THE EQUIVALENCE OF TWO CONDITIONS ON WEIGHTED NORM INEQUALITIES FOR MARTINGALES

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(Received April 12, 1976)

0. Introduction. Let (Ω, F, P) be a complete probability space and let (F_t) be a non-decreasing right continuous family of sub σ -fields of F with $\bigvee_{t \geq 0} F_t = F$ such that F_0 contains all P -null sets. Now, fix a random variable Z such that $Z > 0$ a.s., and $E[Z] = 1$, and define $Z_t = E[Z|F_t]$. In this paper we assume that the martingale Z_t is continuous and $Z_0 = 1$. Then, as is well known, there exists a unique continuous local martingale M with $M_0 = 0$ such that $Z_t = \exp(M_t - (1/2)\langle M \rangle_t)$. Here $\langle M \rangle$ denotes the unique continuous increasing process such that $M^2 - \langle M \rangle$ is a local martingale. This representation is of fundamental importance in our study. Furthermore we assume that the local martingale M is L^2 -bounded. We proved in [1] that M is a *BMO*-martingale if and only if Z satisfies the condition:

$$(A_p) \quad \sup_t \operatorname{ess. sup}_\omega E \left[\left(\frac{Z_t}{Z} \right)^{1/(p-1)} \middle| F_t \right]^{p-1} < \infty$$

for some $p > 1$. We denote this supremum by $A_p(Z)$. On the other hand, M is a *BMO*-martingale if and only if Z satisfies the condition:

$$(B_\lambda) \quad \sup_t \operatorname{ess. sup}_\omega E[\exp \{ \lambda(\langle M \rangle_\infty - \langle M \rangle_t) \} | F_t] < \infty$$

for some $\lambda > 0$. For convenience, the supremum is denoted by $B_\lambda(Z)$. Our aim is to investigate the relation between the conditions (A_p) and (B_λ) . In Section 2 we show that there exists an equivalence relation between these conditions. Namely, set $Z'_t = \exp(-M_t - (1/2)\langle M \rangle_t)$ and $Z' = Z'_\infty$. Then in order that Z satisfies (B_λ) for all $\lambda > 0$, it is necessary and sufficient that Z and Z' satisfy (A_p) for all $p > 1$. In Section 3 we prove: if M is a *BMO*-martingale, there exists a non-decreasing sequence (T_n) of stopping times with $\lim_n T_n = \infty$ a.s., such that the weighted norm inequality

$$\int_{\Omega} \sup_{0 \leq t \leq T_n} |X_t|^p Z dP \leq C_{p,n} \int_{\Omega} |X_{T_n}|^p Z dP, \quad 1 < p < \infty, \quad n = 1, 2, \dots$$

is valid for all uniformly integrable martingale X . Here $C_{p,n}$ is a con-

stant independent of X . Finally, we apply these results to prove a new type martingale inequality. Namely, if M is a BMO -martingale, then for every $p > 1$ there exists a positive constant c_p depending only on p such that the inequality

$$E[\sup_t |X_t - c_p \langle X, M \rangle_t|^p] \leq C_p E[|X_\infty - c_p \langle X, M \rangle_\infty|^p]$$

is valid for all L^2 -bounded continuous martingale X . Particularly, if Z satisfies (B_λ) for all $\lambda > 0$, then c_p may be taken to be 1.

The reader is assumed to be familiar with the martingale theory as expounded in [2]. In what follows we denote by C a positive constant and by C_p a positive constant depending only on the indexed parameter p , both letters are not necessarily the same in each occurrence.

1. The (B_λ) condition. First, observe that Z satisfies (B_λ) if and only if for any stopping time T

$$E[\exp\{\lambda(\langle M \rangle_\infty - \langle M \rangle_T)\} | F_T] \leq C_\lambda \quad \text{a.s.}$$

For simplicity, we denote by (B_λ) the condition such that the (B_λ) condition holds for some $\lambda > 0$.

LEMMA 1. *If Z satisfies (B_λ) , then for any stopping time T*

$$(1) \quad E\left[\exp\left(\sqrt{\frac{\lambda}{2}}|M_\infty - M_T|\right) \middle| F_T\right] \leq 2B_\lambda(Z)^{1/2} \quad \text{a.s.}$$

PROOF. Since $\exp(\sqrt{2\lambda}M_t - \lambda\langle M \rangle_t)$ is a local martingale, a trivial stopping argument and the Fatou lemma give

$$E[\exp\{\sqrt{2\lambda}(M_\infty - M_T) - \lambda(\langle M \rangle_\infty - \langle M \rangle_T)\} | F_T] \leq 1.$$

An application of the Schwarz inequality shows that

$$\begin{aligned} & E\left[\exp\left\{\sqrt{\frac{\lambda}{2}}(M_\infty - M_T)\right\} \middle| F_T\right] \\ &= E\left[\exp\left\{\sqrt{\frac{\lambda}{2}}(M_\infty - M_T) - \frac{\lambda}{2}(\langle M \rangle_\infty - \langle M \rangle_T)\right\} \right. \\ &\quad \left. \times \exp\left\{\frac{\lambda}{2}(\langle M \rangle_\infty - \langle M \rangle_T)\right\} \middle| F_T\right] \\ &\leq E[\exp\{\lambda(\langle M \rangle_\infty - \langle M \rangle_T)\} | F_T]^{1/2} \\ &\leq B_\lambda(Z)^{1/2} \quad \text{a.s.} \end{aligned}$$

The same argument works if M is replaced by $-M$. Thus the lemma is established.

A continuous local martingale X is a BMO -martingale if

$$\|X\|_{BMO}^2 = \sup_t \operatorname{ess. sup}_\omega E[\langle X \rangle_\infty - \langle X \rangle_t | \mathcal{F}_t] < \infty.$$

LEMMA 2. Let X be a continuous martingale with $\|X\|_{BMO} < 1$. Then for any stopping time T

$$(2) \quad E[\exp(\langle X \rangle_\infty - \langle X \rangle_T) | \mathcal{F}_T] \leq \frac{1}{1 - \|X\|_{BMO}^2} \quad \text{a.s.}$$

PROOF. To prove (2), it suffices to show that for every $A \in \mathcal{F}_T$

$$E[\exp(\langle X \rangle_\infty - \langle X \rangle_T); A] \leq \frac{1}{1 - \|X\|_{BMO}^2} P(A) \quad \text{a.s.}$$

We may assume that $P(A) > 0$. Let $dP' = I_A(dP/P(A))$ and $\mathcal{F}'_t = \mathcal{F}_{T+t}$. Then $X'_t = X_{T+t} - X_T$ is a P' -continuous martingale over $(\mathcal{F}'_t)$ such that $\langle X' \rangle_t = \langle X \rangle_{T+t} - \langle X \rangle_T$. Clearly, we have

$$E[\exp(\langle X \rangle_\infty - \langle X \rangle_T); A] = E'[\exp(\langle X' \rangle_\infty)]P(A)$$

where $E'[\cdot]$ denotes expectation over Ω with respect to dP' . Now, let S be any $\mathcal{F}'_t$ -stopping time. Then for any $A \in \mathcal{F}'_S = \mathcal{F}_{T+S}$ we get

$$\begin{aligned} \int_A E'[\langle X' \rangle_\infty - \langle X' \rangle_S | \mathcal{F}'_S] dP' &= \frac{1}{P(A)} \int_{A \cap A} (\langle X \rangle_\infty - \langle X \rangle_{T+S}) dP \\ &= \frac{1}{P(A)} \int_A E[\langle X \rangle_\infty - \langle X \rangle_{T+S} | \mathcal{F}_{T+S}] I_A dP \\ &\leq \int_A \|X\|_{BMO}^2 dP', \end{aligned}$$

from which $E'[\langle X' \rangle_\infty - \langle X' \rangle_S | \mathcal{F}'_S] \leq \|X\|_{BMO}^2$. By the energy inequalities,

$$E'[\langle X' \rangle_\infty^n] \leq n! \|X\|_{BMO}^{2n}, \quad n = 0, 1, 2, \dots$$

As $\|X\|_{BMO} < 1$, we get

$$\begin{aligned} E'[\exp(\langle X' \rangle_\infty)] &= \sum_{n=0}^{\infty} \frac{1}{n!} E'[\langle X' \rangle_\infty^n] \\ &\leq \frac{1}{1 - \|X\|_{BMO}^2}, \end{aligned}$$

completing the proof.

From Lemma 2, Z satisfies the $(B)_\lambda$ condition if $\|M\|_{BMO} < 1$. More generally, if M is a non-trivial BMO -martingale, then Z satisfies the $(B)_\lambda$ condition for $\lambda = 1/(2\|M\|_{BMO}^2)$. Conversely, if Z satisfies $(B)_\lambda$ for some $\lambda > 0$, the Jensen inequality shows that

$$\|M\|_{BMO}^2 \leq \frac{1}{\lambda} \log B_\lambda(Z),$$

from which M is a BMO -martingale. Consequently, Z satisfies the (B_∞) condition if and only if M is a BMO -martingale. For example, Z satisfies (B_λ) for all $\lambda > 0$ if $\langle M \rangle$ is bounded.

2. The relation between (A_p) and (B_λ) . If Z satisfies (A_p) , then the Hölder inequality shows that it satisfies (A_r) for every $r > p$. For convenience, we denote by (A_∞) the condition such that the (A_p) condition holds for some $p > 1$.

The main result of this section is the following.

THEOREM 1.

- (i) Z and Z' satisfy (A_p) for all $p > 1$ if and only if Z satisfies (B_λ) for all $\lambda > 0$.
- (ii) The (A_∞) condition is equivalent to the (B_∞) condition.

This is an immediate consequence of the next lemma.

LEMMA 3. Let $1 < p < \infty$, and let T be any stopping time.

(i) If Z satisfies (A_p) , then

$$(3) \quad E \left[\exp \left\{ \frac{1}{2(\sqrt{p} + 1)^2} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right] \leq A_p(Z)^{\sqrt{p}(\sqrt{p}+1)}$$

$$(4) \quad E \left[\exp \left\{ \frac{1}{2(\sqrt{p} + 1)} |M_\infty - M_T| \right\} \middle| F_T \right] \leq 2A_p(Z)^{\sqrt{p}(\sqrt{p}+1)/2}.$$

(ii) If Z and Z' satisfy (A_p) , then

$$(5) \quad E \left[\exp \left\{ \frac{1}{2(p-1)} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right] \leq \{A_p(Z)A_p(Z')\}^{1/2(p-1)}$$

$$(6) \quad E \left[\exp \left\{ \frac{1}{2\sqrt{p-1}} |M_\infty - M_T| \right\} \middle| F_T \right] \leq 2\{A_p(Z)A_p(Z')\}^{1/4(p-1)}.$$

(iii) Conversely, if the inequality

$$(7) \quad E \left[\exp \left\{ \frac{1}{2(\sqrt{p} - 1)^2} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right] \leq C_p$$

holds, then Z and Z' satisfy (A_p) .

PROOF. According to Lemma 1, (4) follows at once from (3). Similarly, (5) implies (6). We begin with the proof of (3). Set $p_0 = \sqrt{p}/(\sqrt{p} - 1)$. Then, as the exponent conjugate q_0 is $\sqrt{p}$,

$$\frac{1}{2p_0(p-1)} - \frac{q_0}{2p_0^2(p-1)^2} = \frac{1}{2(\sqrt{p}+1)^2}.$$

We apply the Hölder inequality

$$\begin{aligned} & E \left[\exp \left\{ \frac{1}{2(\sqrt{p}+1)^2} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right] \\ &= E \left[\exp \left\{ -\frac{1}{p_0(p-1)} (M_\infty - M_T) + \frac{1}{2p_0(p-1)} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \right. \\ &\quad \times \exp \left\{ \frac{1}{p_0(p-1)} (M_\infty - M_T) - \frac{q_0}{2p_0^2(p-1)^2} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \Big] \\ &\leq E \left[\exp \left\{ -\frac{1}{p-1} (M_\infty - M_T) + \frac{1}{2(p-1)} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right]^{1/p_0} \\ &\quad \times E \left[\exp \left\{ \frac{q_0}{p_0(p-1)} (M_\infty - M_T) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \left(\frac{q_0}{p_0(p-1)} \right)^2 (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right]^{1/q_0}. \end{aligned}$$

As $Z_T/Z = \exp \{ -(M_\infty - M_T) + (1/2)(\langle M \rangle_\infty - \langle M \rangle_T) \}$, the first term on the right side is equal to $E[(Z_T/Z)^{1/(p-1)} | F_T]^{1/p_0}$ which dominated by $A_p(Z)^{\sqrt{p}/(\sqrt{p}+1)}$. The second term is also dominated by 1. Consequently, (3) holds.

Now we are going to show (5). By the Schwarz inequality, we have

$$\begin{aligned} & E \left[\exp \left\{ \frac{1}{2(p-1)} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right] \\ &\leq E \left[\exp \left\{ -\frac{1}{p-1} (M_\infty - M_T) + \frac{1}{2(p-1)} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right]^{1/2} \\ &\quad \times E \left[\exp \left\{ \frac{1}{p-1} (M_\infty - M_T) + \frac{1}{2(p-1)} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \middle| F_T \right]^{1/2}. \end{aligned}$$

The first term on the right side is $E[(Z_T/Z)^{1/(p-1)} | F_T]^{1/2}$, and the second term is $E[(Z'_T/Z')^{1/(p-1)} | F_T]^{1/2}$. Then (5) follows at once.

Finally, we prove (iii). Set $p_0 = \sqrt{p} + 1$. As $q_0 = (\sqrt{p} + 1)/\sqrt{p}$ and

$$\frac{1}{p-1} = \frac{1}{q_0(\sqrt{p}-1)^2} - \frac{p_0}{(p-1)^2},$$

we get by the Hölder inequality

$$\begin{aligned} & E \left[\left(\frac{Z_T}{Z} \right)^{1/(p-1)} \middle| F_T \right] \\ &= E \left[\exp \left\{ -\frac{1}{p-1} (M_\infty - M_T) - \frac{p_0}{2(p-1)^2} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \right] \end{aligned}$$

$$\begin{aligned}
& \times \exp \left\{ \frac{1}{2q_0(\sqrt{p}-1)^2} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \Big| F_T \Big] \\
& = E \left[\exp \left\{ -\frac{p_0}{p-1} (M_\infty - M_T) - \frac{1}{2} \left(\frac{p_0}{p-1} \right)^2 (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \Big| F_T \right]^{1/p_0} \\
& \times E \left[\exp \left\{ \frac{1}{2(\sqrt{p}-1)^2} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \Big| F_T \right]^{1/q_0}.
\end{aligned}$$

The first term on the right side is smaller than 1, and by the assumption the second term is dominated by some constant C_p . Therefore Z satisfies (A_p) . Similarly, Z' satisfies (A_p) . Thus the lemma is established.

PROOF OF THEOREM 1. We begin with the proof of the part (i). Firstly, suppose that Z satisfies (B_λ) for all $\lambda > 0$. Then it follows from Lemma 3 (iii) that Z and Z' satisfy (A_p) for all $p > 1$. On the other hand, for any $\lambda > 0$ we can choose $p > 1$ such that $\lambda \leq 1/(2(p-1))$. Thus from Lemma 3 (ii) the sufficiency follows at once.

Now we are going to show the part (ii). Assume that Z satisfies (B_λ) for some $\lambda > 0$. And choose $p > 1$ such that $1/(2(\sqrt{p}-1)^2) \leq \lambda$. Then Lemma 3 (iii) implies that Z satisfies (A_p) . Conversely, if Z satisfies (A_p) , then from Lemma 3 (i) it satisfies (B_λ) for $\lambda = 1/(2(\sqrt{p}+1)^2)$. The theorem is proved.

As remarked in Section 1, in order that Z satisfies (B_∞) , it is necessary and sufficient that M is a *BMO*-martingale. Thus we get:

COROLLARY 1. Z satisfies (A_∞) if and only if M is a *BMO*-martingale.

COROLLARY 2. Let $p > 9$. If Z satisfies (A_{p_0}) for $p_0 = (\sqrt{p}-2)^2$, then Z' satisfies (A_p) .

PROOF. As $(\sqrt{p_0}+1)^2 = (\sqrt{p}-1)^2$, Lemma 3 shows that for any stopping time T

$$E \left[\exp \left\{ \frac{1}{2(\sqrt{p}-1)^2} (\langle M \rangle_\infty - \langle M \rangle_T) \right\} \Big| F_T \right] \leq C_p.$$

Thus Z' satisfies (A_p) .

It is also easy to check that Z satisfies (A_p) if $\|M\|_{BMO} < \sqrt{2}(\sqrt{p}-1)$.

PROPOSITION. Let T be any stopping time. Suppose that for every $p > 1$ there exists a constant $\varepsilon > 0$ such that $\varepsilon < p-1$ and

$$E \left[\exp \left(\frac{2}{p-1-\varepsilon} |M_\infty - M_T| \right) \Big| F_T \right] \leq C_p.$$

Then Z satisfies (A_p) for all $p > 1$ if and only if Z' satisfies (A_p) for all $p > 1$.

PROOF. We only show the necessity. Assume that Z satisfies (A_p) for all $p > 1$. Set $p_0 = (p-1)/\varepsilon$. Then the exponent conjugate q_0 is $(p-1)/(p-1-\varepsilon)$. As $Z'_T/Z' = \exp\{2(M_\infty - M_T)\}Z_T/Z$, the Hölder inequality shows that

$$\begin{aligned} E\left[\left(\frac{Z'_T}{Z'}\right)^{1/(p-1)} \middle| F_T\right] \\ \leq E\left[\left(\frac{Z_T}{Z}\right)^{p_0/(p-1)} \middle| F_T\right]^{1/p_0} E\left[\exp\left(\frac{2q_0}{p-1}|M_\infty - M_T|\right) \middle| F_T\right]^{1/q_0}. \end{aligned}$$

The first term on the right side is dominated by $A_r(Z)^{1/p_0(r-1)}$ where $r = (p + p_0 - 1)/p_0 = 1 + \varepsilon > 1$. The second term is also dominated by some constant C_p , from the assumption. Consequently, Z' satisfies (A_p) .

Particularly, if the martingale M is bounded, the following are equivalent:

- 1°. Z satisfies (A_p)
- 2°. Z' satisfies (A_p)
- 3°. Z satisfies (B_λ) for $\lambda = \frac{1}{2(p-1)}$.

However we do not know whether this equivalence is true in general.

3. A weighted norm inequality for continuous martingales. We use $d\hat{P}$ to denote the weighted probability measure ZdP , and let $\hat{E}[\cdot]$ be expectation over Ω with respect to $d\hat{P}$. As Z_t is a P -martingale, we get $d\hat{P} = Z_t dP$ on each F_t . Thus an elementary calculation shows that

$$(8) \quad \hat{E}[Y|F_t] = \frac{E[ZY|F_t]}{Z_t} \quad \text{a.s., under } dP \text{ and } d\hat{P}$$

for every $\hat{P}$ -integrable random variable Y .

THEOREM 2. If Z satisfies (A_∞) , then there exists a non-decreasing sequence (T_n) of stopping times with $\lim_n T_n = \infty$ such that we have

$$(9) \quad \hat{E}[(X_{T_n}^*)^p] \leq C_{p,n} \hat{E}[|X_{T_n}|^p], \quad 1 < p < \infty, \quad n = 1, 2, \dots$$

for all P -uniformly integrable martingale X where $X_{T_n}^* = \sup_t |X_{t \wedge T_n}|$.

PROOF. By the assumption, Z satisfies (A_{p_0}) for some $p_0 > 1$. Let X be any P -uniformly integrable martingale, and set $X^* = X_\infty^*$. Firstly, we show that the inequality

$$(10) \quad \hat{E}[(X^*)^p] \leq C_p \hat{E}[|X_\infty|^p]$$

is valid for every $p > p_0$. To see it, we may assume that $\hat{E}[|X_\infty|^p]$ is finite. As $X_t = Z_t \hat{E}[X_\infty/Z | F_t]$, we get

$$|X_t|^{p_0} \leq \{Z_t^{p_0} \hat{E}[(1/Z)^{q_0} | F_t]^{p_0/q_0}\} \hat{E}[|X_\infty|^{p_0} | F_t]$$

from the Hölder inequality. The first term on the right side is $E[(Z_t/Z)^{1/(p_0-1)} | F_t]^{p_0-1}$ which is dominated by $A_{p_0}(Z)$. Then, applying the Doob inequality to the $\hat{P}$ -martingale $\{\hat{E}[|X_\infty|^{p_0} | F_t]\}_{t \geq 0}$, we have

$$\begin{aligned} \hat{E}[(X^*)^p] &\leq A_{p_0}(Z) \hat{E}[\sup_t \hat{E}[|X_\infty|^{p_0} | F_t]^{p/p_0}] \\ &\leq C_p \hat{E}[|X_\infty|^p]. \end{aligned}$$

Therefore, if Z satisfies (A_p) for all $p > 1$, the inequality (10) holds for every $p > 1$.

Now let $T_n = \inf\{t \geq 0; \langle M \rangle_t > n\}$, $n = 1, 2, \dots$. Then clearly $T_n \leq T_{n+1} \rightarrow \infty$ a.s., $\langle M \rangle_{T_n} \leq n$ and $E[Z | F_{T_n}] = Z_{T_n}$. Set

$$Z_t^{(n)} = \exp\left(M_{t \wedge T_n} - \frac{1}{2} \langle M \rangle_{t \wedge T_n}\right).$$

As $\langle M \rangle_{t \wedge T_n} \leq n$, $Z_t^{(n)} = Z_{T_n}$ satisfies (B_λ) for all $\lambda > 0$. Thus it follows from Theorem 1 that each $Z^{(n)}$ satisfies (A_p) for all $p > 1$. Since $|X_{T_n}|$ and $X_{T_n}^*$ are F_{T_n} -measurable, we have

$$\begin{aligned} \hat{E}[(X_{T_n}^*)^p] &= E[Z(X_{T_n}^*)^p] \\ &= E[Z^{(n)}(X_{T_n}^*)^p] \\ &\leq C_{p,n} E[Z^{(n)} | X_{T_n}]^p \\ &= C_{p,n} E[Z | X_{T_n}]^p \\ &= C_{p,n} \hat{E}[|X_{T_n}|^p] \end{aligned}$$

for every $p > 1$, completing the proof.

We proved in [1]: if $(1/Z)^{1/(p-1)} \in L^1(dP)$ and the inequality (10) is valid for all P -martingale X with $\sup_t \hat{E}[|X_t|^p] < \infty$, then Z satisfies (A_p) .

Finally, as an application, we give a new martingale inequality. For that, we need the next two lemmas.

LEMMA 4. *Let X be any P -continuous local martingale, and let $\langle X, M \rangle = (1/2)(\langle X + M \rangle - \langle X \rangle - \langle M \rangle)$. Then $\hat{X} = X - \langle X, M \rangle$ is a $\hat{P}$ -continuous local martingale such that $\langle \hat{X} \rangle = \langle X \rangle$ under either probability measure.*

PROOF. We sketch briefly its proof. Because of (8), for $s < t$

$$\hat{E}[\hat{X}_t | F_s] = \frac{1}{Z_s} E[Z_t \hat{X}_t | F_s].$$

Therefore, to see that $\hat{X} = X - \langle X, M \rangle$ is a $\hat{P}$ -local martingale, it suffices to show that $Z\hat{X}$ is a P -local martingale. Observe that the process $\hat{X}$ is a semi-martingale with respect to dP . As $d\langle X, M \rangle_t = (1/Z_t)d\langle X, Z \rangle_t$, applying the Ito formula we get

$$Z_t \hat{X}_t = Z_0 \hat{X}_0 + \int_0^t \hat{X}_s dZ_s + \int_0^t Z_s dX_s,$$

which is a P -local martingale. It is immediate to check that $\langle \hat{X} \rangle = \langle X \rangle$ under dP and $d\hat{P}$.

We now point out that the converse is true. More precisely, for any $\hat{P}$ -continuous local martingale X' , $X = X' + \langle X', \hat{M} \rangle$ is a P -continuous local martingale. From Lemma 4, $\hat{X} = X - \langle X, M \rangle$ is a $\hat{P}$ -continuous local martingale, so that $\hat{X} - X' = \langle X', \hat{M} \rangle - \langle X, M \rangle$ is also a $\hat{P}$ -continuous local martingale zero at $t = 0$. Thus $\hat{X} = X'$.

Now if Z' satisfies the (A_2) condition, then $\hat{M} = M - \langle M \rangle$ is a BMO -martingale with respect to $d\hat{P}$ since $\hat{E}[\exp(\langle \hat{M} \rangle_\infty - \langle \hat{M} \rangle_T) | F_T] = E[Z'_T/Z' | F_T]$ for any stopping time T . But we do not know whether $\hat{M}$ is a BMO -martingale with respect to $d\hat{P}$ for any BMO -martingale M .

LEMMA 5. For any $\lambda > 0$ and any stopping time T ,

$$(11) \quad \begin{aligned} \hat{E}[\exp\{\lambda(\langle \hat{M} \rangle_\infty - \langle \hat{M} \rangle_T)\} | F_T] \\ \leq E[\exp\{(2\lambda + 1)(\langle M \rangle_\infty - \langle M \rangle_T)\} | F_T]^{1/2}. \end{aligned}$$

PROOF. Since $\exp(2M_t - 2\langle M \rangle_t)$ is a local martingale, an elementary stopping argument gives

$$E[\exp\{2(M_\infty - M_T) - 2(\langle M \rangle_\infty - \langle M \rangle_T)\} | F_T] \leq 1.$$

As

$$\begin{aligned} (Z/Z_T) \exp\{\lambda(\langle \hat{M} \rangle_\infty - \langle \hat{M} \rangle_T)\} &= \exp\{(M_\infty - M_T) - (\langle M \rangle_\infty - \langle M \rangle_T)\} \\ &\quad \times \exp\left\{\frac{2\lambda + 1}{2}(\langle M \rangle_\infty - \langle M \rangle_T)\right\}, \end{aligned}$$

the Schwarz inequality shows (11).

Thus, according to the Jensen inequality, $\hat{M}$ is a BMO -martingale with respect to $d\hat{P}$ if $E[\exp\{\lambda(\langle M \rangle_\infty - \langle M \rangle_T)\} | F_T] \leq C_\lambda$ for some $\lambda > 1$.

THEOREM 3. Suppose that Z satisfies (B_λ) for all $\lambda > 0$. Then the inequality

$$(12) \quad E[\sup_t |X_t - \langle X, M \rangle_t|^p] \leq C_p E[|X_\infty - \langle X, M \rangle_\infty|^p], \quad 1 < p < \infty$$

is valid for all $L^2(dP)$ -bounded continuous martingale X .

PROOF. The process defined by $\hat{Z}_t = \exp(-\hat{M}_t - (1/2)\langle \hat{M} \rangle_t)$, $t \geq 0$, is a $\hat{P}$ -continuous local martingale. But, as $\hat{M} = M - \langle M \rangle$, we get $\hat{Z}_t = 1/Z_t$ and $\hat{Z} = \hat{Z}_\infty = 1/Z \in L^1(d\hat{P})$, so that $\hat{E}[\hat{Z}|F_t] = \hat{Z}_t$ from (8). That is to say, $\hat{Z}_t$ is $\hat{P}$ -uniformly integrable. As $\hat{Z}d\hat{P} = dP$, we also get from Lemma 5

$$E[\exp\{\lambda(\langle M \rangle_\infty - \langle M \rangle_T)\}|F_T] \leq \hat{E}[\exp\{(2\lambda + 1)(\langle \hat{M} \rangle_\infty - \langle \hat{M} \rangle_T)\}|F_T]^{1/2}.$$

Therefore Z satisfies (B_λ) for all $\lambda > 0$ if and only if $\hat{Z}$ satisfies (B_λ) for all $\lambda > 0$ with respect to $d\hat{P}$. Then, from the proof of Theorem 2, if Z satisfies (B_λ) for all $\lambda > 0$, the inequality

$$(13) \quad E[(X'^*)^p] \leq C_p E[|X'_\infty|^p], \quad 1 < p < \infty$$

is valid for all $\hat{P}$ -uniformly integrable martingale X' . Thus, to prove (12), it suffices to show that $\hat{X} = X - \langle X, M \rangle$ is a $\hat{P}$ -uniformly integrable martingale for every $L^2(dP)$ -bounded continuous martingale X . By the Davis inequality and Lemma 4, we have

$$\frac{1}{4\sqrt{2}}\hat{E}[\hat{X}^*] \leq \hat{E}[\langle X \rangle_\infty^{1/2}].$$

By the Schwarz inequality the right side is smaller than $E[Z^2]^{1/2}E[X_\infty^2]^{1/2}$. As Z satisfies (B_λ) for all $\lambda > 0$, Lemma 1 shows that

$$E[Z^2] \leq E[\exp(2|M_\infty|)] < \infty.$$

Thus for any $L^2(dP)$ -bounded continuous martingale X , $\hat{X}^*$ is $\hat{P}$ -integrable. Consequently, $\hat{X}$ is $\hat{P}$ -uniformly integrable. This completes the proof.

More generally, let X be a P -continuous local martingale. Since there exist stopping times $T_n \uparrow \infty$ a.s., such that for each n the process $(X_{t \wedge T_n})$ stopped at T_n is an $L^2(dP)$ -bounded martingale, we obtain

$$(14) \quad E[\sup_t |X_t - \langle X, M \rangle_t|^p] \leq C_p \sup_n E[|X_{T_n} - \langle X, M \rangle_{T_n}|^p],$$

$$1 < p < \infty$$

from the Fatou lemma.

We can weaken the above assumption on Z as follows.

THEOREM 4. Suppose that Z satisfies (A_∞) . Let $1 < p < \infty$. Then there exists a constant $c_p > 0$ depending only on p such that the inequality

$$(15) \quad E[\sup_t |X_t - c_p \langle X, M \rangle_t|^p] \leq C_p E[|X_\infty - c_p \langle X, M \rangle_\infty|^p]$$

is valid for all $L^2(dP)$ -bounded continuous martingale X .

PROOF. If Z satisfies (A_∞) , then it follows from Theorem 1 that Z

satisfies (B_{λ_0}) for some $\lambda_0 > 0$. Then choose $\lambda > 0$ such that

$$\lambda > \text{Max} \left\{ 1 + \frac{1}{(\sqrt{p} - 1)^2}, 8\lambda_0 \|M\|_{BMO}^2 \right\}$$

and set $N = c_p M$, $c_p = (\lambda_0/\lambda)^{1/2}$. The process N is a *BMO*-martingale. As $\langle N \rangle = c_p^2 \langle M \rangle$, we get

$$\|\sqrt{8}N\|_{BMO}^2 = 8c_p^2 \|M\|_{BMO}^2 = \frac{1}{\lambda} 8\lambda_0 \|M\|_{BMO}^2 < 1.$$

Now we define $Z_t^* = \exp(N_t - (1/2)\langle N \rangle_t)$ and $Z^* = Z_\infty^*$. As $\|\sqrt{8}N\|_{BMO}^2 < 1$, Lemma 2 implies that Z^* satisfies the (B_8) condition. Thus by Lemma 1 the process Z_t^* is an $L^2(dP)$ -bounded martingale, so that $d\hat{P} = Z^*dP$ is a probability measure on Ω . Set $\alpha = (\lambda - 1)/2 > 0$. Then from Lemma 5 we get

$$\begin{aligned} \hat{E}[\exp\{\alpha(\langle \hat{N} \rangle_\infty - \langle \hat{N} \rangle_T)\} | F_T] &\leq E[\exp\{\lambda(\langle N \rangle_\infty - \langle N \rangle_T)\} | F_T]^{1/2} \\ &\leq E[\exp\{\lambda_0(\langle M \rangle_\infty - \langle M \rangle_T)\} | F_T]^{1/2} \end{aligned}$$

where $\hat{N} = N - \langle N \rangle$. The right side is dominated by some constant C because Z satisfies (B_{λ_0}) . As $\alpha > 1/2(\sqrt{p} - 1)^2$, we have

$$\hat{E}\left[\exp\left\{\frac{1}{2(\sqrt{p} - 1)^2}(\langle \hat{N} \rangle_\infty - \langle \hat{N} \rangle_T)\right\} \middle| F_T\right] \leq C.$$

Thus from Lemma 3, $\hat{Z}^*$ satisfies (A_p) with respect to $d\hat{P}$ where $\hat{Z}_t^* = \exp(-\hat{N}_t - (1/2)\langle \hat{N} \rangle_t)$ and $\hat{Z}^* = \hat{Z}_\infty^*$ as before. The process $\hat{Z}_t^*$ is a $\hat{P}$ -uniformly integrable continuous martingale such that $\hat{Z}_t^* = 1/Z_t^*$. Furthermore, one can choose $\varepsilon > 0$ such that $\varepsilon < p - 1$ and $\alpha > 1/2(\sqrt{p - \varepsilon} - 1)^2$. Then $\hat{Z}^*$ satisfies the $(A_{p-\varepsilon})$ condition. Therefore the inequality

$$E[(X'^*)^p] \leq C_p E[|X'_\infty|^p]$$

is valid for all $\hat{P}$ -uniformly integrable martingale X' . And, as the martingale Z_t^* is $L^2(dP)$ -bounded, $\hat{X} = X - \langle X, N \rangle = X - c_p \langle X, M \rangle$ is a $\hat{P}$ -uniformly integrable martingale for any $L^2(dP)$ -bounded continuous martingale X . This completes the proof.

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