A DEFORMATION THEOREM ON CONFORMAL MAPPING

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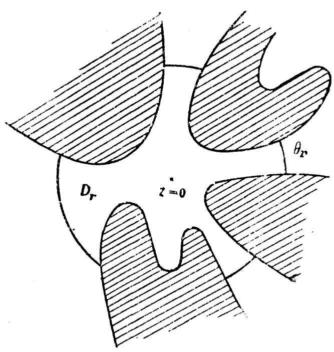
1. Let D be a simply connected domain on z-plane, which contains $z = 0$ and $z = \infty$ belongs to its boundary. The boundary Γ of D consists of at most a countable number of curves C, which extend to infinity in the both directions. Let z_0 ($|z_0| = r_0$) be the point on Γ, which lies nearest to $z = 0$, then a circle $|z| = r(> r_0)$ meets D in a number of cross cuts. We consider only such cross cuts, which separate $z = 0$ from $z = \infty$ in D and denote them by $\theta_r^{(i)}(i = 1, 2, \cdots, n = n(r))$.

We assume that $n(r)$ is finite, but may tend to infinity for $r \to \infty$. We put $\theta_r = \sum_{i=1}^{n} \theta_r^{(i)}$ and $r\theta(r)$ be the total length of θ_r. θ_r divides D into $n+1$ simply connected domains. Let D_r be the simply connected one, which contains $z = 0$, then D_r is bounded by θ_r and a part of Γ.



We will prove the following theorem, which is a generalization of Ahlfors' deformation theorem.

THEOREM 1. *If we map D conformally on $|w| < 1$ by $w = f(z)$ $(f(0) = 0)$, then the image of θ_r in $|w| < 1$ can be enclosed in a finite number of circles $K_r^{(i)}$ $(i = 1, 2, \cdots, \nu(r) \leqq n(r))$, which cut $|w| = 1$ orthogonally, such that the sum of radii is less than*

$$\text{const.} \exp\left(-\pi \int_{r_0}^{kr} \frac{dr}{r\theta(r)} \right), \qquad (0 < k < 1),$$

where k is any positive number less than 1

When D is bounded by only one curve, then by Ahlfors' deformation theorem, we can prove easily that we can take $k = 1$. Hence our theorem is worse than Ahlfors' deformation theorem, but is more general, since D may be bounded by a countable number of curves.

PROOF. First we map D conformally on $\Im \zeta > 0$ by $\zeta = \varphi(z)$ $(\varphi(z_0) = \infty,\ \varphi(0) = i)$, then $z = \infty$ is mapped on a bounded closed set E of masure zero on $\Im \zeta = 0$. Let $\lambda_r^{(i)}$ be the image of $\theta_r^{(i)}$ then $\lambda_r^{(i)}$ is a Jordan arc, whose both end points lie on $\Im \zeta = 0$. Let $\Delta_r^{(i)}$ be the finite domain, which is bounded by $\lambda_r^{(i)}$ and $\Im\zeta = 0$. If we invert $\Delta_r^{(i)}$ and $\lambda_r^{(i)}$ with respect to $\Im\zeta = 0$, then we obtain $\bar{\Delta}_r^{(i)}$, $\bar{\lambda}_r^{(i)}$, which lie in $\Im\zeta < 0$ and are symmetric to $\Delta_r^{(i)}$, $\lambda_r^{(i)}$. We denote the area of $\Delta_r^{(i)}$ by $|\Delta_r^{(i)}|$ and the length of $\lambda_r^{(i)}$ by $|\lambda_r^{(i)}|$, then

$$|\bar{\Delta}_r^{(i)}| = |\Delta_r^{(i)}|, \qquad |\bar{\lambda}_r^{(i)}| = |\lambda_r^{(i)}|.$$

We put $D_r^{(i)} = \Delta_r^{(i)} + \bar{\Delta}_r^{(i)}$, $l_r^{(i)} = \lambda_r^{(i)} + \bar{\lambda}_r^{(i)}$, then

$$(1) \qquad |D_r^{(i)}| = 2|\Delta_r^{(i)}|, \quad |l_r^{(i)}| = 2|\lambda_r^{(i)}|.$$

By the isometric inequality,

$$4\pi|D_r^{(i)}| \leqq |l_r^{(i)}|^2,$$

so that by (1),

$$(2) \qquad 2\pi|\Delta_r^{(i)}| \leqq |\lambda_r^{(i)}|^2.$$

Let $L(r)$ be the total length of $\lambda_r = \sum_{i=1}^{n}\lambda_r^{(i)}$:

$$(3) \qquad L(r) = \sum_{i=1}^{n}|\lambda_r^{(i)}|,$$

then

$$L(r) = \int_{\theta_r}|\varphi'(re^{i\theta})|rd\theta,$$

so that

$$L(r)^2 \leqq r\theta(r)\int_{\theta_r}|\varphi'(re^{i\theta})|^2 rd\theta,$$

hence

$$(4) \qquad \int_r^\infty \frac{L(r)^2}{r\theta(r)}dr \leqq \int_r^\infty\int_{\theta_r}|\varphi'(re^{i\theta})|^2 rdrd\theta.$$

We see easily that the right hand side of (4) is at most $\sum_{i=1}^{n}|\Delta_r^{(i)}|$, so that by (2),

$$\int_r^\infty \frac{L(r)^2}{r\theta(r)}dr \leqq \sum_{i=1}^{n}|\Delta_r^{(i)}| \leqq \frac{1}{2\pi}\sum_{i=1}^{n}|\lambda_r^{(i)}|^2 \leqq \frac{1}{2\pi}\left(\sum_{i=1}^{n}|\lambda_r^{(i)}|\right)^2 = \frac{1}{2\pi}L(r)^2,$$

or

$$(5) \qquad 2\pi\int_r^\infty \frac{L(r)^2}{r\theta(r)}dr \leqq L(r)^2.$$

We put

$$(6) \qquad \lambda(r) = \int_r^\infty \frac{L(r)^2}{r\theta(r)}dr,$$

then

$$L(r)^2 = -r\theta(r)\frac{d\lambda}{dr},$$

so that from (5)

$$2\pi \frac{dr}{r\theta(r)} \leqq -\frac{d\lambda}{\lambda},$$

hence integrating between r_0, r, we have

$$2\pi\int_{r_0}^{r} \frac{dr}{r\theta(r)} \leqq \log\frac{\lambda(r_0)}{\lambda(r)},$$

or

$$(7) \qquad \lambda(r) = \int_{r}^{\infty} \frac{L(r)^2}{r\theta(r)} dr \leqq \text{const. } \exp\left(-2\pi\int_{r_0}^{r} \frac{dr}{r\theta(r)}\right).$$

Since $\lambda_r^{(i)}$ can be enclosed in a circle of radius $|\lambda_r^{(i)}|$, which has its center on $\Im\zeta = 0$, if we denote $\Lambda(r)$ the lower limit of the sum of radii of a finite number of circles, which contain $\{\lambda_r^{(i)}\}$ and each of which has its center on $\Im\zeta = 0$, then

$$(8) \qquad \Lambda(r) \leqq 2L(r).$$

Evidently there exists a finite number of circles $\Lambda_r^{(i)}$ $(i = 1, 2, \cdots, \nu \leqq n)$ which contain $\{\lambda_r^{(i)}\}$ and each of which has its center on $\Im\zeta = 0$ and the sum of radii is $\Lambda(r)$.

From (7), (8), we have

$$(9) \qquad \int_{r}^{\infty} \frac{\Lambda(r)^2}{r\theta(r)} dr \leqq \text{const. } \exp\left(-2\pi\int_{r_0}^{r} \frac{dr}{r\theta(r)}\right).$$

Since by the definition of $\Lambda(r)$, $\Lambda(r)$ decreases, when r increases, we have for any $0 < k < 1$,

$$(10) \qquad \Lambda(r)^2\int_{kr}^{r} \frac{dr}{r\theta(r)} \leqq \int_{kr}^{r} \frac{\Lambda(r)^2}{r\theta(r)} dr \leqq \text{const. } \exp\left(-2\pi\int_{r_0}^{kr} \frac{dr}{r\theta(r)}\right).$$

Since $\theta(r) \leqq 2\pi$, we have

$$(11) \qquad \Lambda(r) \leqq \text{const. } \exp\left(-\pi\int_{r_0}^{kr} \frac{dr}{r\theta(r)}\right), \qquad (0 < k < 1).$$

By $w = \dfrac{\zeta - i}{\zeta + i}$, we map $\Im\zeta > 0$ on $|w| < 1$ and put $w = \dfrac{\varphi(z) - i}{\varphi(z) + i} = f(z)$, then $f(0) = 0$ and $w = f(z)$ maps D conformally on $|w| < 1$. Let $\Lambda_r^{(i)}$ be mapped on a circle $K_r^{(i)}$, then $K_r^{(i)}$ cuts $|w| = 1$ orthogonally and the image of θ_r in $|w| < 1$ by $w = f(z)$ is contained in $\{K_r^{(i)}\}$. The sum of radii of $K_r^{(i)}$ is less than const. $\Lambda(r)$, so that by (11), is less than

$$\text{const. } \exp\left(-\pi\int_{0}^{kr} \frac{dr}{r\theta(r)}\right), \qquad (0 < k < 1),$$

which proves the theorem.

2. With the same notation as § 1, let $u_r(z)$ be a harmonic function in D_r, such that $u_r(z) = 1$ on θ_r and $u_r(z) = 0$ on the remaining part of the boundary of D_r, i. e. $u_r(z)$ is the harmonic measure of θ_r with respect to

D_r. We will prove:

THEOREM 2. *For any point z in D, such that $|z| \leqq \rho$,*

$$u_r(z) \leqq C(\rho)\exp\left(-\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right), \qquad (0 < k < 1),$$

where $C(\rho)$ depends on ρ only.

PROOF. We map D conformally on $|w| < 1$ by $w = f(z)$ $(f(0) = 0)$, then by Theorem 1, the image of θ_r in $|w| < 1$ is contained in a finite number of orthogonal circles $K_r^{(i)}$ $(i = 1, 2, \cdots, n = n(r))$, such that the sum of radii is less than

$$\text{const. } \exp\left(-\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right), \qquad (0 < k < 1). \tag{1}$$

Let $K_r^{(i)}$ meet $|w| = 1$ at α_i, β_i and put

$$\psi_i = \arg(\beta_i/\alpha_i) > 0, \tag{2}$$

then by (1),

$$\sum_{i=1}^{n}\psi_i \leqq \text{const. } \exp\left(-\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)}\right). \tag{3}$$

Now $K_r^{(i)}$ drivides $|w| < 1$ into two parts and let $\Delta_r^{(i)}$ be that part, which contains $z = 0$. $\Delta_r^{(i)}$ is bounded by a part of $K_r^{(i)}$ in $|w| < 1$ and an arc $\widehat{\beta_i\alpha_i}$ on $|w| = 1$.

Consider

$$v_i(w) = \arg\frac{w-\beta_i}{w-\alpha_i}, \tag{4}$$

then it is easily seen that

$$\begin{aligned} v_i(w) &= \pi/2 + \psi_i/2 \quad \text{on the part of } K_r^{(i)} \text{ in } |w| < 1, \\ &= \psi_i/2 \quad \text{on the arc } \widehat{\beta_i\alpha_i} \text{ on } |w| = 1. \end{aligned}$$

Hence if we put

$$U_i(w) = \frac{2}{\pi}(v_i(w) - \psi_i/2), \tag{5}$$

then

$$\begin{aligned} U_i(w) &= 1 \quad \text{on the part of } K_r^{(i)} \text{ in } |w| < 1, \\ &= 0 \quad \text{on the arc } \widehat{\beta_i\alpha_i} \text{ on } |w| = 1. \end{aligned} \tag{6}$$

We put

$$U(w) = \sum_{i=1}^{n} U_i(w), \qquad \Delta_r = \prod_{i=1}^{n}\Delta_r^{(i)}, \tag{7}$$

then $U(w)$ is harmonic in Δ_r and from (6),

$$\begin{aligned} U(w) &\geqq 1 \quad \text{on the boundary of } \Delta_r \text{ in } |w| < 1, \\ &= 0 \quad \text{on the boundary of } \Delta_r \text{ on } |w| = 1. \end{aligned} \tag{8}$$

Let by $w = f(z)$, $u_r(z)$ become $U_r(w)$ in $|w| < 1$, then $U_r(w)$ is harmonic in Δ_r and since the image of θ_r is contained in $\{K_r^{(i)}\}$, we have from (8), $U_r(w) \leqq U(w)$ on the boundary of Δ_r, so that

$$U_r(w) \leqq U(w) \text{ in } \Delta_r.$$

Let $D^{(\rho)}$ be the part of D contained in $|z| \leqq \rho$ and $\Delta^{(\rho)}$ be its image in $|w| < 1$. If w lies in $\Delta^{(\rho)}$, then since $U_i(0) = \dfrac{\psi}{\pi}$,

$$U_i(w) \leqq \text{const.}\,\psi_i,$$

where const. depends on ρ only. Hence by (3),

$$u_r(z) = U_r(w) \leqq U(w) \leqq \text{const.} \sum_{i=1}^{n} \psi_i \leqq \text{const. } \exp\left(-\pi \int_{r_0}^{k} \frac{dr}{r\theta(r)}\right),$$

where const. depends on ρ only, which proves the thorem.

3. Let Δ be a connected domain on z-plane, which contains $z=0$ and $z=\infty$ belongs to its boundary. The boundary of Δ consists of at most a countable number of curves $\{C\}$. We divide $\{C\}$ into two classes $\{C\} = \{C'\} + \{C''\}$, where C' are closed curves and C'' are open curves, which extend to infinity in the both directions.

We add the insides of C' to Δ and D be the resulting domain. D is simply connected and is bounded by $\{C''\}$. We call D the associated domain of Δ. We define θ_r, $r\theta(r)$ for the associated domain D of Δ as in §1.

THEOREM 3. *Let $w = f(z)$ be regular in Δ and $|f(z)| \leqq \lambda$ on its boundary and $M(r)$ be the maximum of $|f(z)|$ on the part of $|z| = r$ contained in Δ. If there exists a point z_0 in Δ, such that $|f(z_0)| > \lambda$, then*

$$\log\log \frac{M(r)}{\lambda} \geqq \pi \int_{r_0}^{kr} \frac{dr}{r\theta(r)} - \text{const.}, \qquad (0 < k < 1).$$

PROOF. $\log^+|f(z)/\lambda|$ is subharmonic in Δ and vanishes on its boundary. We extend the definition of $\log^+|f(z)/\lambda|$ in D by putting $\log^+|f(z)/\lambda| = 0$ insides of C', then $\log^+|f(z)/\lambda|$ is subharmonic in D.

We define $u_r(z)$ as Theorem 2, then $\log^+|f(z)/\lambda| \leqq \log(M(r)/\lambda)\cdot u_r(z)$ on the boundary of D_r. Since $\log^+|f(z)/\lambda|$ is subharmonic in D_r, we have

$$\log^+|f(z)/\lambda| \leqq \log(M(r)/\lambda)\cdot u_r(z) \quad \text{in } D_r,$$

especially at z_0,

$$0 < \log^+|f(z_0)/\lambda| \leqq \log(M(r)/\lambda)\cdot u_r(z_0).$$

Since by Theorem 2,

$$u_r(z_0) \leqq \text{const. } \exp\left(-\pi \int_{r_0}^{kr} \frac{dr}{r\theta(r)}\right), \qquad (0 < k < 1),$$

we have

$$\log\log \frac{M(r)}{\lambda} \geqq \pi \int_{r_0}^{kr} \frac{dr}{r\theta(r)} - \text{const.}$$

which proves the theorem.

From Theorem 3, we can deduce easily Ahlfors' theorem[1] on the num-

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1) L. Ahlfors: Über die asymptotischen Werte der meromorphen Funktionen endlicher Ordnung. Acta Acad. Aboensis. Math. et Phys. 6 Nr. 9 (1932).

ber of asymptotic values of an integral function of finite order. An analogous theorem as Theorem 3 was proved by H. Milloux[2] and A. Dinghas[3], but they assume that the boundary of Δ consists of only one curve.

From Theorem 3, we have:

THEOREM 4. *Let $f(z)$ be regular in a domain Δ, which contains $z=0$ and $z=\infty$ belongs to its boundary and $|f(z)| \leqq \lambda$ on its boundary.*
If

$$\overline{\lim_{r\to\infty}}\left(\pi\int_{r_0}^{kr}\frac{dr}{r\theta(r)} - \log\log\frac{M(r)}{\lambda}\right) = \infty, \qquad (0<k<1),$$

then $|f(z)| \leqq \lambda$ in Δ, where $r\theta(r)$ is defined for the associated domain D of Δ.

As a special case, we have the following theorem, which is an extension of the classiacl theorem of Lindelöf and Phragmén:

COROLLARY. *Let $f(z)$ be regular in a domain Δ and $|f(z)| \leqq \lambda$ on its boundary and let $\theta(r) \leqq \theta$ for $r \geqq r_1$.*
If

$$\lim_{r\to\infty}\frac{\log M(r)}{r^{\pi/\theta}} = 0,$$

then $|f(z)| \leqq \lambda$ in Δ.

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2) H. Milloux: Sur les domaines de détermination infinie des fonctions entières. Acta Math. 61 (1933).
3) A. Dinghas: Bemerkung zu einer Differentialgleichung von Carleman. Math. Zeits. 41 (1936).