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ON THE ABSOLUTE NÖRLUND SUMMABILITY FACTORS OF FOURIER SERIES

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1. Let $\{s_n\}$ denote the n-th partial sum of a given infinite series $\sum a_n$. Let $\{p_n\}$ be a sequence of constants, real or complex, and let

$$P_n = p_0 + p_1 + \cdots + p_n;\ P_{-k} = p_{-k} = 0, \quad \text{for } k \geqq 1 .$$

The sequence $\{t_n\}$, given by

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k = \frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}a_k, \qquad (P_n \neq 0), \tag{1.1}$$

defines the Nörlund means of the sequence $\{s_n\}$ generated by the sequence $\{p_n\}$.

Then, the series $\sum a_n$ is said to be summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is, the series

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| \tag{1.2}$$

is convergent.

In the special cases in which $p_n = \Gamma(n+\alpha)/\Gamma(\alpha)\Gamma(n+1)$, $\alpha > 0$, and $p_n = 1/(n+1)$, summability $|N, p_n|$ are the same as the summability $|C, \alpha|$ and the absolute harmonic summability, respectively.

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. We assume without any loss of generality that the Fourier series of $f(t)$ is given by

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \tag{1.3}$$

and $\int_{-\pi}^{\pi} f(t)dt = 0$.

We use the notations

$$\varphi_x(t) = \varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\};$$

$$\triangle\lambda(n) = \lambda(n) - \lambda(n+1);$$

$$\frac{d\lambda(t)}{dt} = \lambda'(t);$$

$[x]$ denotes the integral part of real number x.

2. Recently, one of the present authors [10] proved the following theorem, which is a generalization of theorems due to M. Izumi and S. Izumi [3] and K. Kanno [5].

THEOREM A. *Let $\{p_n\}$ be non-negative and non-increasing. Suppose that $\{\mu(n)\}$ is a positive bounded sequence and $\lambda(t)$, $t > 0$, is a positive non-decreasing function such that $\{\lambda(n)\mu(n)/(n+1)\}$ is non-increasing.*

If the conditions

$$\sum_{k=n}^{\infty} \frac{\lambda(k)\mu(k)}{kP_k} = O\left(\frac{\lambda(n)}{P_n}\right), \qquad n = 1, 2, \cdots \tag{2.1}$$

and

$$\int_0^{\pi} \lambda(C/t)\,|d\varphi(t)| < \infty \tag{2.2}$$

hold for some constant $C(>2\pi)$, then the series

$$\sum_{n=1}^{\infty} \lambda(n)\mu(n)A_{n+1}(t)$$

is summable $|N, p_n|$, at $t = x$.

If we replace the condition (2.2) by the condition

$$\lambda(C/t)\varphi(t) \in BV(0, \pi) \quad \text{for a constant } C > \pi\,, \tag{2.3}$$

then next theorems are known.

THEOREM B (M. and S. Izumi [4]). *Let $\{p_n\}$ be a positive non-increasing sequence and $\alpha \geqq 0$.*

If the conditions

$$\sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{(\log n)^{\alpha}}{P_n}\right) \quad \text{for all } n \geqq 1 \tag{2.4}$$

and

$$\varphi(t)(\log C/t)^{\alpha} \in BV(0, \pi) \quad \text{for a constant } C(>\pi) \tag{2.5}$$

hold, then the series $\sum_{n=1}^{\infty} A_n(t)$ is summable $|N, p_n|$, at $t = x$.

This theorem is a generalization of the theorems of T. Singh [13] and O. P. Varshney [15].

THEOREM C (M. Mudiraj [9]). *Let $\{p_n\}$ be a positive non-increasing sequence such that*

$$\{\triangle(P_n/p_n)\} \quad \text{is bounded} \tag{2.6}$$

and

(2.7) $$P_m/P_n = O\{(m/n)^\alpha\}, \quad \textit{where } 0 < \alpha \leqq 1, \textit{ for } m \leqq n,$$

uniformly in m, n.

Let $\{\mu(n)\}$ *be a positive non-increasing sequence such that the series* $\sum_{n=1}^{\infty} \mu(n)/n$ *converges. If*

(2.8) $$t^{-\alpha}\varphi(t) \in BV(0, \pi),$$

then the series

$$\sum_{n=1}^{\infty} n^\alpha \mu(n) A_n(t)$$

is summable $|N, p_n|$, *at* $t = x$.

Theorem C is an extension of the theorem due to S. M. Mazhar [7] for the case $\beta = 0$. Putting $\lambda(t) = t^\alpha$, the condition (2.1) is deduced from the condition (2.7) under the hypothesis of $\mu(n)$ in Theorem C.

Now, S. Izumi [2] proved the following theorem.

THEOREM D. *Two conditions*

(2.9) $$(\log 2\pi/t)f(t) \in BV(0, \pi)$$

and

(2.10) $$\int_0^\pi (\log 2\pi/t)\,|df(t)| < \infty$$

are mutually exclusive.

Also, Y. Okuyama [11] proved the following theorem.

THEOREM E. *Let* $\lambda(t)$ *be a non-decreasing function. If the condition*

$$\int_0^\pi \frac{\lambda'(C/t)\,|\varphi(t)|}{t^2}\,dt < \infty$$

holds, then the condition (2.2) *is equivalent to the condition* (2.3).

Thus, by these Theorems D and E, we see that Theorem A does not necessarily contain Theorems B and C.

In this paper, we shall generalize these Theorems B and C in the following form.

THEOREM. *Let* $\{p_n\}$ *be non-negative and non-increasing. Suppose that* $\lambda(t)$, $t>0$, *is a positive non-decreasing function such that* $t\lambda'(t)/\lambda^2(t)$ *is non-increasing,* $t^2\lambda'(t)/\lambda^2(t)$ *is non-decreasing and* $\{\lambda(n)\mu(n)/n\}$ *is non-increasing, where* $\mu(t)$, $t > 0$, *is a positive bounded function.*

If the conditions

(2.11) $$\sum_{k=1}^{\infty} \frac{\lambda'(k)\mu(k)}{\lambda(k)} < \infty ,$$

(2.1) $$\sum_{k=n}^{\infty} \frac{\lambda(k)\mu(k)}{kP_k} = O\left(\frac{\lambda(n)}{P_n}\right), \quad n = 1, 2, \cdots$$

and

(2.3) $$\lambda(C/t)\varphi(t) \in BV(0, \pi) \quad \textit{for a constant } C(>\pi)$$

hold, then the series

$$\sum_{n=1}^{\infty} \lambda(n)\mu(n)A_{n+1}(t)$$

is summable $|N, p_n|$, *at* $t = x$.

If we put in our theorem $\lambda(t) = (\log t)^\alpha$ and $\mu(k) = 1/(\log k)^\alpha$ $(\alpha \geqq 0)$, then $t\lambda'(t)/\lambda^2(t) = \alpha/(\log t)^{\alpha+1}$ is non-increasing and $t^2\lambda'(t)/\lambda^2(t) = \alpha t/(\log t)^{\alpha+1}$ is non-decreasing. Further, we can easily see that

$$\sum_{k=2}^{\infty} \frac{\lambda'(k)\mu(k)}{\lambda(k)} = \sum_{k=2}^{\infty} \frac{\alpha}{k(\log k)^{1+\alpha}} < \infty$$

and

$$\sum_{k=n}^{\infty} \frac{\lambda(k)\mu(k)}{kP_k} = \sum_{k=n}^{\infty} \frac{1}{kP_k} = O\left(\frac{\lambda(n)}{P_n}\right) = O\left(\frac{(\log n)^\alpha}{P_n}\right).$$

Thus our theorem contains Theorem B.

Similarly, if we put $\lambda(t) = t^\alpha$ and $p_n = \Gamma(n+\alpha)/\Gamma(\alpha)\Gamma(n+1)$ $(0 \leqq \alpha \leqq 1)$ then Theorem C is deduced from our theorem.

3. We need some lemmas for the proof of our theorem.

LEMMA 1 (H. P. Dikshit [1]). *Let* $\{p_n\}$ *be a given sequence, then for any* x, *we have*

$$(1-x)\sum_{k=m}^{n} p_k x^k = p_m x^m - p_n x^{n+1} - \sum_{k=m}^{n-1} \Delta p_k x^{k+1}$$

where $n \geqq m \geqq 0$.

LEMMA 2 (L. McFadden [8]). If $\{p_n\}$ *is non-negative and non-increasing, then for* $0 \leqq a \leqq b < \infty$, $0 \leqq t \leqq \pi$, *and for any* n, *we have*

$$\left|\sum_{k=a}^{b} p_k \exp(i(n-k)t)\right| \leqq AP_{[1/t]},$$

where A *is a positive constant, not necessarily the same at each occurences.*

LEMMA 3. *Let* $\lambda(t)$, $t > 0$, *be a positive non-decreasing function. If*

$t\lambda'(t)/\lambda^2(t)$ is non-increasing and $t^2\lambda'(t)/\lambda^2(t)$ is non-decreasing, then we have

$$\left|\int_0^\pi \frac{\cos kt}{\lambda(C/t)}\, dt\right| \leqq A\frac{\lambda'(k)}{\lambda^2(k)} \quad \textit{for a constant } C(>\pi).$$

PROOF. By an integration by parts, we have

$$\begin{aligned} J = \int_0^\pi \frac{\cos kt}{\lambda(C/t)}\, dt &= \left[\frac{\sin kt}{k\lambda(C/t)}\right]_0^\pi - \frac{1}{k}\int_0^\pi \frac{d}{dt}\left\{\frac{1}{\lambda(C/t)}\right\}\sin kt\, dt \\ &= \frac{C}{k}\int_0^\pi \frac{\lambda'(C/t)}{t^2\lambda^2(C/t)}\sin kt\, dt\,. \end{aligned}$$

Since $\lambda'(C/t)/t\lambda^2(C/t)$ is non-decreasing and $\lambda'(C/t)/t^2\lambda^2(C/t)$ is non-increasing, we obtain

$$\begin{aligned} |J| &\leqq \frac{C}{k}\int_0^{\pi/k} \frac{\lambda'(C/t)}{t^2\lambda^2(C/t)}\sin kt\, dt \\ &= \frac{C}{k}\int_0^{\pi/k} \frac{\lambda'(C/t)}{t\lambda^2(C/t)}\,\frac{\sin kt}{t}\, dt \\ &\leqq \frac{C}{k}\,\frac{k\lambda'\left(\frac{C}{\pi}k\right)}{\pi\lambda^2\left(\frac{C}{\pi}k\right)}\int_0^{\pi/k}\frac{\sin kt}{t}\, dt \\ &\leqq A\frac{\lambda'(k)}{\lambda^2(k)}\,. \end{aligned}$$

Hence we complete the proof of Lemma 3.

4. PROOF OF THEOREM. By (1.1), we have

$$t_n = \frac{1}{P_n}\sum_{k=0}^{n} P_{n-k}\lambda(k)\mu(k)A_{k+1}(x) \tag{4.1}$$

where

$$A_{k+1}(x) = \frac{2}{\pi}\int_0^\pi \varphi(t)\cos(k+1)t\,dt\,. \tag{4.2}$$

Hence we have by (4.1) and (4.2)

$$\begin{aligned} t_n - t_{n-1} &= \sum_{k=1}^{n}\left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right)\lambda(k)\mu(k)A_{k+1}(x) \\ &= \frac{2}{\pi}\int_0^\pi \varphi(t)\left\{\sum_{k=1}^{n}\left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right)\lambda(k)\mu(k)\cos(k+1)t\right\}dt\,. \end{aligned} \tag{4.3}$$

Now, we put

$$g(t) = \lambda(C/t)\varphi(t) \qquad \text{for} \quad 0 < t \leqq \pi .$$

Then, by (4.2) and an integration by parts, we have

$$(4.4) \qquad A_{k+1}(x) = \frac{2}{\pi} g(\pi) \int_0^\pi \frac{\cos(k+1)t}{\lambda(C/t)} dt - \frac{2}{\pi} \int_0^\pi dg(t) \int_0^t \frac{\cos(k+1)u}{\lambda(C/u)} du .$$

Putting $\tau = [C/2t]$, we have by (4.3) and (4.4)

$$\begin{aligned}\sum_{n=1}^\infty |t_n - t_{n-1}| &< A\,|g(\pi)| \sum_{n=1}^\infty \left| \sum_{k=1}^n \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \lambda(k)\mu(k) \int_0^\pi \frac{\cos(k+1)t}{\lambda(C/t)} dt \right| \\ &\quad + A \int_0^\pi |dg(t)| \left\{ \sum_{n=1}^{2\tau+1} \left| \sum_{k=1}^n \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \lambda(k)\mu(k) \int_0^t \frac{\cos(k+1)u}{\lambda(C/u)} du \right| \right\} \\ &\quad + A \int_0^\pi |dg(t)| \left\{ \sum_{n=2\tau+2}^\infty \left| \sum_{k=1}^n \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \lambda(k)\mu(k) \int_0^t \frac{\cos(k+1)u}{\lambda(C/u)} du \right| \right\} \\ &= I_1 + I_2 + I_3 ,\end{aligned}$$

say. By Lemma 3, we obtain

$$\begin{aligned} I_1 &\leqq A \sum_{n=1}^\infty \sum_{k=1}^n \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \lambda(k)\mu(k) \frac{\lambda'(k)}{\lambda^2(k)} \\ &= A \sum_{k=1}^\infty \frac{\lambda'(k)\mu(k)}{\lambda(k)} \sum_{n=k}^\infty \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \\ &\leqq A \sum_{k=1}^\infty \frac{\lambda'(k)\mu(k)}{\lambda(k)} < \infty , \end{aligned}$$

because the sequence $\{P_{n-k}/P_n\}(k \geqq 1)$ is bounded, non-decreasing and the hypothesis (2.11) holds.

Since

$$\left| \int_0^t \frac{\cos(k+1)u}{\lambda(C/u)} du \right| \leqq \frac{t}{\lambda(C/t)} ,$$

we have

$$\begin{aligned} I_2 &\leqq A \int_0^\pi |dg(t)| \left\{ \frac{t}{\lambda(C/t)} \sum_{n=1}^{2\tau+1} \sum_{k=1}^n \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \lambda(k)\mu(k) \right\} \\ &= A \int_0^\pi |dg(t)| \left\{ \frac{t}{\lambda(C/t)} \sum_{k=1}^{2\tau+1} \lambda(k)\mu(k) \sum_{n=k}^{2\tau+1} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \right\} \\ &\leqq A \int_0^\pi |dg(t)| \left\{ \frac{t}{\lambda(C/t)} \sum_{k=1}^{2\tau+1} \lambda(k)\mu(k) \right\} \\ &\leqq A \int_0^\pi |dg(t)| \left\{ \frac{t}{\lambda(C/t)} \cdot \lambda(2\tau) \cdot 2\tau \right\} \\ &\leqq A \int_0^\pi |dg(t)| < \infty \end{aligned}$$

by virtue of the hypothesis that $\lambda(k)$ is non-decreasing and $\mu(k)$ is bounded.

By an integration by parts, we have

$$\int_0^t \frac{\cos(k+1)u}{\lambda(C/u)}\,du = \frac{\sin(k+1)t}{(k+1)\lambda(C/t)} + \frac{C}{(k+1)}\int_0^t \frac{\lambda'(C/t)}{t^2\lambda^2(C/t)}\sin(k+1)t\,dt\ .$$

Thus we obtain

$$\begin{aligned} I_3 &\leqq A\int_0^\pi |dg(t)|\left\{\frac{1}{\lambda(C/t)}\sum_{n=2\tau+2}^{\infty}\left|\sum_{k=1}^{n}\left(\frac{P_{n-k}}{P_n}-\frac{P_{n-k-1}}{P_{n-1}}\right)\lambda(k)\mu(k)\frac{\sin(k+1)t}{(k+1)}\right|\right\} \\ &\quad + A\int_0^\pi |dg(t)|\left\{\sum_{n=2\tau+2}^{\infty}\left|\sum_{k=1}^{n}\left(\frac{P_{n-k}}{P_n}-\frac{P_{n-k-1}}{P_{n-1}}\right)\frac{\lambda(k)\mu(k)}{(k+1)}\right|\left|\int_0^t\frac{\lambda'(C/t)}{t^2\lambda^2(C/t)}\sin(k+1)t\,dt\right|\right\} \\ &= I_{31}+I_{32}\ , \end{aligned}$$

say. By the same method as that used by Y. Okuyama [10], we have

$$\sum_{n=2\tau+2}^{\infty}\left|\sum_{k=1}^{n}\left(\frac{P_{n-k}}{P_n}-\frac{P_{n-k-1}}{P_{n-1}}\right)\lambda(k)\mu(k)\frac{\sin(k+1)t}{(k+1)}\right| = O(\lambda(C/t))\ .$$

Thus we proved the finiteness of I_{31}. On the other hand, by the similar estimation as $|J|$ in the proof of Lemma 3, we have

$$\left|\int_0^t \frac{\lambda'(C/t)}{t^2\lambda^2(C/t)}\sin(k+1)t\,dt\right| \leqq A\frac{k\lambda'(k)}{\lambda^2(k)} \qquad \text{for}\quad 0<t\leqq\pi\ .$$

Thus we have

$$\begin{aligned} I_{32} &\leqq A\int_0^\pi |dg(t)|\left\{\sum_{n=2\tau+2}^{\infty}\sum_{k=1}^{n}\left(\frac{P_{n-k}}{P_n}-\frac{P_{n-k-1}}{P_{n-1}}\right)\frac{\lambda(k)\mu(k)}{(k+1)}\frac{k\lambda'(k)}{\lambda^2(k)}\right\} \\ &= A\int_0^\pi |dg(t)|\left\{\sum_{k=1}^{2\tau+2}\frac{\lambda'(k)\mu(k)}{\lambda(k)}\sum_{n=2\tau+2}^{\infty}\left(\frac{P_{n-k}}{P_n}-\frac{P_{n-k-1}}{P_{n-1}}\right)\right. \\ &\quad \left. + \sum_{k=2\tau+3}^{\infty}\frac{\lambda'(k)\mu(k)}{\lambda(k)}\sum_{n=k}^{\infty}\left(\frac{P_{n-k}}{P_n}-\frac{P_{n-k-1}}{P_{n-1}}\right)\right\} \\ &\leqq A\int_0^\pi |dg(t)|\left\{\sum_{k=1}^{\infty}\frac{\lambda'(k)\mu(k)}{\lambda(k)}\right\} \\ &\leqq A\int_0^\pi |dg(t)| < \infty \end{aligned}$$

by virtue of the hypothesis (2.11). Thus, by I_{31} and I_{32}, we see that I_3 is finite.

Collecting the above estimations, we have

$$\sum_{n=1}^{\infty}|t_n - t_{n-1}| < \infty\ .$$

Therefore, we complete the proof of our theorem.

5. In this section, we consider some applications of our theorem.

COROLLARY 1. *If* $1 > \alpha \geqq 0$, $\beta \geqq 0$, $\alpha + \beta < 1$, *and*

$$(\log C/t)^{\beta}\varphi(t) \in BV(0, \pi)\ ,$$

then the series

$$\sum_{n=0}^{\infty} \frac{A_n(t)}{\{\log(n+2)\}^{1-\beta}} \quad \textit{is summable} \quad |N, 1/(n+2)\{\log(n+2)\}^{\alpha}|$$

at $t = x$.

For $\alpha = \beta = 0$, this corollary is due to O.P. Varshney [14].

COROLLARY 2. *If* $\alpha > 0$ *and*

$$(\log C/t)\varphi(t) \in BV(0, \pi)\ ,$$

then the series

$$\sum_{n=0}^{\infty} A_n(t) \quad \textit{is summable} \quad |N, \{\log(n+2)\}^{\alpha}/(n+2)|$$

at $t = x$.

Corollary 2 is due to M. Izumi and S. Izumi [4]. This corollary does not hold for $\alpha = 0$ by Pati's theorem [12], and the case $\alpha = 1$ in the corollary is due to O.P. Varshney [15].

COROLLARY 3. *If*

$$(\log\log C/t)^{\beta}\varphi(t) \in BV(0, \pi) \qquad \textit{for} \quad 0 \leqq \beta < 1\ ,$$

then the series

$$\sum_{n=0}^{\infty} \frac{A_n(t)}{\log(n+2)\{\log\log(n+2)\}^{1-\beta}} \textit{ is summable } |N, 1/(n+2)\log(n+2)|$$

at $t = x$.

COROLLARY 4. *If* $\alpha > 0$ *and*

$$(\log\log C/t)\varphi(t) \in BV(0, \pi)\ ,$$

then the series

$$\sum_{n=0}^{\infty} \frac{A_n(t)}{\log(n+2)} \quad \textit{is summable} \quad |N, \{\log\log(n+2)\}^{\alpha}/(n+2)\log(n+2)|$$

at $t = x$.

COROLLARY 5. *If* $\alpha \geqq 0$ *and*

$$t^{-\alpha}\varphi(t) \in BV(0, \pi)\ ,$$

then the series

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}}{\{\log(n+1)\}^{1+\varepsilon}} A_n(t) \text{ is summable } |C, \alpha|$$

at $t = x$, *where* $\varepsilon > 0$.

For Corollary 5, the reader is also referred to K. Matsumoto [6] for the case $\beta = 0$.

As these corollaries are similarly proved, we shall prove here only Corollary 1.

PROOF OF COROLLARY 1. In our theorem, we put $p_k = 1/(k+2)\{\log(k+2)\}^{\alpha}$, $\lambda(t) = \{\log(t+2)\}^{\beta}$ and $\mu(k) = 1/\log(k+2)$. Then we have

$$P_k = \sum_{l=0}^{k} \frac{1}{(l+2)\{\log(l+2)\}^{\alpha}} \cong \{\log(k+2)\}^{1-\alpha},$$

$$\sum_{k=1}^{\infty} \frac{\lambda'(k)\mu(k)}{\lambda(k)} \leqq A \sum_{k=1}^{\infty} \frac{1}{(k+2)\{\log(k+2)\}^2} < \infty$$

and

$$\sum_{k=n}^{\infty} \frac{\lambda(k)\mu(k)}{kP_k} = O\left(\frac{1}{\{\log(n+2)\}^{1-\alpha-\beta}}\right) = O\left(\frac{\lambda(n)}{P_n}\right).$$

Hence we see that all assumptions of our theorem hold. Therefore, the proof is complete.

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