NOTES ON FOURIER ANALYSIS (XXXIX)

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This paper contains some remarks on the theory of Fourier series and others. They will not contain a great deal that is important, but they may be of some interest. This paper consists of the following articles which may be read separately.

1. The integral of Marcinkiewicz and Zygmund.
2. The summability of the derived Fourier series by Riesz logarithmic means.
3. Maximal theorems on functions of H-class.
4. Applications of the method of Hardy-Littlewood concerning to the proof of a maximal theorem.
5. A one-side localization theorem.
6. A problem of Zalcwasser.
7. Distribution of signs of the terms of conditionally convergent series.

1. The integral of Marcinkiewicz and Zygmund. Let $\varphi(z)$ be analytic in the unit circle and $\varphi(0) = 0$, and let

$$g(\theta) \equiv g_2(\theta) = \left\{\int_0^1 (1-\rho)|\varphi'(\rho e^{i\theta})|^2 d\rho\right\}^{1/2}$$

and

$$g_r(\theta) = \left\{\int_0^1 (1-\rho)^{r-1} |\varphi'(\rho e^{i\theta})|^r d\rho\right\}^{1/r}$$

where $z = \rho e^{i\theta}$. Littlewood-Paley [12] proved the following theorem, whose simple proof was given by Zygmund [32].

THEOREM 1.1. (Littlewood-Paley)

$$(1.1)\quad A_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} g^r(\theta)\, d\theta\right\}^{1/r} \leqq B_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}, \quad (1 < r < \infty),$$

$$(1.2)\quad \left\{\int_0^{2\pi} g_q^q(\theta)\, d\theta\right\}^{1/q} \leqq C_q\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^q d\theta\right\}^{1/q}, \quad (2 \leqq q < \infty),$$

$$(1.3)\quad D_p\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^p d\theta\right\}^{1/p} \leqq \left\{\int_0^{2\pi} g_p^p(\theta) d\theta\right\}^{1/p}, \quad (1 < p \leqq 2),$$

where A_r, B_r, ... are constants depending only on r.

Generalizing the parts of (1. 2) and (1. 3), the following theorem was given by Marcinkiewicz-Zygmund [15] and the author [18].

THEOREM 1.2. (Marcinkiewicz-Zygmund) *If* $1 < r < \infty$, *then*

$$A_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} g^r(\theta)d\theta\right\}^{1/r} \leqq B_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}, \tag{1.4}$$

$$\left\{\int_0^{2\pi} g_q^r(\theta)d\theta\right\}^{1/r} \leqq C_{q,r}\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}, \quad (2 \leqq q < \infty), \tag{1.5}$$

$$D_{p,r}\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} g_p^r(\theta)d\theta\right\}^{1/r}, \quad (1 < p \leqq 2). \tag{1.6}$$

If we put $q = r$ and $p = r$ in (1. 5) and (1. 6) respectively, we get (1.2) and (1. 3). Since the above integral $g_r(\theta)$ depends on the interior values of the unit circle, Marcinkiewicz [14] defined another integral of simple characters depending only on the boundary values, that is,

$$\mu(\theta) \equiv \mu_2(\theta) = \left\{\int_0^{\pi} [F(\theta+t) + F(\theta-t) - 2F(\theta)]^2/t^3 dt\right\}^{1/2}$$

and

$$\mu_r(\theta) = \left\{\int_0^{\pi} |F(\theta+t) + F(\theta-t) - 2F(\theta)|^r/t^{r+1} dt\right\}^{1/r},$$

where

$$F(\theta) = c + \int_0^{\theta} f(u)\, du.$$

Then the following theorem has been proved.

THEOREM 1. 3. (Marcinkiewicz and Zygmund)

$$A_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi} \mu^r(\theta)\, d\theta\right\}^{1/r} \leqq B_r\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^r d\theta\right\}, \quad (1 < r < \infty), \tag{1.7}$$

$$\left\{\int_0^{2\pi} \mu_q^q(\theta)d\theta\right\}^{1/q} \leqq C_q\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^q d\theta\right\}^{1/q}, \quad (2 \leqq q < \infty), \tag{1.8}$$

$$D_p\left\{\int_0^{2\pi} |\varphi(e^{i\theta})|^p d\theta\right\}^{1/p} \leqq \left\{\int_0^{2\pi} \mu_p^p(\theta)d\theta\right\}^{1/p}, \quad (1 < p \leqq 2). \tag{1.9}$$

Marcinkiewicz [14] proved (1.8) and (1.9) and conjectured the validity of (1.7). The proof of (1.7) was given by Zygmund [31]. Since Theorem 1.3 is the type of Theorem 1. 1, we shall generalize it into the form of Theorem 1. 2.

THEOREM 1.4. *If* $1 < r < \infty$, *then*

(1.10) $$A_r\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi}\mu^r(\theta)d\theta\right\}^{1/r} \leqq B_r\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r},$$

(1.11) $$\left\{\int_0^{2\pi}\mu_q^r(\theta)d\theta\right\}^{1/r} \leqq C_{q,r}\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}, \quad (2 \leqq q < \infty),$$

(1.12) $$D_{p,r}\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r} \leqq \left\{\int_0^{2\pi}\mu_p^r(\theta)d\theta\right\}^{1/r}, \quad (1 < p \leqq 2).$$

PROOF. (1.10) is nothing else (1.7) due to Zygmund [31]. If we write

$$\sup_{0\leqq t\leqq 2\pi}|F(\theta+t)-F(\theta)|/t = \Phi(\theta),$$

then we have

$$\left\{\int_0^{2\pi}\mu_q^r(\theta)d\theta\right\}^{1/r} = \left[\int_0^{2\pi}d\theta\left\{\int_0^{\pi}\frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^q}{t^{q-1}}dt\right\}^{r/q}\right]^{1/r}$$

$$= \left[\int_0^{2\pi}d\theta\left\{\int_0^{\pi}\frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^2}{t^3}\,\frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^{q-2}}{t^q}dt\right\}^{r/q}\right]^{1/r}$$

$$\leqq \left[\int_0^{2\pi}d\theta\left\{\mu^2(\theta)(\Phi(\theta))^{q-2}\right\}^{r/q}\right]^{1/r} \leqq \left\{\int_0^{2\pi}(\Phi(\theta))^{(1-2/q)r}(\mu(\theta))^{2r/q}d\theta\right\}^{1/r}$$

$$\leqq \left[\left\{\int_0^{2\pi}\Phi^r(\theta)d\theta\right\}^{1/r}\right]^{1-2/q}\left[\left\{\int_0^{2\pi}\mu^r(\theta)d\theta\right\}^{1/r}\right]^{2/q}$$

$$\leqq C_{q,r}\left[\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}\right]^{1-2/q}\left[\left\{\int_0^{2\pi}\mu^r(\theta)d\theta\right\}^{1/r}\right]^{2/q}$$

$$\leqq C_{q,r}\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}$$

by the successive application of Hölder's inequality, the Hardy-Littlewood maximal theorem and (1.10). Thus we get (1.11).

The proof of (1.12) runs similarly. That is,

$$\left\{\int_0^{2\pi}\mu_2^r(\theta)d\theta\right\}^{1/r}$$

$$\leqq \left[\int_0^{2\pi}d\theta\left\{\int_0^{\pi}\frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^p}{t^{p+1}}\,\frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^{2-p}}{t^{2-p}}dt\right\}^{r/2}\right]^{1/r}$$

$$\leqq \left[\int_0^{2\pi}d\theta\left\{\int_0^{\pi}\frac{|F(\theta+t)+F(\theta-t)-2F(\theta)|^p}{t^{p+1}}A_{p,r}(\Phi(\theta))^{2-p}dt\right\}^{r/2}\right]^{1/r}$$

$$\leqq \left[\int_0^{2\pi}\left\{A_{p,r}(\Phi(\theta))^{2-p}\mu_p^p(\theta)\right\}^{r/2}d\theta\right]^{1/r}$$

$$\leqq B_{p,r}\left\{\int_0^{2\pi}(\Phi(\theta))^{(1-p/2)r}(\mu_p(\theta))^{pr/2}d\theta\right\}^{1/r}.$$

Since $0 < p/2 < 1$, we get applying Hölder's inequality,

$$\leqq C_{p,r}\left[\left\{\int_0^{2\pi}\Phi^r(\theta)\right\}^{1-p/2}\left\{\int_0^{2\pi}\mu_p^r(\theta)d\theta\right\}^{p/2}\right]^{1/r}$$

$$\leqq C_{p,r}\left[\left\{\int_0^{2\pi}\Phi^r(\theta)d\theta\right\}^{1/r}\right]^{1-p/2}\left[\left\{\int_0^{2\pi}\mu_p^r(\theta)d\theta\right\}^{1/r}\right]^{p/2}$$

$$\leqq D_{p,r}\left[\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}\right]^{1-p/2}\left[\left\{\int_0^{2\pi}\mu_p^r(\theta)d\theta\right\}^{1/r}\right]^{p/2}.$$

Thus we get

$$\left\{\int_0^{2\pi}\mu_2^r(\theta)d\theta\right\}^{1/r}\leqq D_{p,r}\left[\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}\right]^{1-p/2}\left[\left\{\int_0^{2\pi}\mu_p^r(\theta)d\theta\right\}^{1/r}\right]^{p/2}.$$

By (1.10), we have

$$\left\{\int_0^{2\pi}|\varphi(e^{i\theta})|^r d\theta\right\}^{1/r}\leqq D_{p,r}\left\{\int_0^{2\pi}\mu_p^r(\theta)d\theta\right\}^{1/r}$$

which is the required.

2. The summability of the derived Fourier series by Riesz logarithmic means. Let

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

$$f(x+t)-f(x-t)=\psi_x(t),\quad \Psi_x(t)=\psi_x(t)\,/\,(2\sin t/2)$$

and

$$n(b_n\cos nx-a_n\sin nx)\equiv nB_n(x),$$

then

$$\mathfrak{S}'[f]\equiv\sum_{n=1}^{\infty}nB_n(x). \tag{2.1}$$

In this article we shall investigate Riesz logarithmic means of the series (2.1). We begin with a preliminary lemma pue to Bosanquet [4].

LEMMA 2.1. (Bosanquet) *Let $\varphi(t)$ be integrable in the Cauchy sense at $t=0$ and $t\varphi(t)$ be Lebesgue integrable in $(0,\pi)$. If we put*

$$\alpha(\mu)=\frac{2}{\pi}\int_{\to 0}^{\pi}\varphi(t)\cos\mu t\,dt, \tag{2.2}$$

then the series

$$\frac{1}{2}\alpha(0)+\sum_{\mu=1}^{\infty}\alpha(\mu)$$

and

$$\sum_{\mu=0}^{\infty} \alpha(\mu + 1/2)$$

converge at the same time.

PROOF. Let

$$\sigma_n \equiv \frac{1}{2}\alpha(0) + \sum_{\mu=1}^{n} \alpha(\mu) = \frac{2}{\pi}\int_{\to 0}^{\pi} \varphi(t) \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{1}{2} t} dt,$$

and

$$\tau_n \equiv \sum_{\mu=0}^{n} \alpha(\mu + 1/2) = \frac{2}{\pi}\int_{\to 0}^{\pi} \varphi(t) \frac{\sin(n+1)t}{2 \sin \frac{1}{2} t} dt,$$

then we have

$$2\tau_{n-1} - \sigma_n - \sigma_{n-1} = \frac{2}{\pi}\int_{0}^{\pi} \varphi(t) \tan \frac{1}{2} t \cdot \sin nt \, dt = o(1)$$

by the Riemann-Lebesgue theorem, since $t\varphi(t) \in L(0, \pi)$. Similarly

$$2\sigma_n - \tau_n - \tau_{n-1} = o(1).$$

Thus we get the lemma.

THEOREM 2.1. *Let $\Psi_x(t)$ be integrable in the Cauchy sense at $t = 0$, and partial sums of $\mathfrak{S}[\Psi_x(t)]$ be $o(\log n)$, and let $\mathfrak{S}[\Psi_x(t)]$ be $(R, \log n, 1)$-summable, then $\mathfrak{S}'[f]$ is $(R, \log n, 1)$-summable at $t = x$.*

PROOF. Let

$$\Psi_x(t) = \psi_x(t)/(2 \sin t/2) \in C$$

and

$$\mathfrak{S}[\Psi_x(t)] = \frac{1}{2}\alpha(0) + \sum_{\mu=1}^{\infty} \alpha(\mu) \cos \mu t. \tag{2.3}$$

If we consider the series (2.3) when $t = 0$, this is equi-summable with

$$\sum_{\mu=1}^{n} \alpha(\mu + 1/2),$$

by the lemma 2.1. If we put

$$u_n = \alpha(n + 1/2),$$

then we have

$$\Delta u_{n-1} = \Delta\alpha(n - 1/2) = \frac{2}{\pi}\int_{\to 0}^{\pi} \Psi_x(t)\, \Delta \cos(n - 1/2)\, t \, dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \psi_x(t) \sin nt \, dt \equiv B_n,$$

that is,

$$n \,\Delta\, u_{n-1} = n(u_{n-1} - u_n) = nB_n \equiv v_n, \quad \text{say.} \tag{2.4}$$

If we consider the relations of Σu_n and Σv_n, we get the relations of $\mathfrak{S}[\Psi_x(t)]$ and $\mathfrak{S}'[\psi_x(t)]$. If we denote by U_n, V_n and $U_n^{(1)}$, $V_n^{(1)}$, the n-th partial sums and $(R, \log n, 1)$-sums of Σu_n and Σv_n, respectively, then by (2.4) we have

$$V_n = U_n - (n+1)\, u_{n+1},$$
$$\sum_{n=0}^{N} V_n/(n+1) = \sum_{n=0}^{N} U_n/(n+1) - \sum_{n=0}^{N} u_{n+1},$$
$$V_n^{(1)} = U_n^{(1)} - U_{n+1} + u_0$$

and

(2.5) $V_n^{(1)}/\log(n+1) = U_n^{(1)}/\log(n+1) - U_{n+1}/\log(n+1) + u_0/\log(n+1).$

Since $U_n = o(\log n)$ and $U_n^{(1)}/\log(n+1) \to s$, we have $U_n^{(1)}/\log(n+1) \to s$. Thus the theorem is proved.

COROLLARY 2.1.1. *If*

$$\int_0^t |\Psi_x(t) - 2s|\,dt = o(t),$$

then $\mathfrak{S}'[f]$ *is* $(R, \log n, 1)$-*summable to* s *at* x.

PROOF. Since

$$\int_0^t |\Psi_x(t) - 2s|\,dt = o(t),$$

we have $S_n = o(\log n)$ and $\mathfrak{S}[\Psi_x(t)]$ is $(C, 1)$-summable.

COROLLARY 2.1.2. *If*

$$\int_0^t |\Psi_x(t) - 2s|\,dt = O(t) \quad and \quad \int_0^t \{\Psi_x(t) - 2s\}\,dt = o(t),$$

then there is a function $f(x)$ *such that* $\mathfrak{S}'[f]$ *is not summable by* $(R, \log n, 1)$-*means.*

PROOF. If $U_n^{(1)}/\log(n+1) \to s$, then, since $V_n^{(1)}/\log(n+1) \to s$ under the hypothesis, we have $U_{n+1} = o(\log n)$ by (2.5).

But there is a function $\Psi_x(t)$ such as $U_{n+1} \neq o(\log n)$ under the conditions of Corollary (cf. Izumi [8]).

For the generalization of Theorem 2.1, we need a lemma due to Hardy and Riesz [7].

LEMMA 2.2. *If we put*

$$C_\lambda(\tau) = \sum_{\lambda_n < \tau} C_n,$$

$$C_\lambda^k(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^k c_n = k\int_0^\omega C_\lambda(\tau)(\omega - \tau)^{k-1} d\tau,$$

then

$$C_{\lambda}^{(k+m)}(\omega) = \frac{\Gamma(k+m+1)}{\Gamma(k+1)\Gamma(m)} \int_0^{\omega} C_{\lambda}^{k}(u)(\omega-u)^{m-1}\,du,$$

where $k>0, \quad m>0.$

THEOREM 2.2. *If* $\Psi_x(t) \in C$ (integrable in the Cauchy sense), $(R, \log n, k)$*-means of* $\mathfrak{S}[\Psi_x(t)]$ *are* $o(\log\omega)$ *and* $\mathfrak{S}[\Psi_x(t)]$ *is* $(R, \log n, k+1)$*-summable to s, then* $\mathfrak{S}'[f]$ *is*$(R, \log n, k+1)$*-summable to s at x.*

PROOF. Put $\lambda_n = \log n$ in the lemma 2. 2, then we write by $U^{(k)}(\omega)$ and $V^{(k)}(\omega)$ omitting λ the $(R, \log n, k)$-sums (see the notation of Theorem 2. 1).

$$V^{(1)}(\omega) = U^{(1)}(\omega) - U^*(\omega) + u_0,$$

where

$$U^*(\omega) = \sum_{\lambda n+1<\omega} u_n.$$

By Lemma 2. 2, we get

$$V^{(k+1)}(\omega) = U^{(k+1)}(\omega) - (k+1)U^{(k)}(\omega) + u_0,$$
$$V^{(k+1)}(\log u) = U^{(k+1)}(\log u) - (k+1)U^{(k)}\{\log(u+1)\} + u_0$$

and

$$\frac{V^{(k+1)}(\log u)}{(\log u)^{k+1}} = \frac{U^{(k+1)}(\log u)}{(\log u)^{k+1}} - (k+1)\frac{U^{(k)}\{\log(u+1)\}}{\{\log(u+1)\}^{k+1}}\,\frac{\{\log(u+1)\}^{k+1}}{\{\log u\}^{k+1}} + \frac{u_0}{(\log u)^{k+1}}.$$

If

$$U^{(k+1)}(\log u)/(\log u)^{k+1} \to s,$$

and

$$U^{(k)}(\log u)/(\log u)^{k} = o(\log u),$$

then we have

$$V^{(k+1)}(\log u)/(\log u)^{k+1} \to s,$$

which is the required.

COROLLARY 2. 2. 1. *If* $\Psi_x(t) \in C$ *and* $\mathfrak{S}[\Psi_x(t)]$ *is* $(R, \log n, k)$*-summable, then* $\mathfrak{S}'[f]$ *is* $(R, \log n, k+1)$*-summable* where $k \geqq 0$.

COROLLARY 2. 2. 2. *If* $\Psi_x(t) \in C$, *and* $\Psi_x(t) \to s$ $(R, \log n, k)$ *as* $s \to 0$, *then* $\mathfrak{S}'[f]$ *is* $(R, \log n, k+1+\varepsilon)$*-summable, where* $k>1$, $\varepsilon>0$. ($k \geqq 1$, $\varepsilon>0$, *provided that* $\Psi_x(t) \in L$ *instead of* C).

PROOF Since $\Psi_x(t) \to s$ $(R, \log n, k)$ as $s \to 0$ implies the $(R, \log n, k+\varepsilon)$-summability of $\mathfrak{S}[\Psi_x(t)]$, (cf. Wang [24]).

COROLLARY 2. 2. 3. *If* $\Psi_x(t) \in C$ *and*

$$\int_0^t \{\Psi_x(t) - 2s\}\,dt = o(t),$$

then $\mathfrak{S}'[f]$ *is* $(R, \log n, 2)$*-summable to s. More generally if* $\Psi_x(t)$ *is* (C, α)-

summable $(\alpha \geqq 1)$ *as* $t \to 0$, *then* $\mathfrak{S}'[f]$ *is* $(R, \log n, 1+\alpha)$*-summable.*

PROOF This is evident from the Zygmund theorem [27, p. 62].

COROLLARY 2.2.4. *If*

$$\int_0^t |\Psi_x(t) - 2s|\, dt = o\left(t \log \frac{1}{t}\right) \text{ and } \int_t^\pi \Psi_x(t) \frac{dt}{t} = o\left(\log \frac{1}{t}\right),$$

then $\mathfrak{S}'[f]$ *is* $(R, \log n, 2)$*-summable to* s *at* x.

PROOF This is immediate from Hardy's theorem [5] and Theorem 2.2.

Corollary 2.2.2. is proved by Matsuyama [16] under the more restricted condition $\Psi_x(t) \in L$ and the case $\alpha = 1$ of Corollary 2.2.3 is given by Wang [23]. But they prove these theorems by the direct calculation.

3. Maximal theorems on functions of H-class.

If

$$\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leqq M, \quad r \leqq 1,$$

then we call the function $f(z)$ belongs to H-class. Let

$$f(e^{i\theta}) \sim \sum_{k=0}^{\infty} c_k e^{ik\theta},$$

then we denote by $s_n(\theta)$ and $\sigma_n^{(r)}(\theta)$ the partial sums and the r-th Cesàro means of the above series

Then we have the following theorem.

THEOREM 3.1. *If* $f(z) \in H$, *then*

$$\int_0^{2\pi} \max_n |\sigma_n^{(\varepsilon)}(\theta)|\, d\theta \leqq A \int_0^{2\pi} |f(e^{i\theta})|\, d\theta, \quad (\varepsilon > 0), \tag{3.1}$$

$$\int_0^{2\pi} \max_n |s_n(\theta)/\log(n+2)|\, d\theta \leqq B \int_0^{2\pi} |f(e^{i\theta})|\, d\theta, \tag{3.2}$$

$$\int_0^{2\pi} \max_n |\sigma_n^{(-\varepsilon)}(\theta)/n^\varepsilon|\, d\theta \leqq C \int_0^{2\pi} |f(e^{i\theta})|\, d\theta, \quad (1 > \varepsilon > 0). \tag{3.3}$$

PROOF. (3.2) has been proved by Zygmund [30]. Without loss of generality we can assume

$$f(z) = g^2(z), \qquad g(z) \in H^2.$$

Then we have

$$f(z)/(1-z)^{k+1} = \sum_{n=0}^{\infty} S_n^{(k)} z^n,$$

where $S_n^{(k)}$ denotes (C, k)-sum of the Taylor series of $f(z)$, and

$$f(z)/(1-z)^{k+1} = \{g(z)/(1-z)^{(k+1)/2}\}^2.$$

Consequently

$$|S_n^{(\varepsilon)}(f)| = \left|\sum_{\nu=0}^{n} S_{n-\nu}^{((-1+\varepsilon)/2)}(g)\, S_\nu^{((-1+\varepsilon)/2)}(g)\right|$$
$$\leqq \sum_{\nu=0}^{n} |S_\nu^{((-1+\varepsilon)/2)}(g)|^2$$
$$= \sum_{\nu=0}^{n} |\sigma_\nu^{((-1+\varepsilon)/2)}(g)|^2 \cdot \nu^{-1+\varepsilon}$$
$$\leqq \sum_{\nu=0}^{n} |\sigma_\nu^{((-1+\varepsilon)/2)}(g) - \sigma_\nu^{((1+\varepsilon)/2)}(g)|^2\, \nu^{-1+\varepsilon}$$
$$+ \sum_{\nu=0}^{n} |\sigma_\nu^{((1+\varepsilon)/2)}(g)\,|^2 \nu^{-1+\varepsilon}.$$

$$|\sigma_n^{(\varepsilon)}(f)| = C|S_n^{(\varepsilon)}(f)/n^\varepsilon| \leqq C\sum_{\nu=0}^{n} |\sigma_\nu^{((-1+\varepsilon)/2)}(g) - \sigma_\nu^{((1+\varepsilon)/2)}(g)|^2/\nu$$
$$+ C|\max \sigma_\nu^{((1+\varepsilon)/2)}(g)|^2.$$

Applying Zygmund's theorem [28], we have

$$\int_0^{2\pi} \max_n |\sigma_n^{(\varepsilon)}(f)| \leqq C\int_0^{2\pi} \sum_{n=1}^{\infty} |\sigma_\nu^{((-1+\varepsilon)/2)}(g) - \sigma_\nu^{((1+\varepsilon)/2)}(g)|^2/\nu \, d\theta$$
$$+ C\int_0^{2\pi} |\max \sigma_\nu^{((1+\varepsilon)/2)}(g)|^2 d\theta$$
$$\leqq A\int_0^{2\pi} g^2(\theta)\, d\theta \leqq B\int_0^{2\pi} |f(\theta)|\, d\theta.$$

Thus we get (3.1).

The proof of (3.3) is almost identical. Since

$$|\sigma_n^{(-\varepsilon)}(f)| \leqq \sum_{\nu=0}^{n} |\sigma_\nu^{((-1-\varepsilon)/2)}(g) - \sigma_\nu^{((1-\varepsilon)/2)}(g)|^2/\nu$$
$$+ |\max \sigma_\nu^{((1-\varepsilon)/2)}(g)|^2,$$

we have

$$\int_0^{2\pi} |\max \sigma_n^{(-\varepsilon)}(f)/n^\varepsilon|\, d\theta \leqq A\int_0^{2\pi} \sum_{\nu=0}^{n} |\sigma_\nu^{((-1-\varepsilon)/2)}(g) - \sigma_\nu^{((1-\varepsilon)/2}(g)|^2/\nu^{1+\varepsilon}\, d\theta$$
$$+ A\int_0^{2\pi} \max_n |\sigma_\nu^{((1-\varepsilon)/2)}(g)|^2/n^\varepsilon\, d\theta$$
$$\leqq B\int_0^{2\pi} |g(\theta)|^2\, d\theta \leqq C\int_0^{2\pi} |f(\theta)|\, d\theta.$$

THEOREM 3.2. *If $f(z) \in H$, then* (3.1) *is (C, ε)-summable a.e.*, $(\log n)$ *is convergence factors of* (3.1) *a.e. and* (n^ε) $(1 > \varepsilon > 0)$ *is $(C, -\varepsilon)$-summability factors of* (3.1) *a.e.*

PROOF. This is immediate from Theorem 3.1, except the last proposition. Since trigonometrical polynomials are dense in H-class, by the well-known method from (3.3) we can conclude

$$\sigma_n^{(-\varepsilon)} = o(n^{\varepsilon}), \qquad \text{a.e.}$$

Then applying the generalized Hardy-Littlewood theorem of the author [20], (n^{ε}) is $(C, -\varepsilon)$-summability factors of the series (3.1) a.e.

4. Applications of the method of Hardy-Littlewood concerning the proof of a maximal theorem. Hardy and Littlewood [6] have proved a maximal theorem by the very elegant method. Their method seems to have comprehensive applications. We shall give here some. Theorem 4.1 is sharper than Titchmarsh's [22, p. 87] and Theorem 4.2 is sharper than Kaczmarz's [10], but the proofs of them are easier.

THEOREM 4.1. *If* $f(t) \in L^2(-\infty, \infty)$, *then*

$$\int_{-\xi}^{\xi} \left| \max \int_{-n}^{n} f(t) e^{-ixt}\, dt / \sqrt{\log(n+1)} \right|^2 dx \leqq A(\xi) \int_{-\infty}^{\infty} |f(t)|^2 dt$$

and

$$\int_{-\xi}^{\xi} \left| \max_n \int_{-n}^{n} f(t) e^{-ixt} / \sqrt{\log(|t|+2)}\, dt \right|^2 dx \leqq A(\xi) \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

PROOF. Let $F(u)$ be the Fourier transform of $f(t) \in L^2$, then we have

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(u)|^2 du,$$

and if we put

$$G_n(t) = \begin{cases} -e^{-ixt}, & (|t| \leqq n) \\ 0, & (|t| > n) \end{cases}$$

then the transform of $G_n(t)$ is

$$g_n(u) = \sqrt{\frac{2}{\pi}} \frac{\sin n(x+u)}{x+u}.$$

By Plancherel's theorem, we have

$$\int_{-n}^{n} f(t) e^{-ixt}\, dt = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} F(u) \frac{\sin n(x-u)}{x-u}\, du.$$

On the other hand, if we assume

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = 1, \qquad \lambda(t) = \{\log(|t|+2)\}^{1/2},$$

$$\lambda\{n(x)\} = \lambda_N, \qquad n(x) = N,$$

and $\varphi(u)$ be any function such that

$$\int_{-\xi}^{\xi} |\varphi(u)|^2 du = 1,$$

then

$$J \equiv \int_{-\xi} \lambda_N \varphi(x) dx \int_{-N}^{N} f(t) e^{-ixt}\, dt$$

$$= \sqrt{\frac{2}{\pi}} \int_{-\xi}^{\xi} \lambda_N \varphi(x) dx \int_{-\infty}^{\infty} F(u) \frac{\sin N(x-u)}{x-u}\, du$$

$$= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} F(u)\, du \int_{-\xi}^{\xi} \lambda_N \varphi(x) \frac{\sin N(x-u)}{x-u}\, dx.$$

$$J^2 \leqq \frac{2}{\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\xi}^{\xi} \lambda_N \varphi(x) \frac{\sin N(x-u)}{x-u}\, dx \right\}^2 du \int_{-\infty}^{\infty} |F(u)|^2\, du$$

$$\leqq \frac{2}{\pi} \int_{-\infty}^{\infty} du \int_{-\xi}^{\xi} \lambda_{N_1} \varphi(x_1) \frac{\sin N_1(x_1-u)}{x_1-u}\, dx_1 \int_{-\xi}^{\xi} \lambda_{N_2} \varphi(x_2) \frac{\sin N_2(x_2-u)}{x_2-u}\, dx_2,$$

where $N_1 = n(x_1)$ and $N_2 = n(x_2)$.

On the other hand, since

$$\sqrt{\frac{2}{\pi}} \frac{\sin(x+u)}{x+u}$$

is the transform of $G_n(t)$, we have

$$\left(\sqrt{\frac{2}{\pi}}\right)^2 \int_{-\infty}^{\infty} \frac{\sin N_1(x_1-u)}{x_1-u} \frac{\sin N_2(x-u)}{x_2-u}\, du$$

$$= \int_{-N_{1,2}}^{N_{1,2}} e^{ix_1 t} e^{-ix_2 t}\, dt = \sqrt{\frac{2}{\pi}} \frac{\sin N_{1,2}(x_1-x_2)}{x_1-x_2}$$

where $N_{1,2} = \min(N_1, N_2)$.

$$J^2 \leqq \sqrt{\frac{2}{\pi}} \int_{-\xi}^{\xi} \lambda_{N_1} \varphi(x_1) \int_{-\xi}^{\xi} \lambda_{N_2} \varphi(x_2) \frac{\sin N_{1,2}(x_1-x_2)}{x_1-x_2}\, dx_1\, dx_2$$

$$\leqq \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_{-\xi}^{\xi} \int_{-\xi}^{\xi} \{\lambda_{N_1}^2 \varphi^2(x_1) + \lambda_{N_2}^2 \varphi^2(x_2)\} \left| \frac{\sin N_{1,2}(x_1-x_2)}{x_1-x_2} \right| dx_1\, dx_2$$

$$= (J_1 + J_2)/\sqrt{2\pi},$$

say. Since

$$\left| \frac{\sin N_{1,2}(x_1-x_2)}{x_1-x_2} \right| \leqq \frac{C}{|x_1-x_2| + (N_{1,2}+1)^{-1}},$$

we have

$$J_1 \leqq C \int_{-\xi}^{\xi} \lambda_{N_1}^2 \varphi^2(x_1) dx_1 \int_{-\xi}^{\xi} \frac{dx_2}{|x_1-x_2| + (N_{1,2}+1)^{-1}}$$

$$\leqq C \int_{-\xi}^{\xi} \lambda_{N_1}^2 \varphi^2(x_1) dx_1 \int_{-2\xi}^{2\xi} \frac{dx_2}{|x_2| + (N_1+1)^{-1}}$$

$$\leqq 2C\int_{-\xi}^{\xi} \lambda_{N_1}^2 \varphi^2(x_1)dx_1 \log\{2\xi(N_1+1)+1\}$$

$$\leqq A(\xi)\int_{-\xi}^{\xi} \varphi^2(x)dx \leqq A(\xi).$$

Similarly we have $|J_2| \leqq A(\xi)$. The remaing part of the proof is analogous to the Fourier series case.

THEOREM 4.2. *If we denote by* $S_{m,n}(x,y)$ *the partial sum of the double Fourier series, then*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \max_{m,n} \frac{S_{m,n}(x,y)}{\sqrt{\log(m+2)}\sqrt{\log(n+2)}} \right|^2 dx\, dy \leqq K\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |f(x,y)|^2 dxdy.$$

PROOF. Let

$$c_n(x) = \sin(n+1/2)x/(2\sin x/2),$$

then

$$S_{mn}(x,y) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(u,v)c_m(x-u)c_n(y-v)du\, dv.$$

Let

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^2(x,y)dx\, dy = 1,$$

and $\varphi(x,y)$ be any function such as

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \varphi^2(x,y)\, dx\, dy = 1,$$

and put

$$J = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} \lambda_{M,N}\varphi(x,y)dx\, dy\int_0^{2\pi}\int_0^{2\pi} f(u,v)c_M(u-x)c_N(v-y)du\, dv,$$

where

$$\lambda_{m,n} = \{\log(m+2\}^{1/2}\{\log(n+2\}^{1/2},$$
$$M = m(x,y), \qquad N = n(x,y),$$

then

$$J = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(u,v)du\, dv\int_0^{2\pi}\int_0^{2\pi} \lambda_{M,N}\varphi(x,y)\, c_M(u-x)c_N(v-y)\, dx\, dy$$

and

$$J^2 \leqq \frac{1}{\pi^4}\int_0^{2\pi}\int_0^{2\pi} du\, dv\left\{\int_0^{2\pi}\int_0^{2\pi} \lambda_{M,N}\varphi(x,y)c_M(u-x)c_N(v-y)\, dx\, dy\right\}^2 du\, dv$$

$$= \frac{1}{\pi^4}\int_0^{2\pi}\int_0^{2\pi} du\, dv\left\{\int_0^{2\pi}\int_0^{2\pi} \lambda_{M_1,N_1}\varphi(x_1,y_1)\, c_{M_1}(u-x_1)c_{N_1}(v-y_1)\, dx_1dy_1\right\}$$

$$\cdot\left\{\int_0^{2\pi}\int_0^{2\pi}\lambda_{M_2,N_2}\varphi(x_2,y_2)c_{M_2}(u-x_2)c_{N_2}(v-y_2)dx_2dy_2\right\}$$

$$\leqq\frac{1}{\pi^4}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\lambda_{M_1,N_1}\lambda_{M_2,N_2}|\varphi(x_1,y_1)|\cdot|\varphi(x_2,y_2)|\cdot$$

$$|c_{M_{1,2}}(x_1-x_2)|\cdot|c_{N_{1,2}}(y_1-y_2)|dx_1dx_2dy_1dy_2,$$

where

$$M_{1,2}=\min(M_1,\ M_2).$$

Since

$$\lambda_{M_1,N_1}\lambda_{M_2,N_2}|\varphi(x_1,y_1)|\cdot|\varphi(x_2,y_2)|\leqq\frac{1}{2}\left\{\lambda^2_{M_1,N_1}\varphi^2(x_1,y_1)+\lambda^2_{M_2,N_2}\varphi^2(x_2,y_2)\right\},$$

we have

$$J^2\leqq C(J_1+J_2),$$

where

$$J_1\leqq\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\lambda^2_{M_1,N_1}\varphi^2(x_1,y_1)dx_1dy_1\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dx_2}{|x_1-x_2|+(M_{1,2}+2)^{-1}}\,\frac{dy_2}{|y_1-y_2|+(N_{1,2}+2)^{-1}}\cdot$$

$$\leqq\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\lambda^2_{M_1,N_1}\varphi^2(x_1,y_1)dx_1dy_1\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dx_2}{|x_2|+(M_1+2)^{-1}}\,\frac{dy_2}{|y_2|+(N_1+2)^{-1}}$$

$$\leqq\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\lambda^2_{M_1,N_1}\varphi^2(x_1,y_1)\,dx_1\,dy_1C\log(M_1+2)\log(N_1+2)$$

$$\leqq C\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varphi^2(x,y)\,dx\,dy\leqq C.$$

The analogous result concerning simple Fourier series, can be gotten applying Abel's transformation twice to Theorem 4.2.

5. A one-side localization theorem. A. Zygmund [29] has proved the following localization theorem.

THEOREM 5.1. (Zygmund) *If the Fourier coefficients of $f(x)$ are $o(1/n)$ and $f(x)=0$ in $x_0<x\leqq x_0+\varepsilon$, then $\mathfrak{S}[f]$ converges to zero at the point x_0.*

It is evident that the condition $o(1/n)$ cannot be transposed by $O(1/n)$, but there is a continuous function of the bounded variation, which Fourier coefficients are not $o(1/n)$. (cf. Zygmund [27, p. 293]). Therefore the condition $o(1/n)$ is not best possible.

THEOREM 5.2. *If $\sum_{n=1}^{N} n\rho_n/N\to 0$ as $N\to\infty$, where $\rho_n=(a_n^2+b_n^2)^{1/2}$ and a_n, b_n are Fourier coefficients of $f(x)$, and $f(x)=0$ in $x_0<x\leqq x+\varepsilon$, then $\mathfrak{S}[f]$ converges to zero at x_0.*

PROOF. Since

$$\sum_{n=1}^{N} (nb_n \cos nx_0 - na_n \sin nx_0)/N \to 0$$

as $N \to \infty$, $f(x)$ has not generalized jump at x_0, that is

$$(C, 1+\varepsilon)\{f(x_0+t) - f(x_0-t)\} \to 0$$

as $t \to 0$. (cf. Sunouchi [19]). From the one-side vanishing, we have

$$(C, 1+\varepsilon) f(x_0+t) \to 0, \qquad \text{as } t \to 0.$$

Therefore

$$(C, 1+\varepsilon) f(x_0-t) \to 0, \qquad \text{as } t \to 0,$$

and then

$$(C, 1+\varepsilon)\{f(x_0+t) + f(x_0-t)\} \to 0.$$

Thus $\mathfrak{S}[f]$ is $(C, 1+2\varepsilon)$-summable to zero at x_0. On the other hand, since

$$\sum_{n=1}^{N} n\,\rho_n/N \to 0,$$

as $N \to \infty$, $\mathfrak{S}[f]$ converges to zero at x from the well-known Tauberian theorem. Thus we get the theorem.

REMARK. $\sum_{n=1}^{N} n\rho_n/N \to 0$ is a necessary and sufficient condition for the function of the bounded variation to be continuous, from Wiener's theorem [25]. Therefore the condition of the theorem 5.2 is best possible in a sense.

6. A problem of Zalcwasser. Zalcwasser [26] has proved the following:

If $f(x) \in L^2(0, 2\pi)$, and $s_n(x)$ is n-th partial sum of the $\mathfrak{S}[f]$, then

$$\lim_{n\to\infty} \sum_{\nu=1}^{N} |s_{p_\nu}(x) - f(x)|/N = 0, \qquad \text{a. e.}$$

provided that

$$p_k^2 \sum_{\nu=k}^{\infty} 1/\nu p_\nu^2 = O(1),$$

and he proposed the problem: whether $(C, 1)$-mean is transposed by (C, α) $(0 < \alpha < 1)$ mean or not. If the strong summablility is replaced by ordinary summability, then the problem is answered affirmatively. *)

THEOREM 6.1. *If $f(x) \in L^2$ $(0, 2\pi)$, then $s_{p_\nu}(x)$ is (C, α)-summable $(\alpha > 0)$ a.e., provided that*

$$p_k^2 \sum_{\nu=k}^{\infty} 1/\nu p_\nu^2 = O(1).$$

PROOF. Let us put

$$\varphi_\nu(x) = \{c_{p_{\nu-1}+1} e^{i(p_{\nu-1}+1)x} + \cdots + c_{p_\nu} e^{ip_\nu x}\}/\gamma_\nu,$$

where

*) *Added in the proof.* We have gotten the complete solution of Zalcwasser.

$$\gamma_\nu = (c_{p_{\nu-1}+1}^2 + \cdots + c_{p_\nu}^2)^{1/2},$$

and consider the series

$$\sum_{\nu=0}^{n} \gamma_\nu \varphi_\nu(x).$$

It is evident $\{\varphi_\nu(x)\}$ is normalized orthogonal system and

$$\sum_{\nu=0}^{\infty} \gamma_\nu^2 = \sum_{n=0}^{\infty} c_n^2 = \int_0^{2\pi} \{f(x)\}^2 dx < \infty.$$

Since

$$p_k^2 \sum_{\nu=k}^{\infty} 1/\nu p_\nu^2 = O(1),$$

$\sum_{\nu=0}^{n} \gamma_\nu \varphi_\nu(x) \equiv S_{p_\nu}(x)$ is $(C,1)$-summable by Zalcwasser's theorem.

Since the normalized orthogonal series of a function of L^2 which is $(C,1)$-summable is already (C,α)-summable $(\alpha > 0)$ (cf. Kaczmarz-Steinhaus [11, p. 182]), the theorem is valid.

7. Distribution of signs of the terms of conditionally convergent series. E. Cesàro (cf. Pólya and Szegö [17, p. 25]) proved the following theorem regarding to the distribution of signs of the terms of a conditionally convergent series.

THEOREM 7.1. (Cesàro) *If* $\sum a_n = \infty$ *where* $a_n \downarrow 0$ *and* $\sum a_n \varepsilon_n$ *converges where* $\varepsilon_n = \pm 1$, *then*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} \varepsilon_\nu \leqq 0 \leqq \limsup_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} \varepsilon_\nu.$$

H. Auerbach [3] proved the partial converse of this which reads as follows:

THEOREM 7.2. (Auerbach) *If* $\sum a_n = \infty$ $(a_n \geqq 0)$, *and* $a_n \to 0$, *then there is an distribution of signs* $\{\varepsilon_\nu\}$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} \varepsilon_\nu = 0 \quad (\varepsilon_\nu = \pm 1)$$

and

$$\sum_{n=1}^{\infty} a_n \varepsilon_n$$

has an inferior limit l *and superior* L $(l \leqq L)$ *which are preassigned arbitrarily.*

On the other hand Izumi-Sunouchi [9] generalized Theorem 7.1.

THEOREM 7.3. (Izumi-Sunouchi) *If* $\sum a_n = \infty$ $(a_n \geqq 0)$ *where* $a_n/\mu_n \downarrow 0$ *and* $\sum a_n \varepsilon_n$ *converges* $(\varepsilon_n = \pm 1)$, *then*

$$\liminf_{n\to\infty} \sum_{\nu=1}^{n} \mu_\nu \varepsilon_\nu \Big/ \sum_{\nu=1}^{n} \mu_\nu \leqq 0 \leqq \limsup_{n\to\infty} \sum_{\nu=1}^{n} \varepsilon_\nu \mu_\nu \Big/ \sum_{\nu=1}^{n} \mu_\nu.$$

The object of this paragraph is to give a converse of this theorem analogous to Theorem 7.2.

THEOREM 7.4. *If* $\sum a_n = \infty (a_n \geqq 0)$, $a_n \to 0$ *and* $\mu_n/\mu_{n+1} \to 1$, $\sum \mu_n = \infty$ $(\mu_n \geqq 0)$, *then there is a distribution of signs* $\{\varepsilon_\nu\}$ *such that*

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \mu_\nu \varepsilon_\nu \Big/ \sum_{\nu=1}^{n} \mu_\nu = 0$$

and $\sum a_n \varepsilon_n$ *has an inferior limit* l *and superior* L $(l \leqq L)$ *which are preassigned arbitrarily.*

For the proof of this theorem, we need a lemma due to Agnew [1].

LEMMA 7.1. (Agnew) *If* $\sum a_n = \infty$ $(a_n \geqq 0)$, *then there is a sequence* $\{n_k\}$ *such as* $n_{k+1} - n_k \to \infty$ *and* $\sum a_{n_k} = \infty$.

PROOF OF THE THEOREM. If we denote by $\sum a'_n$ the series which is obtained by putting $a_{n_k} = 0$ in the series $\sum a_n$, then we can select $\bar{\varepsilon}_\nu$ ($+1$ or -1) such that

$$\sum_{\nu=1}^{2n} \bar{\varepsilon}_\nu a'_\nu \equiv \pm (a'_1 - a'_2) \pm (a'_3 - a'_4) \pm \cdots$$

converges. Since $a_n \to 0$, we have

$$\sum_{\nu=1}^{\infty} \bar{\varepsilon}_\nu a'_\nu = s.$$

If $\{n_k\}$ is the sequence of Lemma 7. 1, then since $\sum a_{n_k} = \infty$, we can select ε_{n_k} such that

$$\limsup_{m\to\infty} \sum_{k=1}^{m} a_{n_k} \varepsilon_{n_k} = L - s$$

$$\liminf_{m\to\infty} \sum_{k=1}^{m} a_{n_k} \varepsilon_{n_k} = l - s,$$

by Riemann's theorem.

Let $\{\varepsilon_\nu\}$ be the sequence derived from $\{\bar{\varepsilon}_\nu\}$, by replacing the n_k-th term by ε_{n_k}. Then since

$$\sum a_n \varepsilon_n = \sum a'_n \bar{\varepsilon}_n + \sum a_{n_k} \varepsilon_{n_k},$$

we have

$$\limsup_{m\to\infty} \sum_{n=1}^{m} a_n \varepsilon_n = L, \qquad \liminf_{m\to\infty} \sum_{n=1}^{m} a_n \varepsilon_n = l.$$

Since $\{\bar{\varepsilon}_\nu\}$ takes $+1$ and -1 alternatively and $\mu_{n+1}/\mu_n \to 1$, we have

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \mu_\nu \bar{\varepsilon}_\nu \Big/ \sum_{\nu=1}^{n} \mu_\nu = 0.$$

On the other hand since $n_{k+1} - n_k \to \infty$ and $\mu_{n+1}/\mu_n \to 1$, from the theorem of Cauchy, we have

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{k} \mu_{n_i}}{\sum_{i=1}^{n_k} \mu_i} = \lim_{k\to\infty} \frac{\mu_{n_k}}{\sum_{i=n_{k-1}+1}^{n_k} \mu_i} = 0.$$

Consequently

$$\lim_{n\to\infty}\sum_{\nu=1}^{n}\mu_\nu\varepsilon_\nu\Big/\sum_{\nu=1}^{n}\mu_\nu=0,$$

which is the required.

REMARK. If we introduce the Lebesgue measure in the space of the dyadic sequences in the usual manner, then for almost all sequences $\{\varepsilon_\nu\}$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n}\varepsilon_\nu=0.$$

On the other hand,

$$\lim_{n\to\infty}\sum_{\nu=1}^{n}\mu_\nu\varepsilon_\nu\Big/\sum_{\nu=1}^{n}\mu_\nu=0$$

only in a null set for some $\{\mu_\nu\}$ (cf. Maruyama [13] and Tsuchikura [22]). Theorem 7.2 gives an example from a full set, but Theorem 7.4 from a null set.

In analogous manner we can generalize another theorem of Agnew [2].

THEOREM 7.5. *If a sequence $\{s_n\}$ is terminating in* 0, 1, 0, 1, ..., *then let* $\lim_{n\to\infty} As_n = 1/2$ *where*

$$As_n=\sum_{k=1}^{\infty}a_k(n)s_k.$$

If $\{s_n\}$ is any sequence from 0 *or* 1 *only, then let $As_n = 0$ whenever* $\lim\sum_{\nu=1}^{n}\mu_\nu s_\nu\Big/\sum\mu_\nu=0$ $\left(\mu_\nu\geqq 0,\ \sum_{\nu=1}^{\infty}\mu_\nu=\infty,\ \mu_\nu/\mu_{\nu+1}\to 1\right)$. *On these hypothesis we can conclude that A is regular.*

Since the proof is analogous to Theorem 7.4 and Agnew [2], we omit the proof.

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