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ON THE CONSTRUCTION OF A PROBABILITY MEASURE ON THE SPACE OF BOREL-RADON MEASURES

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1. THEOREM. The purpose of this paper is to give a simple proof of a fundamental theorem for the existence of a probability measure on the space of all Borel-Radon measures on a locally compact separable metric space.

In this paper, let S be a locally compact separable metric space. Let $\mathscr{D}$ be a countable basis for the topology of S and let $\{K_j: j = 1, 2, \cdots\}$ be a class of compact subsets satisfying that $K_j \subset K_{j+1}^{\circ}$ (the interior of K_{j+1}) and $\bigcup_j K_j = S$. Let $\mathscr{A}$ be the algebra generated by $\mathscr{D} \cup \{K_j\}$, and let $\mathscr{B}(S)$ be the Borel class, i.e., the σ-algebra generated by the topology. Let $M(S)$ be the set of all measures μ on $\mathscr{B}(S)$ such that $\mu(A) < \infty$ for any bounded set $A \in \mathscr{B}(S)$. Such a measure μ is called a *Borel-Radon measure* on S. We consider the vague topology of $M(S)$ generated by the class of all sets

$$\left\{\mu \in M(S): \left| \int_S f_j d\mu - \int_S f_j d\nu \right| < \varepsilon \ (1 \leqq j \leqq n)\right\}$$

where $\nu \in M(S)$, $n \in Z_+$[1], $f_j \in C_c(S)$[2] $(1 \leqq j \leqq n)$ and $\varepsilon > 0$. Let $\mathscr{B}(M(S))$ be the Borel class in $M(S)$ with the vague topology. Let $\Omega = \Omega(\mathscr{A})$ be the product space $[0, \infty]^{\mathscr{A}}$ with the product topology. Then the product σ-algebra $\mathscr{B}([0, \infty])^{\mathscr{A}}$ coincides with the Borel class $\mathscr{B}(\Omega)$ generated by the product topology, because the topology of $[0, \infty]$ has a countable basis and $\mathscr{A}$ is countable by Lemma 2.2 below.

Any projective system $\{P_{\{A_1,\cdots,A_n\}}: n \in Z_+, \{A_1, \cdots, A_n\} \in \mathscr{A}\}$ of finite dimensional probability measures which is consistent in usual sense determines a probability measure P_0 on $\mathscr{B}(\Omega)$, and so we shall state the existence theorem in terms of P_0.

THEOREM. *Let P_0 be a probability measure on $\mathscr{B}(\Omega)$ satisfying that*
(a) *if $A_1, A_2 \in \mathscr{A}$ and $A_1 \cap A_2 = \varnothing$,*

$$P_0\{\omega: \omega(A_1 \cup A_2) = \omega(A_1) + \omega(A_2)\} = 1 \ ;$$

[1] Z_+ denotes the set of all positive integers.
[2] $C_c(S)$ denotes the space of all continuous functions on S with compact support.

(b) *if* $A_n \in \mathscr{A}(n = 1, 2, \cdots)$, A_1 *is bounded and* $A_n \searrow \emptyset$,

$$P_0\{\omega: \omega(A_n) \leqq \varepsilon\} \to 1 \quad \textit{for any} \quad \varepsilon > 0 ;$$

(c) *if* $A \in \mathscr{A}$ *is bounded,*

$$P_0\{\omega: \omega(A) < \infty\} = 1 .$$

Then there exists a unique probability measure P *on* $\mathscr{B}(M(S))$ *such that*

(d) *if* $A_1, \cdots, A_n \in \mathscr{A}$ *are bounded and* $E \in \mathscr{B}([0, \infty]^n)$,

$$P\{\mu: (\mu(A_1), \cdots, \mu(A_n)) \in E\} = P_0\{\omega: (\omega(A_1), \cdots, \omega(A_n)) \in E\} .$$

The theorem is due to P. Jagers (Theorem 1 in [2]). Since the proof depends on making use of the corresponding result in compact case and is rather complicated, so we shall give a simple direct proof.

2. LEMMA. A non-negative finitely additive set function ω on $\mathscr{A}$ is called a *content* if $\omega(A) < \infty$ for any bounded $A \in \mathscr{A}$. A content ω on $\mathscr{A}$ is called *σ-additive* [*boundedly σ-additive*] if $\omega(\bigcup_j A_j) = \sum_j \omega(A_j)$ whenever $A_j \in \mathscr{A} (j = 1, 2, \cdots)$ are mutually disjoint and $\bigcup_j A_j \in \mathscr{A}$ [and moreover $\bigcup_j A_j$ is bounded].

LEMMA 2.1. *Let* ω *be a boundedly σ-additive content on* $\mathscr{A}$. *Then there exists a unique* $\mu_\omega \in M(S)$ *such that*

$$(1) \qquad \mu_\omega(A) = \omega(A) \quad \textit{for any bounded} \quad A \in \mathscr{A}.$$

PROOF. Let $\mathscr{A}_j$ and $\mathscr{B}_j$ be, respectively, the restriction of $\mathscr{A}$ and of $\mathscr{B}(S)$ to K_j, and let ω_j be the restriction of ω to $\mathscr{A}_j$. Note that $\mathscr{B}_j$ is generated by $\mathscr{A}_j$ and coincides with the Borel class $\mathscr{B}(K_j)$ in K_j. Since ω_j is a σ-additive content on $\mathscr{A}_j$, there exists a unique measure μ_j on $\mathscr{B}(K_j)$ such that

$$\mu_j(A) = \omega_j(A) \quad \text{for all} \quad A \in \mathscr{A}_j .$$

Then a measure $\mu_\omega \in M(S)$ is well-defined as

$$\mu_\omega(A) = \lim_j \mu_j(A \cap K_j) \quad \text{for} \quad A \in \mathscr{B}(S) ,$$

and satisfies (1).

Now, let $\mu, \mu' \in M(S)$ satisfy (1). Then, for any $A \in \mathscr{A}$,

$$\mu(A) = \lim_j \mu(A \cap K_j) = \lim_j \omega(A \cap K_j) = \lim_j \mu'(A \cap K_j) = \mu'(A) ,$$

which implies $\mu = \mu'$.

LEMMA 2.2. *Let* $\mathscr{D}_*$ *be* $\mathscr{D} \cup \{K_j\}$, $\mathscr{D}_s$ *the class of all finite unions of sets in* $\mathscr{D}_*$, $\mathscr{D}_{sd}$ *the class of all finite intersections of sets in* $\mathscr{D}_s$

and $\mathscr{A}_*$ *the class of all finite unions of proper differences of sets in* $\mathscr{D}_{sd}$. *Then* $\mathscr{A}_*$ *is countable and coincides with* $\mathscr{A}$.

The proof is easy and omitted.

LEMMA 2.3. 1°. *For any bounded* $A \in \mathscr{A}$,

(2) *there exists* $\{A(n)\}_1^\infty$ *such that each* $A(n)$ *is a closed set in* $\mathscr{A}$ *and* $A(n) \nearrow A$.

2°. *Let* ω *be a content on* $\mathscr{A}$. *Then the following properties* (i), (ii), (iii) *are mutually equivalent*:

(i) ω *is boundedly* σ*-additive on* $\mathscr{A}$.

(ii) *if* $A_j \in \mathscr{A}$ $(j = 1, 2, \cdots)$, A_1 *is bounded and* $A_j \searrow \varnothing$, then $\omega(A_j) \to 0$.

(iii) *if* $A \in \mathscr{A}$ *is bounded,* $\omega(A \backslash A(n)) \to 0$, *where* $A(n)$*'s are the sets in* (2).

The lemma is a slight generalization of a part seen in the proof of Proposition 1.3 in [2] to the case S is locally compact, and is contained substantially in Lemma 6.1 in [1].

PROOF. 1°. Since $\mathscr{A}$ is an algebra, for the proof of (2) it is sufficient to show that

(2′) there exist $\{A(n)\}_1^\infty$ and $\{A'(n)\}_1^\infty$ such that each $A(n)$ is a closed set in $\mathscr{A}$, each $A'(n)$ is an open set in $\mathscr{A}$ and $\bigcup_n A(n) = A$, $\bigcap_n A'(n) = A$.

We shall first show that

(3) any bounded open set $A \in \mathscr{A}$ satisfies (2′).

Since A is bounded, we can choose $K_{j_0} \supset A$. Since $K_{j_0} \backslash A$ is compact, for any $x \in A$ there exist two open sets U_x, $V_x \in \mathscr{D}_s$ (in Lemma 2.2) such that $x \in U_x$, $K_{j_0} \backslash A \subset V_x$ and $U_x \cap V_x = \varnothing$. Then

$$\bigcup_{x \in A} (K_{j_0} \backslash V_x) \subset A = \bigcup_{x \in A} (A \cap U_x) \subset \bigcup_{x \in A} (A \cap V_x^c) \subset \bigcup_{x \in A} (K_{j_0} \backslash V_x) ,$$

which implies $A = \bigcup_{x \in A} (K_{j_0} \backslash V_x)$. Since $\mathscr{A}$ is countable, the number of sets $K_{j_0} \backslash V_x$'s appearing in the union above is countable. We denote the sets by $A(n)$ $(n=1, 2, \cdots)$. On the other hand, let $A'(n) = A$ $(n=1, 2, \cdots)$. Then $\{A(n)\}$ and $\{A'(n)\}$ are the desired.

(4) Each K_j satisfies (2′).

Let $G_n = \{x \colon d(x, K_j) < 1/n\}$ $(n = 1, 2, \cdots)$, where d stands for the metric of S. Since $K_j \subset G_n = \bigcup_{D \in \mathscr{D}, D \subset G_n} D$ and K_j is compact, there exist finitely many $D_{n1}, \cdots, D_{nk_n} \in \mathscr{D}$ such that $D_{n1}, \cdots, D_{nk_n} \subset G_n$, $K_j \subset \bigcup_{i=1}^{k_n} D_{ni}$. Let $A'(n) = \bigcup_{i=1}^{k_n} D_{ni}$ and $A(n) = K_j$ $(n = 1, 2, \cdots)$. Then $\{A(n)\}$ and $\{A'(n)\}$ are the desired.

We shall next prove that

(5) if $A_1, \cdots, A_k \in \mathscr{A}$ satisfy (2′), so do $\bigcup_{i=1}^k A_i$, $\bigcap_{i=1}^k A_i$ and $A_1 \backslash A_2$. For A_i, let $A_i(n)$ and $A_i'(n)$ be the sets in (2′). Then $\bigcup_{i=1}^k A_i = \bigcup_{n=1}^\infty \bigcup_{i=1}^k A_i(n)$ and $\bigcup_{i=1}^k A_i(n)$ is a closed set in $\mathscr{A}$. Also, $\bigcup_{i=1}^k A_i = \bigcap_{(n_1,\cdots,n_k)\in Z_+^k} \bigcup_{i=1}^k A_i'(n_i)$ and $\bigcup_{i=1}^k A_i'(n_i)$ is an open set in $\mathscr{A}$. Hence $\bigcup_{i=1}^k A_i$ satisfies (2′). Similarly it follows that $\bigcap_{i=1}^k A_i$ satisfies (2′). Further, $A_1 \backslash A_2 = \bigcup_{(n,m)\in Z_+^2} (A_1(n) \backslash A_2'(m))$ and $A_1(n) \backslash A_2'(m)$ is a closed set in $\mathscr{A}$. Also, $A_1 \backslash A_2 = \bigcap_{(n,m)\in Z_+^2} (A_1'(n) \backslash A_2(m))$ and $A_1'(n) \backslash A_2(m)$ is an open set in $\mathscr{A}$. Hence $A_1 \backslash A_2$ satisfies (2′).

Now we can easily see, by virtue of Lemma 2.2, that (3), (4) and (5) imply (2′) for any bounded $A \in \mathscr{A}$.

2°: The implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) are obvious. The proof of the implication (iii) $\Rightarrow$ (ii) is the same as that in the proof of Proposition 1.3 in [2], because each F_j appearing in the proof in [2] is compact also by the boundedness of A_1.

LEMMA 2.4. *Let $\mathscr{A}_0$ be a class of Borel sets being closed under finite unions and including a countable basis for the topology of S. Then $\mathscr{B}(M(S))$ coincides with the σ-algebra generated by the class of all sets $\{\mu: \mu(A) \in E\}$ with $A \in \mathscr{A}_0$, $E \in \mathscr{B}([0, \infty])$.*

The lemma is just Proposition 1.2 in [2].

3. PROOF OF THEOREM. Let Ω_0 be the set of all boundedly σ-additive contents on $\mathscr{A}$. Then

(6) $\Omega_0 \in \mathscr{B}(\Omega)$ and $P_0(\Omega_0) = 1$.

Because, by 2° of Lemma 2.3 and by the fact that $\mathscr{A}$ is countable,

$$\begin{aligned}\Omega_0 = \{\omega: \omega(A_1 \cup A_2) = \omega(A_1) + \omega(A_2) \\ \text{for any } A_1, A_2 \in \mathscr{A} \text{ with } A_1 \cap A_2 = \emptyset\} \\ \cap \{\omega: \lim_n \omega(A \backslash A(n)) = 0 \text{ for any bounded } A \in \mathscr{A}\} \\ \cap \{\omega: \omega(A) < \infty \text{ for any bounded } A \in \mathscr{A}\} \in \mathscr{B}(\Omega)\,,\end{aligned}$$

and, further, (a), (b) and (c) imply $P_0(\Omega_0) = 1$.

Consider a mapping T from Ω_0 to $M(S)$ defined by $\omega \mapsto \mu_\omega$, where μ_ω is the measure corresponding uniquely to ω by Lemma 2.1. Then T maps Ω_0 onto $M(S)$, even though it may not be one-to-one. we shall prove that

(7) if $B \in \mathscr{B}(M(S))$, $T^{-1}B \in \mathscr{B}(\Omega)$.

Consider a class $\mathscr{B} = \{B \in \mathscr{B}(M(S)): T^{-1}B \in \mathscr{B}(\Omega)\}$. Then $\mathscr{B}$ is a σ-algebra in $M(S)$. We denote the class of all bounded sets in $\mathscr{A}$ by $\mathscr{A}_0$. Then $\mathscr{A}_0$ is closed under finite unions and includes a countable basis for the topology consisting of all bounded sets in $\mathscr{D}$. For $A \in \mathscr{A}_0$ and $E \in \mathscr{B}([0, \infty])$, set $B = \{\mu: \mu(A) \in E\}$. Then, by (6),

$$T^{-1}B = \{\omega: \omega \in \Omega_0, \omega(A) \in E\} \in \mathscr{B}(\Omega) .$$

Hence $B \in \mathscr{B}$. Therefore, by Lemma 2.4, $\mathscr{B} = \mathscr{B}(M(S))$, which proves (7).

Now, considering (6) and (7), we define

$$P(B) = P_0(T^{-1}B) \qquad \text{for} \quad B \in \mathscr{B}(M(S)) .$$

Then P is a probability measure on $\mathscr{B}(M(S))$.

We shall show that P satisfies (d). Let $A_1, \cdots, A_n \in \mathscr{A}_0$ and $E \in \mathscr{B}([0, \infty]^n)$. Then, by Lemma 2.4,

$$\{\mu: (\mu(A_1), \cdots, \mu(A_n)) \in E\} \in \mathscr{B}(M(S)) ,$$

and

$$\begin{aligned} P\{\mu: (\mu(A_1), \cdots, \mu(A_n)) \in E\} \\ &= P_0\{\omega: \omega \in \Omega_0, (\omega(A_1), \cdots, \omega(A_n)) \in E\} \\ &= P_0\{\omega: (\omega(A_1), \cdots, \omega(A_n)) \in E\} . \end{aligned}$$

Finally we shall prove the uniqueness of such a probability measure P. Suppose that P and P' are two probability measures on $\mathscr{B}(M(S))$ satisfying (d). Consider a class $\mathscr{B} = \{B \in \mathscr{B}(M(S)): P(B) = P'(B)\}$. Then $\mathscr{B}$ is a σ-algebra in $M(S)$. Further, if $A \in \mathscr{A}_0$ and $E \in \mathscr{B}([0, \infty])$, then (d) implies

$$P\{\mu: \mu(A) \in E\} = P'\{\mu: \mu(A) \in E\} ,$$

so that $\{\mu: \mu(A) \in E\} \in \mathscr{B}$. Hence, by Lemma 2.4, $\mathscr{B} = \mathscr{B}(M(S))$ which means $P = P'$.

References

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