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REMARKS ON COMPACT COMPLEX HYPERSURFACES OF COMPACT COMPLEX PARALLELISABLE NILMANIFOLDS

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1. Introduction. Let F be a holomorphic vector bundle over a compact complex manifold M and $\Gamma(F)$ denote the complex vector space of all holomorphic sections of F. A holomorphic vector bundle F over M is called weakly ample if the evaluation map $e_x: \Gamma(F) \to F_x$, which assigns to each section s its value $s(x)$ at x, is surjective at every point $x \in M$. In recent works of Matsushima [3], [4], compact connected Kähler manifolds with weakly ample holomorphic cotangent bundle are studied in connection with manifolds which can be immersed into a complex torus. It is known that if the holomorphic cotangent bundle of a compact complex manifold is weakly ample, so is the cotangent bundle of a compact complex submanifold [3]. In particular, a compact complex submanifold of a compact complex parallelisable manifold has the weakly ample holomorphic cotangent bundle.

In this note we study compact complex hypersurfaces of compact complex parallelisable nilmanifolds as examples of compact complex manifolds with weakly ample cotangent bundle. Let G be a simply connected complex nilpotent Lie group and Γ be a lattice of G, that is, a discrete subgroup of G such that G/Γ is compact. The compact complex manifold G/Γ is called a parallelisable nilmanifold. Let $[G, G]$ be the commutator group of G and $\pi: G \to G/[G, G]$ be the projection. Then G/Γ is the total space of a holomorphic fiber bundle $\pi': G/\Gamma \to (G/[G, G])/\pi(\Gamma)$ with fiber $[G, G]/([G, G] \cap \Gamma)$. We denote by T the complex torus $(G/[G, G])/\pi(\Gamma)$.

THEOREM 1. *Let M be a compact connected complex submanifold of codimension 1 of G/Γ. Let N be the image $\pi'(M)$ of M by the projection $\pi': G/\Gamma \to T$. Then N is a compact connected complex submanifold of codimension 1 of T. Moreover M is the total space of a holomorphic fiber bundle over N with the fiber $F = [G, G]/([G, G] \cap \Gamma)$.*

We shall denote by $H^{1,0}(M)$ the vector space of all holomorphic 1-forms on a compact connected complex manifold M and put $h^{1,0}(M) =$

dim $H^{1,0}(M)$.

THEOREM 2. *Under the same notation as in Theorem* 1, *suppose that* dim $F=1$.
Then

$$h^{1,0}(M)=h^{1,0}(N)+h^{1,0}(F)\ .$$

2. **Proof of Theorem 1.** Let X and Y be connected complex manifolds and $\phi: X \rightarrow Y$ be an onto holomorphic map. Let $\tilde{D}$ be a divisor on Y and D be the induced divisor on X of $\tilde{D}$ by the holomorphic map ϕ (cf. [7] Appendix n° 7).

We recall the following theorem on divisors of a compact complex parallelisable nilmanifold.

THEOREM. *Let D be a positive divisor on G/Γ. Then there exists a positive divisor $\tilde{D}$ on the complex torus T such that the divisor D is the induced divisor of the divisor $\tilde{D}$ by the projection $\pi': G/\Gamma \rightarrow T$.*

PROOF. See [5] Theorem 2.

Since M is a compact connected complex submanifold of codimension 1 of G/Γ, M defines a positive non-singular divisor $D(M)$ on G/Γ. This means that there exist an open covering $\{U_\alpha\}_{\alpha\in A}$ of G/Γ and a family of holomorphic functions $\{F_\alpha\}_{\alpha\in A}$ such that F_α is defined on U_α and if $U_\alpha\cap U_\beta \neq \emptyset$, F_α/F_β and F_β/F_α are both non-vanishing holomorphic functions on $U_\alpha\cap U_\beta$; moreover $U_\alpha\cap M$ coincides with the set $\{z\in U_\alpha | F_\alpha(z)=0\}$ and the differential dF_α is not zero on $U_\alpha\cap M$. By the above Theorem, there is a positive divisor $\tilde{D}$ on T such that the divisor $D(M)$ is the induced divisor of $\tilde{D}$. We may assume that $\tilde{D}$ is given by $\tilde{D}=\{(V_\beta, f_\beta)\}_{\beta\in B}$ and $F_\alpha=f_\alpha\circ\pi$ on U_α by taking suitable refinements of covering. Note that M is the support of the divisor $D(M)$. Since $N=\pi(M)$, N is the support of the divisor $\tilde{D}$ and $M=\pi^{-1}(N)$. Now we claim that N is a complex submanifold of codimension 1 of T. Let w_0 be an arbitrary point of N. Choose a point $z_0\in M$ such that $\pi(z_0)=w_0$. If $w_0\in V_\alpha$ and $z_0\in U_\alpha$, $(dF_\alpha)_{z_0}=(df_\alpha)_{w_0}\cdot(d\pi)_{z_0}$. Since $(dF_\alpha)_{z_0}\neq 0$, $(df_\alpha)_{w_0}\neq 0$. Hence, there exists an open neighborhood V of w_0 such that $V\cap N=\{w\in V | f_\alpha(w)=0\}$ and $(df_\alpha)_w\neq 0$ for any $w\in V\cap N$. Thus N is a complex submanifold of codimension 1 of T. Since $M=\pi^{-1}(N)$, the second assertion is obvious.

3. **Proof of Theorem 2.** Since dim $F=1$, the holomorphic fiber bundle (M, π, N, F) is a principal fiber bundle whose fiber F is a complex torus. By the spectral sequence due to A. Borel ([2] Appendix 2),

$$h^{1,0}(N)\leqq h^{1,0}(M)\leqq h^{1,0}(N)+h^{1,0}(F)\ .$$

Suppose that $h^{1,0}(N)=h^{1,0}(M)$. Since $\pi^*: H^{1,0}(N) \to H^{1,0}(M)$ is an injective homomorphism and N is a Kähler manifold, every holomorphic 1 form on M is d-closed. Let $\{\theta_1, \cdots, \theta_q\}$ be a basis of $H^{1,0}(M)$. Since M has the weakly ample holomorphic cotangent bundle, we can define a hermitian metric g on M by

$$g = \theta_1 \cdot \bar{\theta}_1 + \cdots + \theta_q \cdot \bar{\theta}_q .$$

Since θ_j $(j = 1, \cdots, q)$ are d-closed, g is a Kähler metric on M. We recall a theorem of Blanchard ([1] Chapter II §2).

THEOREM. *Let (M, π, N, F) be a holomorphic fiber bundle. Suppose that the fundamental group $\pi_1(N)$ of N operates trivially on the first real cohomology $H^1(F)$ of F. If M is a Kähler manifold, the transgression which maps $H^1(F)$ into $H^2(N)$ is zero, that is, $b_1(M) = b_1(N)+b_1(F)$ where $b_1(X)$ denotes the first Betti number of X.*

Note that the assumption of Blanchard's theorem is always satisfied if the structure group of a fiber bundle is connected (See [6] Proposition 3, p. 445.). Since M, N and F are Kähler manifolds, we have $h^{1,0}(M) = h^{1,0}(N) + h^{1,0}(F)$ by Blanchard's theorem. This is a contradiction, since $h^{1,0}(F) = 1$. Hence, $h^{1,0}(N) \neq h^{1,0}(M)$, so that $h^{1,0}(M) = h^{1,0}(N) + h^{1,0}(F)$.

4. **An example and remarks.** Let G be a simply connected complex nilpotent Lie group defined by

$$G = \left\{ \left. \begin{pmatrix} 1 & z_1 & \cdots & z_{n-1} & w \\ & 1 & & 0 & y_{n-1} \\ & & \ddots & & \vdots \\ & 0 & & 1 & y_1 \\ & & & & 1 \end{pmatrix} \right| z_j, y_j, w \in \boldsymbol{C},\ j = 1, \cdots, n-1 \right\} \quad \text{for} \quad n \geqq 2$$

and Γ be a lattice of G defined by

$$\Gamma = \left\{ \left. \begin{pmatrix} 1 & a_1 & \cdots & a_{n-1} & c \\ & 1 & & 0 & b_{n-1} \\ & & \ddots & & \vdots \\ & 0 & & 1 & b_1 \\ & & & & 1 \end{pmatrix} \right| a_j, b_j, c \in \boldsymbol{Z} + \sqrt{-1}\boldsymbol{Z},\ j = 1, \cdots, n-1 \right\} .$$

Then G/Γ is a compact complex manifold which satisfies the assumption of Theorem 2.

We remark that there is an example of holomorphic fiber bundle (E, π, B, T) whose fiber is a complex torus and such that $h^{1,0}(E) \neq h^{1,0}(B) +$

$h^{1,0}(T)$. Let E be a Hopf manifold. Then E is a holomorphic fiber bundle over the complex projective space $P^n(C)$ with 1 dimensional complex torus as the standard fiber. It is well-known that $h^{1,0}(M) = 0$, so that $h^{1,0}(M) \neq h^{1,0}(P^n(C)) + h^{1,0}(T)$(cf. [2] Appendix 2).

We finish this paper by a few remarks on $h^{1,0}(N)$. If the Euler number $E(N)$ of N is not zero and if $\dim N \geqq 2$, $h^{1,0}(N) = \dim N + 1$ by Matsushima's theorem ([4] §4). If the Euler number $E(N) = 0$, N is a holomorphic principal fiber bundle over V whose structure group is $\mathrm{Aut}_0(N)$, where V is a complex submanifold of codimension 1 of a complex torus such that the Euler number $E(V) \neq 0$ (cf. [4] Proposition 1 and Theorem 2). Thus $h^{1,0}(N) = h^{1,0}(V) + \dim \mathrm{Aut}_0(N) = \dim N + 1$ if $\dim V \geqq 2$. If $\dim V = 1$, then $h^{1,0}(N) = g + \dim N - 1$ where g is the genus of V.

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