sum \varepsilon_i a_i(t)$ diverges almost everywhere in I

Let us define $\{S_n(x, t)\}$.

For $x = \{\lambda_i\} \in \Omega_1$ or Ω_2 , put

$$(3.8) \quad S_n(x, t) = \sum_{i=1}^n \lambda_i a_i(t).$$

For $x = \{n_i\} \in \Omega_3$, put

$$(3.9) \quad S_n(x, t) = \sum_{n_i < n} a_{n_i}(t).$$

For $x = \{n_i\} \in \Omega_4$, put

$$(3.10) \quad S_n(x, t) = \sum_{i=1}^n a_{n_i}(t).$$

For $x = \{\beta_i\} \in \Omega_5$ or Ω_6 , put

$$(3.11) \quad S_n(x, t) = \sum_{i=1}^n \beta_i a_i(t).$$

The sequence $\{S_n(x, t)\}$ thus defined satisfies the condition (a) and (c) of Theorems 1 and 2 taking $E = \Omega_\mu$ ($\mu = 1, 2, \dots, 6$). If x and $x' \in \Omega_\mu$ ($\mu \neq 4$) are j -equal we see

$$(3.12) \quad |S_n(x, t) - S_n(x'; t)| \leq K_1 \sum_{i=1}^j |a_i(t)| \quad (K_1: \text{constant}),$$

and the condition (b) is easily verified. If $|a_i(t)| < K_2$ for all $i = 1, 2, \dots$ and $t \in I$, then (b) is also satisfied for Ω_4 .

Now applying Theorem 1 we obtain the set $A \subset I$, and which cannot be of measure positive in the case (I). In fact, if $\text{meas } A \geq \alpha_0 > 0$, then by (i) of Theorem 1 there is an $L > 0$ such that

$$(3.13) \quad \text{meas } E \left[t \in A; \sup_n |S_n(x, t)| \geq L \right] < \frac{\alpha_0}{2} \quad \text{for all } x \in \Omega_\mu.$$

On the other hand by Lemmas 1, 2, 3, or 4 there is an $x_0 \in \Omega_\mu$ such that

$$(3.14) \quad \limsup_{n \rightarrow \infty} |S_n(x_0, t)| = \infty \quad \text{for almost all } t \in I.$$

Evidently (3.13) and (3.14) make a contradiction, hence we have $\text{meas } A = 0$.

Similarly applying Theorem 2 for $E = \Omega_4, \Omega_5$ and Ω_6 , we obtain $B \subset I$, and which is of measure zero in the case (II).

When $\text{meas } I = \infty$, let $I = \sum_{j=1}^{\infty} I_j$, $\text{meas } I_j < \infty$ and discussing as above on each I_j we obtain the following theorem.

THEOREM 3. Let $\sum_{i=1}^{\infty} a_i(t)$ be a series whose terms are finite measurable functions defined in a measurable set I of measure > 0 .

Then in the case (I), for all $x \in \Omega_\mu$ ($\mu = 1, 2, \dots, 6$; for Ω_4 suppose that $|a_i(t)| < K_2$ for all $i = 1, 2, \dots$, and $t \in I$) except for a set of the first category, the modified series with partial sum $S_n(x, t)$ is divergent to $\pm \infty$ or oscillates infinitely for almost all $t \in I$.

And in the case (II), for all $x \in \Omega_\mu$ ($\mu = 4, 5, 6$) except for a set of the

first category, the modified series with partial sum $S_n(x, t)$ is divergent for almost all $t \in I$.

REMARK. Let T be a Toeplitz summation method. Then the analogue of Theorem 3 holds concerning the T -mean $\sigma_n(x, t)$ of the series instead of $S_n(x, t)$.

In the second part of Theorem 3 we cannot replace the condition (II) by the condition that the series is not absolutely convergent almost everywhere. Indeed, the series

$$\sum_{n=1}^{\infty} \frac{\cos nt}{n}$$

is not absolutely convergent for all $t \in (0, 2\pi)$, but the series

$$\sum_k \frac{\cos n_k t}{n_k} \quad (x = \{n_k\}),$$

$$\sum_n \beta_n \frac{\cos nt}{n} \quad (x = \{\beta_n\}),$$

$$\sum_n r_n(u) \frac{\cos nt}{n} \quad (x = \{r_n(u)\})$$

are all convergent almost everywhere in $(0, 2\pi)$ for every $x \in \Omega_\mu$ ($\mu = 4, 5, 6$ respectively).

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