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PLANAR GEODESIC IMMERSIONS

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0. Introduction. Let M^n and $\bar{M}^{n+p}$ be connected complete Riemannian manifolds of dimension n and $n+p$ respectively. An isometric immersion of M^n into $\bar{M}^{n+p}$ is called a *planar geodesic immersion* if every geodesic in M^n is mapped locally into a 2-dimensional totally geodesic submanifold of $\bar{M}^{n+p}$. We can see that such an immersion is an isotropic immersion in the sense of B. O'Neill [10] with parallel second fundamental tensor and vice versa. Planar geodesic immersions into Euclidean space has studied by S. L. Hong [7] who stated that if $f: M^n \to E^{n+p}$ is a planar geodesic immersion, then the sectional curvature of M^n is 1/4-pinched except for the totally geodesic case and moreover if M^n has constant positive sectional curvature, then $f(M^n)$ is an n-dimensional sphere or Ω-sphere which is usually called a Veronese manifold. On the other hand, T. Itoh and K. Ogiue [9] has showed that if $f: M^n(c) \to \bar{M}^{n+p}(\bar{c})$ $(p = n(n+1)/2 - 1)$ is an isotropic immersion with parallel second fundamental tensor and $\bar{c} > c$, then $c = n\bar{c}/2(n+1)$ and the immersion is rigid, where $M^n(c)$ (resp. $\bar{M}^{n+p}(\bar{c})$) denotes a Riemannian manifold of constant curvature c (resp. $\bar{c}$). These results lead the conjecture that if $f: M^n \to \bar{M}^{n+p}(\bar{c})$ is a planar geodesic immersion, then M^n is isometric to a symmetric space of rank one or Euclidean space and the immersion is rigid. In the present paper, we shall give the affirmative answer.

In § 1, basic equations of immersions that we need are given. In § 2, the accurate definition of a planar geodesic immersion and its models for compact symmetric spaces of rank one will be given. As will be shown in the later section, we must construct algebraically minimal immersions of compact symmetric spaces of rank one into spheres which are usually constructed by using eigenfunctions of the Laplacian with respect to the invariant metric (cf. [4] and [15]). However, our construction is due to S. S. Tai [12] who gave examples of tight imbeddings for compact symmetric spaces of rank one (in [12], minimum imbeddings mean tight imbeddings and for the definition, see [12] and [14]). S. Kobayashi and M. Takeuchi [14] obtained tight imbeddings for a certain class of compact symmetric spaces containing spaces of rank one and

proved that the height functions are eigenfunctions of the Laplacian by showing that the mean curvature normal vanishes. In § 2, we show more directly that the height functions of the imbeddings constructed by S. S. Tai are eigenfunctions. We also consider how geodesics are mapped into spheres. In § 3, we obtain various properties of the second fundamental form of a planar geodesic immersion especially in the case where the ambient manifold is a space form. In § 4, we reduce our problem to the minimal, full and planar geodesic immersions of compact symmetric spaces of rank one into spheres. In § 5, we state our main theorems and corollaries.

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1. Preliminaries. Let $f: M^n \to \bar{M}^{n+p}$ be an isometric immersion of an n-dimensional Riemannian manifold M^n into an $(n+p)$-dimensional Riemannian manifold $\bar{M}^{n+p}$. For all local formulas and computations we may assume f as an imbedding and thus, in this section, we identify $x \in M^n$ with $f(x) \in \bar{M}^{n+p}$. The tangent space T_xM^n is identified with a subspace of $T_x\bar{M}^{n+p}$. Letters V, W, X, Y and Z (resp. ξ, η and ζ) will be vectors at x or vector fields on a neighborhood of x tangent (resp. normal) to M^n. If we denote the covariant differentiation of the Riemannian manifold $\bar{M}^{n+p}$ by $\bar{\nabla}$, then we may write

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y) \tag{1.1}$$

where $\nabla_X Y$ and $H(X, Y)$ denote the components of $\bar{\nabla}_X Y$ tangent and normal to M^n respectively. Then ∇ becomes the covariant differentiation of the Riemannian manifold M^n. The symmetric bilinear form H valued in the normal space is called the second fundamental form of the immersion f. If ξ is a normal vector field on a neighborhood of x, then we can also write

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{1.2}$$

where $\nabla^\perp$ is the covariant differentiation with respect to the induced connection in the normal bundle NM which will be called the normal connection. The tangential component $A_\xi X$ is related to the second fundamental form H as follows:

$$\langle A_\xi X, Y \rangle = \langle H(X, Y), \xi \rangle \tag{1.3}$$

for any $Y \in T_xM^n$, where $\langle\ ,\ \rangle$ denotes the inner product of vectors with respect to the Riemannian metric of $\bar{M}^{n+p}$. Thus A_ξ is a symmetric linear transformation of T_xM^n. Given an orthonormal normal frame

$\{\xi_{n+1}, \cdots, \xi_{n+p}\}$, we write $A_\alpha = A_{\xi_\alpha}(\alpha = n+1, \cdots, n+p)$. In the sequel, indices α, β and γ run over the range $\{n+1, \cdots, n+p\}$.

Let $Proj_{TM}$ and $Proj_{NM}$ be the projections of $T_x\bar{M}^{n+p}$ to the tangent space T_xM^n and the normal space N_xM^n respectively. Let ∇' be the covariant differentiation with respect to the induced connection in the direct sum $TM + NM$. We denote curvature tensors for the connections $\bar{\nabla}$, ∇ and $\nabla^\perp$ by $\bar{R}$, R and $R^\perp$ respectively. If we take an orthonormal normal frame $\{\xi_{n+1}, \cdots, \xi_{n+p}\}$, then we have the following structure equations of Gauss, Codazzi and Ricci:

$$(1.4)\quad Proj_{TM}\bar{R}(X, Y)Z = R(X, Y)Z - \sum_\alpha \{\langle A_\alpha Y, Z\rangle A_\alpha X - \langle A_\alpha X, Z\rangle A_\alpha Y\},$$

$$(1.5)\quad Proj_{NM}\bar{R}(X, Y)Z = (\nabla'_X H)(Y, Z) - (\nabla'_Y H)(X, Z),$$

$$(1.6)\quad Proj_{NM}\bar{R}(X, Y)\xi = R^\perp(X, Y)\xi - \sum_\alpha \langle [A_\xi, A_\alpha]X, Y\rangle \xi_\alpha .$$

In later sections, we mainly deal with cases where the ambient manifold is a space form $\bar{M}^{n+p}(\bar{c})$, i.e., a simply connected complete Riemannian manifold with constant sectional curvature $\bar{c}$. Thus we must give basic formulas in those cases. Structure equations (1.4), (1.5) and (1.6) can be written as

$$(1.7)\quad R(X, Y)Z = \bar{c}\{\langle Y, Z\rangle X - \langle X, Z\rangle Y\} + \sum_\alpha \{\langle A_\alpha Y, Z\rangle A_\alpha X - \langle A_\alpha X, Z\rangle A_\alpha Y\},$$

$$(1.8)\quad (\nabla'_X H)(Y, Z) = (\nabla'_Y H)(X, Z),$$

$$(1.9)\quad R^\perp(X, Y)\xi = \sum_\alpha \langle [A_\xi, A_\alpha]X, Y\rangle \xi_\alpha .$$

Equation (1.7) gives a formula for the Ricci curvature tensor S:

$$(1.10)\quad S = \bar{c}(n-1)I + \sum_\alpha (\text{trace } A_\alpha)A_\alpha - \sum_\alpha A_\alpha^2$$

where I is the identity transformation on the tangent space of M^n. Let η be the mean curvature normal defined by

$$(1.11)\quad \eta = \frac{1}{n}\sum (\text{trace } A_\alpha)\xi_\alpha .$$

Then the scalar curvature ρ satisfies

$$(1.12)\quad \rho = \bar{c}n(n-1) + n^2\|\eta\|^2 - \|H\|^2$$

where $\|\eta\|^2 = \sum (\text{trace } A_\alpha)^2/n^2$ and $\|H\|^2 = \sum \text{trace } A_\alpha^2$.

We are now listing up complete totally geodesic submanifolds and totally umbilical submanifolds in space forms. For each real number $\bar{c}$ and each integer $n>1$ there is (up to isometry) exactly one n-dimen-

sional (real) space form of constant curvature $\bar{c}$. Thus we may assume that $(n+p)$-dimensional space forms are

(E) Euclidean space E^{n+p}: $\boldsymbol{R}^{n+p}$ with usual inner product, $\bar{c}=0$;

(S) Euclidean sphere $S^{n+p}(\bar{c})$: $\{(x_1, \cdots, x_{n+p+1}) \in \boldsymbol{R}^{n+p+1}$: $(x_1)^2+\cdots+(x_{n+p+1})^2 = 1/\bar{c}\}$

with the metric induced from E^{n+p+1}, $\bar{c}>0$;

(H) Hyperbolic space $H^{n+p}(\bar{c})$: $\{(x_1, \cdots, x_{n+p+1}) \in \boldsymbol{R}^{n+p+1}$: $(x_1)^2+\cdots+(x_{n+p})^2-(x_{n+p+1})^2 = 1/\bar{c}\}$

with the metric induced from the metric $ds^2=(dx_1)^2+\cdots+(dx_{n+p})^2-(dx_{n+p+1})^2$ in $\boldsymbol{R}^{n+p+1}$, $\bar{c}<0$. The following is a list of n-dimensional complete totally geodesic submanifolds and totally umbilical submanifolds up to congruence in $(n+p)$-dimensional space forms. For the space form (E), we have

(i) Planes E^n: $\{(x_1, \cdots, x_{n+p}) \in E^{n+p}$: $x_{n+1}=\cdots=x_{n+p}=0\}$, $H=0$;

(ii) Spheres $S^n(c)$: $\{(x_1, \cdots, x_{n+p}) \in E^{n+p}$: $(x_1)^2+\cdots+(x_{n+1})^2 = 1/c$, $x_{n+2}=\cdots=x_{n+p}=0\}$.

For the space form (S), we have

(i) Great spheres $S^n(\bar{c})$: $\{(x_1, \cdots, x_{n+p+1}) \in S^{n+p}(\bar{c})$: $x_{n+2}=\cdots=x_{n+p+1}=0\}$, $H=0$;

(ii) Small spheres $S^n(c)$: $\{(x_1, \cdots, x_{n+p+1}) \in S^{n+p}(\bar{c})$: $x_{n+2}=\sqrt{1/\bar{c}-1/c}$, $x_{n+3}=\cdots=x_{n+p+1}=0\}$,

where $0<\bar{c}<c$. For the space form (H), we have also

(i) Great spheres $H^n(\bar{c})$: $\{(x_1, \cdots, x_{n+p+1}) \in H^{n+p}(\bar{c})$: $x_{n+1}=\cdots=x_{n+p}=0\}$, $H=0$;

(ii) Small spheres:

(e) $E^n=\{(x_1, \cdots, x_{n+p+1}) \in H^{n+p}(\bar{c})$: $x_{n+p+1}=x_{n+1}+t$, $x_{n+2}=\cdots=x_{n+p}=0$, $t>0\}$;

(s) $S^n(c)=\{(x_1, \cdots, x_{n+p+1}) \in H^{n+p}(\bar{c})$: $(x_1)^2+\cdots+(x_{n+1})^2=1/c$, $x_{n+2}=\cdots=x_{n+p}=0$, $c>0\}$;

(h) $H^n(c)=\{(x_1, \cdots, x_{n+p+1}) \in H^{n+p}(\bar{c})$: $x_{n+1}=\sqrt{1/\bar{c}-1/c}$, $x_{n+2}=\cdots=x_{n+p}=0$, $\bar{c}<c<0\}$.

Above all, a totally geodesic (resp. umbilical) submanifold in $S^{n+p}(\bar{c})$ is the intersection of an $(n+1)$-dimensional plane passing through the origin (resp. not passing through the origin) in E^{n+p+1} with $S^{n+p}(\bar{c})$ and similarly for $H^{n+p}(\bar{c})$.

2. Planar geodesic immersions and their models. Let $f: M^n \to \bar{M}^{n+p}$ be an isometric immersion and $\sigma: (t_1, t_2) \to M^n$ be arbitrary geodesic in M^n. If there exist an open interval I_t and 2-dimensional totally geodesic submanifold P_t for each $t \in (t_1, t_2)$ such that $t \in I_t \subset (t_1, t_2)$ and $f(\sigma(I_t)) \subset P_t$, then f is called a *planar geodesic immersion.* In the sequel, we assume that $n \geqq 2$, because this definition is worthless when $n = 1$.

Well, we shall construct models of planar geodesic immersions. Let $\boldsymbol{F}$ be the field $\boldsymbol{R}$ of real numbers, the field $\boldsymbol{C}$ of complex numbers or the field $\boldsymbol{Q}$ of quaternions. In a natural way, $\boldsymbol{R} \subset \boldsymbol{C} \subset \boldsymbol{Q}$. The conjugate of each element $x \in \boldsymbol{Q}$ is defined as follows:

$$\bar{x} = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3 \quad \text{for} \quad x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 \in \boldsymbol{Q}$$

where $\{1, e_1, e_2, e_3\}$ is usual basis for $\boldsymbol{Q}$. Define a number d by

$$d = \begin{cases} 1 & \text{if } \boldsymbol{F} = \boldsymbol{R}, \\ 2 & \text{if } \boldsymbol{F} = \boldsymbol{C}, \\ 4 & \text{if } \boldsymbol{F} = \boldsymbol{Q}. \end{cases}$$

Let $\boldsymbol{x}$ be a column vector $(x_i) \in \boldsymbol{F}^{m+1}$ and $\mathfrak{M}(m+1, \boldsymbol{F})$ be the vector space of all $(m+1) \times (m+1)$ matrices over $\boldsymbol{F}$. In this section, we shall make use of the following convention on the range of indices: $1 \leqq i \leqq m+1$, $0 \leqq a \leqq d-1$. Let

$$\mathfrak{H}(m+1, \boldsymbol{F}) = \{A \in \mathfrak{M}(m+1, \boldsymbol{F}): A^* = A\},$$
$$U(m+1, \boldsymbol{F}) = \{U \in \mathfrak{M}(m+1, \boldsymbol{F}): U^* U = I\}$$

where $A^* = {}^t\bar{A}$ and I is the identity matrix. The usual inner product on $\boldsymbol{F}^{m+1} = \boldsymbol{R}^{(m+1)d}$ is given by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \mathrm{Re}\,(\boldsymbol{x}^* \boldsymbol{y}) \quad \text{for} \quad \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{F}^{m+1}$$

where $\mathrm{Re}\,(\boldsymbol{x}^* \boldsymbol{y})$ denotes the real part of $\boldsymbol{x}^* \boldsymbol{y}$. The inner product on $\mathfrak{M}(m+1, \boldsymbol{F}) = \boldsymbol{R}^{(m+1)^2 d}$ is also defined as

$$\langle A, B \rangle = \frac{1}{2} \mathrm{Re}\,\mathrm{trace}\,(AB^*) \quad \text{for} \quad A, B \in \mathfrak{M}(m+1, \boldsymbol{F}).$$

If $A, B \in \mathfrak{H}(m+1, \boldsymbol{F})$, then $\mathrm{trace}\,(AB^*) = \mathrm{trace}\,(AB)$ is real and hence

$$\langle A, B \rangle = \frac{1}{2} \mathrm{trace}\,(AB) \quad \text{for} \quad A, B \in \mathfrak{H}(m+1, \boldsymbol{F}).$$

Let $\boldsymbol{F}P^m$ denote the projective space over $\boldsymbol{F}$. $\boldsymbol{F}P^m$ is considered as the quotient space of the unit $((m+1)d-1)$-dimensional sphere $S^{(m+1)d-1}(1) = \{\boldsymbol{x} \in \boldsymbol{F}^{m+1}: \boldsymbol{x}^* \boldsymbol{x} = 1\}$ obtained by identifying $\boldsymbol{x}$ with $\boldsymbol{x}\lambda$ where

$\lambda \in \boldsymbol{F}$ such that $|\lambda| = 1$. The canonical metric g_0 in $\boldsymbol{FP}^m$ is the invariant metric such that the fibering $\pi: S^{(m+1)d-1}(1) \to \boldsymbol{FP}^m$ is a Riemannian submersion. Thus the sectional curvature of $\boldsymbol{RP}^m$ is 1, the holomorphic sectional curvature of $\boldsymbol{CP}^m$ is 4 and the Q-sectional curvature of $\boldsymbol{QP}^m$ is 4 with respect to the metric g_0 (cf. [8]). Define a map $\tilde{\psi}: S^{(m+1)d-1}(1) \to \mathfrak{H}(m+1, \boldsymbol{F})$ as follows:

$$\tilde{\psi}(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^* = \begin{pmatrix} |x_1|^2 & x_1\bar{x}_2 \cdots x_1\bar{x}_{m+1} \\ \cdots\cdots\cdots\cdots \\ x_{m+1}\bar{x}_1 & x_{m+1}\bar{x}_2 \cdots |x_{m+1}|^2 \end{pmatrix} \tag{2.1}$$

for $\boldsymbol{x} = (x_i) \in S^{(m+1)d-1}(1)$. Then it is easily verified that $\tilde{\psi}$ gives a map $\psi: \boldsymbol{FP}^m \to \mathfrak{H}(m+1, \boldsymbol{F})$ such that $\tilde{\psi} = \psi \circ \pi$. Define a hyperplane $\mathfrak{H}_1(m+1, \boldsymbol{F})$ and a vector subspace $\mathfrak{H}_0(m+1, \boldsymbol{F})$ in $\mathfrak{H}(m+1, \boldsymbol{F})$ by

$$\mathfrak{H}_1(m+1, \boldsymbol{F}) = \{A \in \mathfrak{H}(m+1, \boldsymbol{F}): \text{trace } A = 1\},$$
$$\mathfrak{H}_0(m+1, \boldsymbol{F}) = \{A \in \mathfrak{H}(m+1, \boldsymbol{F}): \text{trace } A = 0\}.$$

Then we have

$$\dim \mathfrak{H}_1(m+1, \boldsymbol{F}) = \dim \mathfrak{H}_0(m+1, \boldsymbol{F}) = \frac{m(m+1)}{2}d + m.$$

Since trace $\tilde{\psi}(\boldsymbol{x})=1$ for any $\boldsymbol{x} \in S^{(m+1)d-1}(1)$, ψ maps $\boldsymbol{FP}^m$ into $\mathfrak{H}_1(m+1, \boldsymbol{F})$.

$U(m+1, \boldsymbol{F})$ can be orthogonally represented on $\mathfrak{H}(m+1, \boldsymbol{F})$ by

$$U(A) = UAU^* \quad \text{for} \quad U \in U(m+1, \boldsymbol{F}), \quad A \in \mathfrak{H}(m+1, \boldsymbol{F}).$$

The 1-dimensional subspace spanned by I is orthogonal to $\mathfrak{H}_1(m+1, \boldsymbol{F})$ and $\mathfrak{H}_0(m+1, \boldsymbol{F})$ and pointwise fixed by the action of $U(m+1, \boldsymbol{F})$. Thus the representation of $U(m+1, \boldsymbol{F})$ on $\mathfrak{H}(m+1, \boldsymbol{F})$ induces orthogonal representations on $\mathfrak{H}_1(m+1, \boldsymbol{F})$ and $\mathfrak{H}_0(m+1, \boldsymbol{F})$ respectively. The following two lemmas are well-known.

LEMMA 2.1. *For each $A \in \mathfrak{H}(m+1, \boldsymbol{F})$, there exists an $U \in U(m+1, \boldsymbol{F})$ such that U^*AU is a diagonal matrix whose elements are real numbers.*

LEMMA 2.2. $\psi(\boldsymbol{FP}^m) = \{A \in \mathfrak{H}_1(m+1, \boldsymbol{F}): A^2 = A\}$.

For later use, we shall prove the following lemma due to S.S. Tai [12].

LEMMA 2.3. *The map ψ is an isometric, full and equivariant imbedding of $\boldsymbol{FP}^m$ into $\mathfrak{H}_1(m+1, \boldsymbol{F})$ where the Riemannian metric of $\boldsymbol{FP}^m$ is g_0.*

PROOF. The map ψ is equivariant, since

$$\psi(U\cdot\pi(\boldsymbol{x})) = \tilde{\psi}(U\boldsymbol{x}) = (U\boldsymbol{x})(U\boldsymbol{x})^* = U\tilde{\psi}(\boldsymbol{x})U^* \quad \text{for any } \boldsymbol{x}\in S^{(m+1)d-1}(1)\,.$$

Take an element $\boldsymbol{x}$ of $S^{(m+1)d-1}(1)$ and let $\boldsymbol{y}$ be an element of $S^{(m+1)d-1}(1)$ which is orthogonal to $\{\boldsymbol{x}\lambda\colon \lambda\in\boldsymbol{F},\ |\lambda|=1\}$. This condition is equivalent to $\boldsymbol{y}^*\boldsymbol{x}=0$. If we identify $\boldsymbol{y}$ with an unit tangent vector at $\boldsymbol{x}$, then $\boldsymbol{y}$ is orthogonal to the fiber passing through $\boldsymbol{x}$. The curve $\boldsymbol{x}\cos t+\boldsymbol{y}\sin t$ is an unit speed geodesic in $S^{(m+1)d-1}(1)$. Then the curve $\pi(\boldsymbol{x}\cos t+\boldsymbol{y}\sin t)$ is an unit speed geodesic tangent to $\pi_*(\boldsymbol{y})$ at $\pi(\boldsymbol{x})$ in $\boldsymbol{F}P^m$. We have now

$$\psi(\pi(\boldsymbol{x}\cos t+\boldsymbol{y}\sin t)) = \boldsymbol{x}\boldsymbol{x}^*\cos^2 t+\boldsymbol{y}\boldsymbol{y}^*\sin^2 t + (\boldsymbol{x}\boldsymbol{y}^*+\boldsymbol{y}\boldsymbol{x}^*)\cos t\,\sin t\,. \tag{2.2}$$

Thus we obtain

$$\begin{aligned}\psi_*(\pi_*(\boldsymbol{y})) &= \frac{d}{dt}(\boldsymbol{x}\boldsymbol{x}^*\cos^2 t+\boldsymbol{y}\boldsymbol{y}^*\sin^2 t+(\boldsymbol{x}\boldsymbol{y}^*\\ &\qquad +\boldsymbol{y}\boldsymbol{x}^*)\cos t\sin t)|_{t=0}\\ &= \boldsymbol{x}\boldsymbol{y}^*+\boldsymbol{y}\boldsymbol{x}^*\end{aligned}$$

which is an element of $\mathfrak{H}(m+1,\boldsymbol{F})$. The square of length of this vector is equal to

$$\|\psi_*(\pi_*(\boldsymbol{y}))\|^2 = \frac{1}{2}\operatorname{trace}((\boldsymbol{x}\boldsymbol{y}^*+\boldsymbol{y}\boldsymbol{x}^*)(\boldsymbol{x}\boldsymbol{y}^*+\boldsymbol{y}\boldsymbol{x}^*)) = 1\,.$$

Therefore we see that ψ is an isometric immersion. If $\tilde{\psi}(\boldsymbol{x})=\tilde{\psi}(\boldsymbol{y})$, then $\boldsymbol{x}\boldsymbol{x}^*=\boldsymbol{y}\boldsymbol{y}^*$ which implies that $\boldsymbol{y}=\boldsymbol{x}\lambda$ for some $\lambda\in\boldsymbol{F}$ such that $|\lambda|=1$ and hence ψ is an imbedding. To prove that ψ is full, we assume that there be a hyperplane P in $\mathfrak{H}_1(m+1,\boldsymbol{F})$ such that $\psi(\boldsymbol{F}P^m)\subset P$. Let N be an unit vector in $\mathfrak{H}(m+1,\boldsymbol{F})$ which is normal to the hyperplane containing P and passing through the origin 0 of $\mathfrak{H}(m+1,\boldsymbol{F})$. Then Lemma 2.1 implies that there is an $U\in U(m+1,\boldsymbol{F})$ such that U^*NU is a diagonal matrix whose i-th diagonal element is $\lambda_i\in\boldsymbol{R}$. From Lemma 2.2, we have

$$\psi(\boldsymbol{F}P^m) = \{U^*AU\colon A\in\mathfrak{H}_1(m+1,\boldsymbol{F}),\ A^2=A\}$$

and since

$$\operatorname{trace}(U^*AUU^*NU) = \operatorname{trace}(AN) = 0 \quad \text{for} \quad A\in\psi(\boldsymbol{F}P^m)\,,$$

we see that U^*NU is orthogonal to the vector space spanned by the

set $\psi(FP^m)$. Noting that $\psi(FP^m)$ contains the matrix E_i whose i-th diagonal element is 1 and the others are 0, we have $N = 0$ which is a contradiction. Therefore ψ must be full. q.e.d.

LEMMA 2.4. *Let* $l_d = m(m+1)d/2 + m - 1$. *Then*

$$\psi(FP^m) \subset S^{l_d+1}(2) \cap \mathfrak{H}_1(m+1, F) = S^{l_d}(2(m+1)/m) .$$

PROOF. For each $A \in \psi(FP^m)$, we obtain $\langle A, A\rangle = 1/2$ and hence

$$\psi(FP^m) \subset S^{l_d+1}(2) \cap \mathfrak{H}_1(m+1, F) .$$

The matrix $I/(m+1)$ is contained in $\mathfrak{H}_1(m+1, F)$. Hence we see that the above intersection is a sphere centered at $I/(m+1)$ with radius $\sqrt{m/2(m+1)}$. q.e.d.

Define $\phi: FP^m \to \mathfrak{H}_0(m+1, F)$ by

$$\phi(x) = \psi(x) - \frac{I}{m+1} \quad \text{for} \quad x \in FP^m .$$

Then Lemmas 2.3 and 2.4 imply that ϕ is an isometric, full and equivariant imbedding into the sphere $S^{l_d}(2(m+1)/m)$ with center 0. Let $A \in \mathfrak{H}_0(m+1, F)$ and h_A be the height function defined over FP^m in the direction A. Then

$$h_A(x) = \langle A, \phi(x)\rangle = \frac{1}{2}\operatorname{trace}(A\phi(x)) \tag{2.3}$$

at $x \in FP^m$. Since ϕ is equivariant, we have

$$h_{U(A)} = h_A \circ U^{-1} \tag{2.4}$$

for every $U \in U(m+1, F)$ and $A \in \mathfrak{H}_0(m+1, F)$. In the following lemma, it can be shown that h_A is an eigenfunction of the Laplacian Δ with respect to g_0 for all $A \in \mathfrak{H}_0(m+1, F)$.

LEMMA 2.5. *For each* $A \in \mathfrak{H}_0(m+1, F)$, *we have*

$$\Delta h_A = 2d(m+1)h_A .$$

PROOF. By virture of Lemmas 2.1 and (2.4), we may assume that A is a diagonal matrix whose i-th diagonal element is λ_i. Since π is a Riemannian submersion, it suffices to compute the Laplacian of $h_A \circ \pi$ on $S^{(m+1)d-1}(1)$. Extend $h_A \circ \pi$ to a homogeneous polynomial $\tilde{h}_A$ on F^{m+1}. From (2.1) and (2.3), we have

$$h_A(\pi(x)) = \frac{1}{2}\operatorname{trace}\left(A\left(\tilde{\psi}(x) - \frac{1}{m+1}I\right)\right) = \frac{1}{2}\sum \lambda_i |x_i|^2$$

where $\boldsymbol{x} \in S^{(m+1)d-1}(1)$ and ${}^t\boldsymbol{x} = (x_1, \cdots, x_{m+1})$. Thus we obtain

$$\tilde{h}_A(\boldsymbol{x}) = \frac{1}{2}\sum \lambda_i |x_i|^2 \quad \text{for any} \quad \boldsymbol{x} \in \boldsymbol{F}^{m+1} .$$

Let Δ^S and Δ^F denote the Laplacians on $S^{(m+1)d-1}(1)$ and $\boldsymbol{F}^{m+1}$ respectively. Then we know

$$\Delta^S(h_A \circ \pi) = \Delta^F \tilde{h}_A + \frac{\partial^2}{\partial r^2}\tilde{h}_A + \{d(m+1) - 1\}\frac{\partial}{\partial r}\tilde{h}_A$$

on $S^{(m+1)d-1}(1)$ where $r = \|\boldsymbol{x}\|$. Since $\sum \lambda_i = 0$, we see that $\tilde{h}_A$ is a harmonic homogeneous polynomial of degree 2 and hence $\Delta^F \tilde{h}_A = 0$. We have also

$$\frac{\partial}{\partial r}\tilde{h}_A = 2(h_A \circ \pi) , \qquad \frac{\partial^2}{\partial r^2}\tilde{h}_A = 2(h_A \circ \pi)$$

on $S^{(m+1)d-1}(1)$. Thus we obtain

$$\Delta^S(h_A \circ \pi) = 2d(m+1)(h_A \circ \pi)$$

which shows our assertion. q.e.d.

In the following theorem and corollary, models for planar geodesic immersions will be given.

THEOREM 1. *Let ϕ be the isometric imbedding explained above. Change the metric g_0 for $g = (2(m+1)/m\tilde{c})\, g_0$ in $\boldsymbol{FP}^m$, so that the sectional curvature of $\boldsymbol{RP}^m$ is $m\tilde{c}/2(m+1)$ and the holomorphic sectional curvature (resp. Q-sectional curvature) of $\boldsymbol{CP}^m$ (resp. $\boldsymbol{QP}^m$) is $2m\tilde{c}/(m+1)$. Making use of ϕ, we obtain an isometric imbedding $f\colon \boldsymbol{FP}^m \to S^{1d}(\tilde{c})$ which is minimal, full and equivariant. This imbedding f is also planar geodesic. Moreover we obtain an isometric immersion $f \circ \pi\colon S^m(c) \to S^{1_1}(\tilde{c})$ with the same property, where $c = m\tilde{c}/2(m+1)$.*

PROOF. By using T. Takahashi's result [13] and Lemma 2.5, we can prove the minimality of f. Thus, for the rest we have only to prove that f is planar geodesic. To prove that, it suffices to show that ψ is planar geodesic. Let σ be arbitrary geodesic in $\boldsymbol{FP}^m$. Then σ is

$$\sigma(t) = \pi(\boldsymbol{x}\cos t + \boldsymbol{y}\sin t) \text{ for some } \boldsymbol{x}, \boldsymbol{y} \in S^{(m+1)d-1}(1)$$

where t is the arc-length parameter. By ψ, σ is mapped to the curve (2.2) in the proof of Lemma 2.3, which can be rewritten as

$$\psi(\sigma(t)) = \frac{1}{2}(\boldsymbol{x}\boldsymbol{x}^* + \boldsymbol{y}\boldsymbol{y}^*) + \frac{1}{2}(\boldsymbol{x}\boldsymbol{x}^* - \boldsymbol{y}\boldsymbol{y}^*)\cos 2t$$

$$+ \frac{1}{2}(\boldsymbol{x}\boldsymbol{y}^* + \boldsymbol{y}\boldsymbol{x}^*)\sin 2t .$$

Therefore $\psi\circ\sigma$ is a circle with center $(1/2)(\boldsymbol{x}\boldsymbol{x}^* + \boldsymbol{y}\boldsymbol{y}^*)$ and radius 1/2. q.e.d.

COROLLARY. *Let ι be a totally geodesic or umbilical immersion of $S^{ld}(\tilde{c})$ into a space form $\bar{M}^{md+p}(\bar{c})$ where $p \geqq (m-1)(md+2)/2$ and $\tilde{c} \geqq \bar{c}$. Then the composite $\iota\circ f$ (resp. $\iota\circ f\circ\pi$) of FP^m (resp. $S^m(c)$) into $\bar{M}^{md+p}(\bar{c})$ is a planar geodesic immersion.*

REMARK 2.1. Let σ be a geodesic in FP^m with respect to the metric g. Then $f\circ\sigma$ is a circle with radius $((m+1)/2m\tilde{c})^{1/2}$.

Next, we shall construct a model for planar geodesic imbeddings of Cayley projective plane $CayP^2$ into a sphere. Let Cay denote the Cayley algebra over $\boldsymbol{R}$. For m and d used in the preceding consideration we promise here to be $m=2$ and $d=8$. Thus indices i runs over the range $\{1, 2, 3\}$ and a over the range $\{0, 1, \cdots, 7\}$. Notice that $l_d=25$. The conjugate of $x \in Cay$ is defined as follows:

$$\bar{x} = x_0 - x_1e_1 - \cdots - x_7e_7 \quad \text{for} \quad x = \sum x_ae_a \in Cay$$

where $\{e_0 = 1, e_1, \cdots, e_7\}$ is the usual base for Cay. The usual inner product $Cay = \boldsymbol{R}^8$ is

$$\langle x, y\rangle = \mathrm{Re}\,(x\bar{y}) = \sum x_ay_a \quad \text{for} \quad x = \sum x_ae_a\,, \quad y = \sum y_ae_a \in Cay$$

and the norm of x is defined as $|x| = \langle x, x\rangle^{1/2}$. Let $\mathfrak{H}(3, Cay)$ be the vector space consisting of 3×3 Hermitian matrices, i.e.,

$$\mathfrak{H}(3, Cay) = \{A \in \mathfrak{M}(3, Cay): A^* = A\}\,.$$

Then $\mathfrak{H}(3, Cay)$ is a Jordan algebra under the multiplication

$$A\circ B = \frac{1}{2}(AB + BA) \quad \text{for} \quad A, B \in \mathfrak{H}(3, Cay)\,.$$

Define an inner product in $\mathfrak{H}(3, Cay) = \boldsymbol{R}^{27}$ by

$$\langle A, B\rangle = \frac{1}{2}\,\mathrm{trace}\,(A\circ B) \quad \text{for} \quad A, B \in \mathfrak{H}(3, Cay).$$

Each element $A \in \mathfrak{H}(3, Cay)$ can be written as

$$A = \begin{pmatrix} \lambda_1 & u_3 & \bar{u}_2 \\ \bar{u}_3 & \lambda_2 & u_1 \\ u_2 & \bar{u}_1 & \lambda_3 \end{pmatrix};\ \lambda_i \in \boldsymbol{R},\ u_i \in Cay\ (i = 1, 2, 3)\,,$$

which will be denoted by $\{\boldsymbol{\lambda}, \boldsymbol{u}\}$. If $A = \{\boldsymbol{\lambda}, \boldsymbol{u}\}$ and $B = \{\boldsymbol{\mu}, \boldsymbol{v}\}$, then the inner product of A and B becomes

$$\langle A, B\rangle = \sum\left(\frac{1}{2}\lambda_i\mu_i + \langle u_i, v_i\rangle\right). \tag{2.5}$$

Let $\mathfrak{H}_1(3, Cay) = \{A \in \mathfrak{H}(3, Cay)\colon \text{trace } A = 1\}$ and $\mathfrak{H}_0(3, Cay) = \{A \in \mathfrak{H}(3, Cay)\colon \text{trace } A = 0\}$. Then the Cayley projective plane $CayP^2$ is defined as the subset of $\mathfrak{H}_1(3, Cay)$:

$$CayP^2 = \{A \in \mathfrak{H}_1(3, Cay)\colon A^2 = A\} .$$

The exceptional Lie group F_4 can be defined as the group of automorphisms of the Jordan algebra $\mathfrak{H}(3, Cay)$. Let E_i be the diagonal matrix whose i-th diagonal element is 1 and the others are 0 for each i. The following lemmas are well-known (cf. [6]).

LEMMA 2.6. *The exceptional Lie group F_4 preserves the inner product of $\mathfrak{H}(3, Cay)$.*

LEMMA 2.7. *For each $A \in \mathfrak{H}(3, Cay)$, there exists $\theta \in F_4$ such that $\theta(A)$ is a diagonal matrix, i.e., $\theta(A) = \sum \lambda_i E_i$, $\lambda_i \in \boldsymbol{R}$.*

LEMMA 2.8. $Spin\,(9) \approx \{\theta \in F_4\colon \theta(E_2) = E_2\}$.

LEMMA 2.9. $CayP^2 = \{\theta(E_2)\colon \theta \in F_4\}$.

Lemma 2.9 means that F_4 leaves $CayP^2$ invariant and acts on it transitively. Thus we see from Lemma 2.8 that $CayP^2 = F_4/Spin\,(9)$. Moreover, since F_4 leaves the 1-dimensional subspace spanned by I pointwise fixed which is orthogonal to $\mathfrak{H}_0(3, Cay)$, F_4 has an orthogonal representation on $\mathfrak{H}_0(3, Cay)$.

Let ψ be the inclusion of $CayP^2$ into $\mathfrak{H}_1(3, Cay)$ and let $\phi\colon CayP^2 \to \mathfrak{H}_0(3, Cay)$ be the imbedding such that $\phi(A) = A - I/3$ for any $A \in CayP^2$. Using the similar method to Lemmas 2.3 and 2.4, we have

LEMMA 2.10. *The imbedding $\phi\colon CayP^2 \to \mathfrak{H}_0(3, Cay)$ is full and equivariant, where $\phi(CayP^2) \subset S^{25}(3)$ holds.*

Let $T_{E_2}(CayP^2)$ denote the tangent space of $CayP^2$ at E_2. $T_{E_2}(CayP^2)$ is the set $\{X \in \mathfrak{H}(3, Cay)\colon X \circ E_2 + E_2 \circ X = X\}$. If $X = \{\lambda, \boldsymbol{u}\}$ is a tangent vector of $CayP^2$ at E_2, then the straightforward computation shows that $\lambda = 0$ and $u_2 = 0$, from which we find

$$T_{E_2}(CayP^2) = \{X \in \mathfrak{H}_0(3, Cay)\colon X = \{0, (u_1, 0, u_3)\},\ u_1,\ u_3 \in Cay\} . \tag{2.6}$$

Identifying $\{0, (u_1, 0, u_3)\} \in T_{E_2}(CayP^2)$ with $(u_1, u_3) \in Cay \times Cay = \boldsymbol{R}^{16}$, the induced metric g_0 on $CayP^2$ by the inclusion ψ coincides with the usual metric in $\boldsymbol{R}^{16}$ at E_2. In fact we have from (2.5)

$$g_0(X, Y) = \langle X, Y\rangle = \langle u_1, v_1\rangle + \langle u_3, v_3\rangle$$

for $X = (u_1, u_3)$ and $Y = (v_1, v_3)$. Thus, in the sequel, we continue the above identification. Here we note that the induced metric g_0 is an

invariant metric on $CayP^2$ because of the fact that ψ is equivariant. Let $N_{E_2}(CayP^2)$ be the normal space in $\mathfrak{F}_0(3, Cay)$ with respect to ϕ. Then it is easily verified that $N_{E_2}(CayP^2)$ is given by

$$N_{E_2}(CayP^2) = \{\xi \in \mathfrak{F}_0(3, Cay): \xi = \{\lambda, (0, u_2, 0)\}, \quad \lambda \in \boldsymbol{R}^3,\ u_2 \in Cay\}. \tag{2.7}$$

Well, we shall find two geodesics in $CayP^2$ with respect to the metric g_0 which intersect orthogonally at E_2. Let J_1, J_2 and J_3 be defined respectively as

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $E_{1,a} = \{0, (e_a, 0, 0)\}$, $E_{2,a} = \{0, (0, e_a, 0)\}$ and $E_{3,a} = \{0, (0, 0, e_a)\}$. Then $E_i (i = 1, 2, 3)$ and $E_{i,a} (i = 1, 2, 3;\ a = 0, 1, \cdots, 7)$ form a basis for $\mathfrak{F}(3, Cay)$. Define $\hat{J}_i$ by

$$\hat{J}_i(A) = [J_i, A] \equiv J_iA - AJ_i \quad \text{for any} \quad A \in \mathfrak{F}(3, Cay),$$

which is a linear transformation of $\mathfrak{F}(3, Cay)$. By straightforward computation, we have

$$\begin{cases} \hat{J}_i(E_i) = 0, \quad \hat{J}_i(E_{i+1}) = E_{i,0}, \quad \hat{J}_i(E_{i+2}) = -E_{i,0}, \\ \hat{J}_i(E_{i,0}) = 2(E_{i+2} - E_{i+1}), \quad \hat{J}_i(E_{i,a}) = 0 \quad \text{for} \quad a \neq 0, \\ \hat{J}_i(E_{i+1,0}) = -E_{i+2,0}, \quad \hat{J}_i(E_{i+1,a}) = E_{i+2,a} \quad \text{for} \quad a \neq 0, \\ \hat{J}_i(E_{i+2,0}) = E_{i+1,0}, \quad \hat{J}_i(E_{i+2,a}) = -E_{i+1,a} \quad \text{for} \quad a \neq 0, \end{cases} \quad (\text{mod } 3). \tag{2.8}$$

The Lie algebra of F_4 denoted by $\mathfrak{f}_4$ consists of derivations of the Jordan algebra $\mathfrak{F}(3, Cay)$ and the Lie algebra $\mathfrak{b}_4$ of $Spin\ (9)$ is the subset $\{\delta \in \mathfrak{f}_4: \delta(E_2) = 0\}$. Using (2.8), it can be shown that $\hat{J}_i (i = 1, 2, 3)$ is contained in $\mathfrak{f}_4$, but the author can not examine directly for $\hat{J}_1$ and $\hat{J}_3$ to be orthogonal to $\mathfrak{b}_4$ with respect to the Killing form of $\mathfrak{f}_4$. However we can verify that $\sigma_1(t) = (\exp t\hat{J}_1)(E_2)$ and $\sigma_3(t) = (\exp t\hat{J}_3)(E_2)$ are geodesics in $CayP^2$.

LEMMA 2.11. *The above curves σ_1 and σ_3 in $CayP^2$ are unit speed geodesics with respect to the metric g_0 and can be written as*

$$\begin{aligned} \sigma_1(t) &= \frac{1}{2}(E_2 + E_3) + \frac{1}{2}(\sin 2t)E_{1,0} + \frac{1}{2}(\cos 2t)(E_2 - E_3) \\ \sigma_3(t) &= \frac{1}{2}(E_2 + E_1) + \frac{1}{2}(\sin 2t)E_{3,0} + \frac{1}{2}(\cos 2t)(E_2 - E_1). \end{aligned} \tag{2.9}$$

These geodesics intersect orthogonally at E_2. We have also

$$\dot{\sigma}_1(0) = (1, 0), \quad \dot{\sigma}_3(0) = (0, 1), \quad H((1, 0), (1, 0)) = \{(0, -2, 2), 0\}, \tag{2.10}$$
$$H((0, 1), (0, 1)) = \{(2, -2, 0), 0\}$$

where H is the second fundamental form of the imbedding ϕ.

PROOF. We shall prove (2.9) for only σ_1, since a similar computation shows (2.9) for σ_3. Thus we must calculate $(t\hat{J}_1)^j(E_2)/j!$ for any $j = 0, 1, 2, \cdots$. If $j = 0$, then this is equal to E_2. By the induction, we can easily prove from (2.8) that if $j = 2k - 1$, then

$$\frac{(t\hat{J}_1)^j(E_2)}{j!} = \frac{1}{2}(-1)^{k-1}\frac{(2t)^{2k-1}}{(2k-1)!}E_{1,0} \quad \text{for} \quad k = 1, 2, \cdots,$$

and if $j = 2k$, then

$$\frac{(t\hat{J}_1)^j(E_2)}{j!} = \frac{1}{2}(-1)^{k-1}\frac{(2t)^{2k}}{(2k)!}(E_3 - E_2) \quad \text{for} \quad k = 1, 2, \cdots.$$

Therefore we obtain

$$\begin{aligned}(\exp t\hat{J}_1)(E_2) &= \sum_{j=0}^{\infty}\frac{(t\hat{J}_1)^j(E_2)}{j!} \\ &= E_2 + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2t)^{2k-1}}{(2k-1)!}E_{1,0} + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2t)^{2k}}{(2k)!}(E_3 - E_2) \\ &= \frac{1}{2}(E_2 + E_3) + \frac{1}{2}(\sin 2t)E_{1,0} + \frac{1}{2}(\cos 2t)(E_2 - E_3).\end{aligned}$$

The velocity vectors $\dot{\sigma}_1$ and $\dot{\sigma}_3$ are given by

$$\begin{aligned}\dot{\sigma}_1(t) &= (\cos 2t)E_{1,0} - (\sin 2t)(E_2 - E_3), \\ \dot{\sigma}_3(t) &= (\cos 2t)E_{3,0} - (\sin 2t)(E_2 - E_1)\end{aligned}$$

because of (2.9). Since $E_{1,0}$, $E_{3,0}$ and $E_2 - E_3$ (or $E_2 - E_1$) are orthonormal vectors in $\mathfrak{H}(3, Cay)$, it follows that $\|\dot{\sigma}_1\| = \|\dot{\sigma}_3\| = 1$ and these two curves intersect orthogonally at E_2.

Next, we shall prove that σ_1 and σ_3 are geodesics in $CayP^2$. Let D be the covariant differentiation in $\mathfrak{H}_0(3, Cay)$ with respect to $\langle\ ,\ \rangle$ and ∇ be the covariant differentiation in $CayP^2$ with respect to the induced metric g_0. Then we have from (1.1) and the equations above

$$D_{\dot{\sigma}_1(0)}\dot{\sigma}_1 = \nabla_{\dot{\sigma}_1(0)}\dot{\sigma}_1 + H(\dot{\sigma}_1(0), \dot{\sigma}_1(0)) = 2(E_3 - E_2).$$

Similarly we have

$$\nabla_{\dot{\sigma}_3(0)}\dot{\sigma}_3 + H(\dot{\sigma}_3(0), \dot{\sigma}_3(0)) = 2(E_1 - E_2).$$

On the other hand, (2.7) implies that $E_3 - E_2$ and $E_1 - E_2$ are normal vectors at E_2. Thus we have $\nabla_{\dot{\sigma}_1(0)}\dot{\sigma}_1 = \nabla_{\dot{\sigma}_3(0)}\dot{\sigma}_3 = 0$. We have also

$$H(\dot{\sigma}_1(0), \dot{\sigma}_1(0)) = 2(E_3 - E_2), \quad H(\dot{\sigma}_3(0), \dot{\sigma}_3(0)) = 2(E_1 - E_2) .$$

Since curves σ_1 and σ_3 are orbits of 1-parameter groups of isometry passing through E_2, σ_1 and σ_3 are geodesics in $CayP^2$ with respect to g_0. q.e.d

In the following lemma, the maximal sectional curvature of $CayP^2$ (with respect to g_0) is given, which will be called *C-sectional curvature* in this paper.

LEMMA 2.12. *The C-sectional curvature of $CayP^2$ with respect to the metric g_0 is equal to* 4.

PROOF. Using the equation (1.7), we have

$$s((1, 0), (0, 1)) = \langle H((1, 0), (1, 0)), H((0, 1), (0, 1))\rangle - \|H((1, 0), (0, 1))\|^2$$

where $s((1, 0), (0, 1))$ denotes the sectional curvature corresponding to the plane section spanned by tangent vectors (1, 0) and (0, 1). On the other hand, since ϕ is equivariant and the linear isotropy group acts transitively on a hypersphere of $T_{E_2}(CayP^2)$, we see that ϕ is isotropic. By B. O'Neill [10] Lemma 2 (cf. § 3) and (2.10), we have

$$\langle H((1, 0), (1, 0)), H((0, 1), (0, 1))\rangle + 2\|H((1, 0), (0, 1))\|^2 = \|\{(0, -2, 2), 0\}\|^2 = 4 .$$

Thus we obtain

$$s((1, 0), (0, 1)) = \frac{3}{2}\langle\{(0, -2, 2), 0\}, \{(2, -2, 0), 0\}\rangle - 2 = 1 .$$

However this is the minimal sectional curvature. In fact, R.B. Brown and A. Gray [2] showed that the sectional curvature corresponding to the plane section spanned by orthonormal vectors (1, 0) and (u_1, u_3) is equal to $c(|u_1|^2 + |u_3|^2/4)$ for some $c \in \boldsymbol{R}$. Thus we have $c = 4$. q.e.d.

Let h_A be the height function defined over $CayP^2$ in the direction $A \in \mathfrak{H}_0(3, Cay)$. Since ϕ is equivariant, we have

$$h_{\theta(A)} = h_A \circ \theta^{-1} \tag{2.11}$$

for any $\theta \in F_4$ and $A \in \mathfrak{H}_0(3, Cay)$. Let F_4 act on the space $C^\infty(CayP^2)$ of all C^∞ functions over $CayP^2$ via $\theta \cdot f = f \circ \theta^{-1}$. Equation (2.11) means that the subspace $\{h_A : A \in \mathfrak{H}_0(3, Cay)\}$ of $C^\infty(CayP^2)$ is invariant by this action of F_4. Define $\tilde{\phi}$: $\mathfrak{H}_0(3, Cay) \to C^\infty(CayP^2)$ by $\tilde{\phi}(A) = h_A$. Then $\tilde{\phi}$ is

injective because ϕ is full. Since the representation of F_4 on $\mathfrak{H}_0(3, Cay)$ is irreducible, using [15] Corollary 7.2, we see that $\{h_A: A \in \mathfrak{H}_0(3, Cay)\}$ is contained in the eigenspace V_λ corresponding to some eigenvalue λ of the Laplacian with respect to g_0. On the other hand, [1] Proposition C. I. 8 states that the representation of F_4 on V_λ is irreducible and hence $\tilde{\phi}$ is a linear isomorphism to V_λ. From T. Takahashi's result [13], we obtain $\lambda = 48$.

We are now giving models for planar geodesic immersions of $CayP^2$ into spheres in the following theorem and corollary.

THEOREM 2. *Let ϕ be the isometric imbedding explained above. Change the metric g_0 for $g = (3/\tilde{c})g_0$ in $CayP^2$, so that the C-sectional curvature is $4\tilde{c}/3$. Making use of ϕ, we obtain an isometric imbedding $f: CayP^2 \to S^{25}(\tilde{c})$ which is minimal, full and equivariant. This imbedding is also planar geodesic.*

PROOF. It suffices to show that the inclusion ψ is planar geodesic. This will be easily verified from (2.9) and the fact that ψ is equivariant. Indeed, σ_1 is a circle with center $(E_2 + E_3)/2$ and with radius 1/2. q.e.d.

COROLLARY. *Let ι be a totally geodesic or umbilical immersion of $S^{25}(\tilde{c})$ into a space form $\bar{M}^{16+p}(\bar{c})$ where $p \geqq 9$ and $\tilde{c} \geqq \bar{c}$. Then the composite $\iota \circ f: CayP^2 \to \bar{M}^{16+p}(\bar{c})$ is a planar geodesic immersion.*

REMARK 2.2. Let σ be a geodesic in $CayP^2$ with respect to the metric g. Then $f \circ \sigma$ is a circle with radius $(3/4\tilde{c})^{1/2}$.

REMARK 2.3. Let M be FP^m or $CayP^2$ and let V_λ be the eigenspace with eigenvalue λ of the Laplacian with respect to g_0 where $\lambda = 2d \times (m+1)$ or $\lambda = 48$ according as $M = FP^m$ or $M = CayP^2$. We regard V_λ (with the global inner product) as an Euclidean space. Let dim $V_\lambda = p_\lambda + 1$ and $\{f_1, \cdots, f_{p_\lambda+1}\}$ be an orthonormal basis for V_λ. Define $f': M \to V_\lambda$ via $f'(x) = (f_1(x), \cdots, f_{p_\lambda+1}(x))$. Then the image is contained in a sphere S^{p_λ}. The map f' is an immersion and, suitably changing the metric g_0 on M, we have an isometric minimal immersion of M into a sphere S^{p_λ} (in detail, see [4], [15]). This construction of minimal immersions into spheres coincides with our construction. In fact, the map $\tilde{\phi}: \mathfrak{H}_0(m+1, F) \to V_{2d(m+1)}$ (resp. $\mathfrak{H}_0(3, Cay) \to V_{48}$) defined by $\tilde{\phi}(A) = h_A$ is a homothety because of the irreducibility of the representation of $U(m+1, F)$(resp. F_4) on $\mathfrak{H}_0(m+1, F)$(resp. $\mathfrak{H}_0(3, Cay)$).

3. Properties of the second fundamental form. Let M^n and $\bar{M}^{n+p}$ be connected complete Riemannian manifolds of dimension n and $n+p$ respectively. Let $f: M^n \to \bar{M}^{n+p}$ be a planar geodesic immersion. Then we have

LEMMA 3.1. (S. L. Hong). *If X and Y are orthonormal vectors tangent to M^n, then*

$$\langle H(X, X), H(X, Y)\rangle = 0 . \tag{3.1}$$

PROOF. We may assume $H(X, X) \neq 0$. Let $\sigma: (-t_1, t_1) \to M^n$ be an unit speed geodesic such that $\dot{\sigma}(0) = X$ and $f(\sigma(-t_1, t_1)) \subset P$ where P is a 2-dimensional totally geodesic submanifold in $\bar{M}^{n+p}$. Then, from (1.1), we have

$$\bar{\nabla}_{f\dot{\sigma}} f\dot{\sigma} = H(\dot{\sigma}, \dot{\sigma}) . \tag{3.2}$$

On the other hand, regarding $f\circ\sigma$ as a curve in P, we have $\bar{\nabla}_{f\dot{\sigma}} f\dot{\sigma} \in T_\sigma P$, because P is a totally geodesic submanifold in $\bar{M}^{n+p}$. If we take t_2 small enough, then $H(\dot{\sigma}, \dot{\sigma}) \neq 0$ on $(-t_2, t_2)$. Thus $f\dot{\sigma}$ and $H(\dot{\sigma}, \dot{\sigma})$ span $T_\sigma P$ on $(-t_2, t_2)$. Since $\bar{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})) \in T_\sigma P$, we can write

$$\bar{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})) = uf(\dot{\sigma}) + vH(\dot{\sigma}, \dot{\sigma}) \tag{3.3}$$

for smooth functions $u = u(t)$ and $v = v(t)$ defined on $(-t_2, t_2)$. Extending Y to a vector field Z along σ tangent to M^n, we thus have

$$\begin{aligned}\langle H(X, X), H(X, Y)\rangle &= \langle H(\dot{\sigma}, \dot{\sigma}), H(\dot{\sigma}, Z)\rangle|_{t=0} = \langle H(\dot{\sigma}, \dot{\sigma}), \bar{\nabla}_{f\dot{\sigma}} fZ\rangle|_{t=0} \\ &= -\langle \bar{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})), fZ\rangle|_{t=0} = 0 . \qquad \text{q.e.d.}\end{aligned}$$

The equation (3.1) is equivalent to the condition that f is isotropic, i.e.,

$$\|H(X, X)\|^2 = \lambda^2 \tag{3.4}$$

for all unit vector X tangent to M^n where λ is a function on M^n.

LEMMA 3.2. *Let X, Y be orthonormal vectors tangent to M^n and $\bar{s}(X, Y)$ (resp. $s(X, Y)$) denote the sectional curvature of $\bar{M}^{n+p}$ (resp. M^n) corresponding to the plane section spanned by X and Y. Then*

$$\langle H(X, X), H(Y, Y)\rangle + 2\|H(X, Y)\|^2 = \lambda^2 , \tag{3.5}$$

$$3\|H(X, Y)\|^2 + s(X, Y) - \bar{s}(X, Y) = \lambda^2 , \tag{3.6}$$

$$n^2\|\eta\|^2 + 2\|H\|^2 = n(n + 2)\lambda^2 . \tag{3.7}$$

PROOF. Equations (3.5) and (3.6) are due to B. O'Neill [10]. Choosing an orthonormal basis $\{X_1, \cdots, X_n\}$ on the tangent space T_xM^n, we have from (3.6)

$$\begin{aligned}n(n-1)\lambda^2 &= 3\Big(\sum_{i,j=1}^{n}\|H(X_i, X_j)\|^2 - \sum_{i=1}^{n}\|H(X_i, X_i)\|^2\Big) \\ &\quad + \sum_{i\neq j}(s(X_i, X_j) - \bar{s}(X_i, X_j)) \\ &= 3(\|H\|^2 - n\lambda^2) + \rho - \sum_{i\neq j}\bar{s}(X_i, X_j) .\end{aligned}$$

On the other hand, (1.4) implies

$$\sum_{i\neq j} \bar{s}(X_i, X_j) = \rho - n^2\|\eta\|^2 + \|H\|^2 .$$

Thus we obtain (3.7). q.e.d.

There is a situation under which the equation (1.8) holds although $\bar{M}^{n+p}$ is not a space form, for example, complex submanifold in a complex space form. Under such situation we have

LEMMA 3.3. *If the equation* (1.8) *holds, then* λ *is constant.*

PROOF. Let x be arbitrary fixed point of M^n and X be any unit vector at x tangent to M^n. Take the normal coordinate neighborhood around x in M^n and an unit vector Y at x tangent to M^n which is orthogonal to X. Let σ be the unit speed geodesic in M^n such that $\sigma(0) = x$ and $\dot{\sigma}(0) = Y$. Assume $H(Y, Y) \neq 0$. Then we have the equation (3.3) on a small open interval containing 0. If we parallel translate X and Y along the unique geodesic from x to each point in the normal coordinate neighborhood, then we obtain locally defined vector fields $\tilde{X}$ and $\tilde{Y}$ extending X and Y respectively such that $\nabla_X\tilde{X}=\nabla_Y\tilde{Y}=\nabla_X\tilde{Y}=\nabla_Y\tilde{X}=0$ at x. We find

$$\begin{aligned} X\cdot\lambda^2 &= X\cdot\langle H(\tilde{Y}, \tilde{Y}), H(\tilde{Y}, \tilde{Y})\rangle = 2\langle\bar{\nabla}_X(H(\tilde{Y}, \tilde{Y})), H(Y, Y)\rangle \\ &= 2\langle\nabla^{\perp}_X(H(\tilde{Y}, \tilde{Y})), H(Y, Y)\rangle = 2\langle(\nabla'_X H)(Y, Y), H(Y, Y)\rangle . \end{aligned}$$

Using the equations (1.8) and (3.3), we have

$$\begin{aligned} X\cdot\lambda^2 &= 2\langle(\nabla'_Y H)(X, Y), H(Y, Y)\rangle = 2\langle\nabla^{\perp}_Y(H(\tilde{X}, \tilde{Y})), H(Y, Y)\rangle \\ &= 2\langle\bar{\nabla}_Y(H(\tilde{X}, \tilde{Y})), H(Y, Y)\rangle = -2\langle H(X, Y), \bar{\nabla}_Y(H(\tilde{Y}, \tilde{Y}))\rangle \\ &= -2\langle H(X, Y), u(0)fY + v(0)H(Y, Y)\rangle . \end{aligned}$$

Thus (3.1) implies that $X\cdot\lambda^2 = 0$. If $H(Y, Y) = 0$, then $\lambda(x) = 0$ and so clearly $X\cdot\lambda^2 = 0$. q.e.d.

LEMMA 3.4. *If the equation* (1.8) *holds, then the second fundamental form is parallel, i.e.,* $\nabla' H = 0$. *We have also*

$$\mathfrak{S}\sum_{\alpha}\langle A_\alpha X, Y\rangle A_\alpha Z = \lambda^2\mathfrak{S}\langle X, Y\rangle Z \tag{3.8}$$

where $\mathfrak{S}$ *denotes the cyclic sum with respect to* X, Y *and* Z.

PROOF. If $\lambda = 0$, then $H = 0$. Thus we may assume $\lambda \neq 0$. Let X be any unit vector at a point x tangent to M^n and σ be the unit speed geodesic such that $\sigma(0) = x$ and $\dot{\sigma}(0) = X$. Now we have equations (3.2) and (3.3). Since $\langle H(\dot{\sigma}, \dot{\sigma}), H(\dot{\sigma}, \dot{\sigma})\rangle = \lambda^2$ is constant, we find

$$\lambda^2 v = \langle \bar{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})), H(\dot{\sigma}, \dot{\sigma})\rangle = 0 .$$

We also have

$$u = \langle \bar{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})), f\dot{\sigma}\rangle = -\langle H(\dot{\sigma}, \dot{\sigma}), \bar{\nabla}_{f\dot{\sigma}} f\dot{\sigma}\rangle = -\lambda^2 .$$

Therefore (3.3) reduces to

$$\bar{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})) = -\lambda^2 f\dot{\sigma} . \tag{3.9}$$

On the other hand, by (1.2) we obtain

$$\bar{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})) = -fA_{H(\dot{\sigma},\dot{\sigma})}\dot{\sigma} + \nabla^{\perp}_{\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})) ,$$

from which

$$(\acute{\nabla}_X H)(X, X) = (\acute{\nabla}_{\dot{\sigma}} H)(\dot{\sigma}, \dot{\sigma})|_{t=0} = \nabla^{\perp}_{\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma}))|_{t=0} = 0$$

and

$$\sum_{\alpha} \langle A_\alpha X, X\rangle A_\alpha X = A_{H(\dot{\sigma},\dot{\sigma})}\dot{\sigma}|_{t=0} = \lambda^2 X .$$

Since the equation (1.8) holds, we see that $\nabla' H = 0$. The second equation above is equivalent to $\sum_\alpha \langle A_\alpha X, X\rangle A_\alpha X = \lambda^2\langle X, X\rangle X$ for any vector X tangent to M^n. Symmetrizing this equation, we obtain (3.8). q.e.d.

In the sequel, the ambient space $\bar{M}^{n+p}$ will be a space form $\bar{M}^{n+p}(\bar{c})$ with curvature $\bar{c}$. The Laplacian of the second fundamental form is given by

$$\Delta H_{ij}{}^{\alpha} = \acute{\nabla}^k \acute{\nabla}_k H_{ij}{}^{\alpha}$$

in terms of local coordinates. In the following lemma, we make use of the formula

$$\begin{aligned} \Delta A_\alpha &= (\acute{\nabla}^k \acute{\nabla}_k A_i{}^j{}_\alpha) \\ &= \acute{\nabla}\acute{\nabla}(\text{trace } A_\alpha) + \bar{c}nA_\alpha - \bar{c}(\text{trace } A_\alpha)I \\ &\quad + \sum_\beta (\text{trace } A_\beta)A_\beta A_\alpha - \sum_\beta (\text{trace } A_\alpha A_\beta)A_\beta \\ &\quad + 2\sum_\beta A_\beta A_\alpha A_\beta - \sum_\beta A_\beta^2 A_\alpha - \sum_\beta A_\alpha A_\beta^2 \end{aligned} \tag{3.10}$$

which was calculated in [3].

LEMMA 3.5. *Let $\bar{M}^{n+p}$ be a space form with curvature $\bar{c}$. Then f is pseudo umbilical, i.e.,*

$$A_\eta = \|\eta\|^2 I . \tag{3.11}$$

In particular, if $-\lambda^2 = \bar{c} \leqq 0$, then f is a totally geodesic immersion into an Euclidean space (i.e., (E) (i) exhibited in §1) or a totally um-

bilical immersion of an Euclidean space into a hyperbolic space (*i.e.*, (H) (ii) (e)) *according as* $\bar{c} = 0$ *or* $\bar{c} < 0$.

PROOF. By means of Lemma 3.4, (3.10) can be written as

$$0 = \bar{c}nA_\alpha - \bar{c}(\operatorname{trace} A_\alpha)I + \sum_\beta (\operatorname{trace} A_\beta)A_\beta A_\alpha - \sum_\beta (\operatorname{trace} A_\alpha A_\beta)A_\beta + 2\sum_\beta A_\beta A_\alpha A_\beta - \sum_\beta A_\beta^2 A_\alpha - \sum_\beta A_\alpha A_\beta^2 . \tag{3.12}$$

Let x be arbitrary fixed point and $\{X_1, \cdots, X_n\}$ be an orthonormal basis for T_xM. From (3.8) we have

$$\sum_\alpha \sum_i \{\langle A_\alpha X_i, X_i\rangle A_\alpha Z + 2\langle A_\alpha X_i, Z\rangle A_\alpha X_i\} = \lambda^2 \sum_i \{\langle X_i, X_i\rangle Z + 2\langle X_i, Z\rangle X_i\}$$

which shows

$$\sum_\beta \{(\operatorname{trace} A_\beta)A_\beta + 2A_\beta^2\} = \lambda^2(n+2)I . \tag{3.13}$$

Putting $Y = A_\beta X_i$, $X = X_i$ in (3.8), we have

$$\sum_\alpha \sum_i \{\langle A_\alpha X_i, A_\beta X_i\rangle A_\alpha Z + \langle A_\alpha A_\beta X_i, Z\rangle A_\alpha X_i + \langle A_\alpha Z, X_i\rangle A_\alpha A_\beta X_i\} = \lambda^2 \sum_i \{\langle X_i, A_\beta X_i\rangle Z + \langle A_\beta X_i, Z\rangle X_i + \langle Z, X_i\rangle A_\beta X_i\}$$

which implies

$$\sum_\beta \{(\operatorname{trace} A_\alpha A_\beta)A_\beta + 2A_\beta A_\alpha A_\beta\} = \lambda^2\{(\operatorname{trace} A_\alpha)I + 2A_\alpha\} . \tag{3.14}$$

Since the mean curvature normal η is parallel with respect to the normal connection $\nabla^\perp$, equation (1.9) implies $[A_\eta, A_\alpha] = 0$ for each α which means that $\sum (\operatorname{trace} A_\beta)A_\beta$ and A_α are commutative for each α. Thus from (3.13)

$$(\sum_\beta A_\beta^2)A_\alpha = A_\alpha(\sum_\beta A_\beta^2) . \tag{3.15}$$

Making use of (3.13), (3.14) and (3.15), equation (3.12) can be rewritten as

$$(\bar{c} + \lambda^2)\{nA_\alpha - (\operatorname{trace} A_\alpha)I\} = 4\sum_\beta (A_\beta^2 A_\alpha - A_\beta A_\alpha A_\beta) . \tag{3.16}$$

It follows that

$$(\bar{c} + \lambda^2)\sum_\alpha \{n(\operatorname{trace} A_\alpha)A_\alpha - (\operatorname{trace} A_\alpha)^2 I\} = 4\sum_{\alpha,\beta} \{A_\beta^2(\operatorname{trace} A_\alpha)A_\alpha - A_\beta(\operatorname{trace} A_\alpha)A_\alpha A_\beta\} = 0$$

where we have used the fact that $\sum_\alpha (\operatorname{trace} A_\alpha)A_\alpha$ and A_β are commutative for each β. Thus we see that if $\lambda^2 \neq -\bar{c}$, then $A_\eta = ||\eta||^2 I$.

Now suppose that $\lambda^2 + \bar{c} = 0$, then $\sum_\beta A_\beta^2 A_\alpha = \sum_\beta A_\beta A_\alpha A_\beta$ for each α

because of (3.16). The square of the length of the normal curvature tensor denoted by $||R^{\perp}||^2$ is given as follows:

$$||R^{\perp}||^2 = 2\,\mathrm{trace}\sum_{\alpha,\beta}(A_\beta A_\alpha^2 A_\beta - A_\beta A_\alpha A_\beta A_\alpha)\,,$$

where we used (1.9). From (3.15), we thus have $R^{\perp}=0$, so that A_α and A_β are commutative for every α, β. By taking suitable orthonormal basis $\{Y_1, \cdots, Y_n\}$ of T_xM, we can diagonalize A_α for all α, i.e., $A_\alpha Y_i = \lambda_i^\alpha Y_i$. Then

$$H(Y_i, Y_j) = \sum_\alpha \langle A_\alpha Y_i, Y_j\rangle \xi_\alpha = 0 \quad \text{for any } i \neq j\,,$$

which implies that $s(Y_i, Y_j) = 0$ for every $i \neq j$ owing to (3.6). Hence the submanifold M^n is locally flat. Since (3.5) yields

$$\langle H(Y_i, Y_i), H(Y_j, Y_j)\rangle = \lambda^2 \quad \text{for any } i \neq j\,,$$

we conclude that f is an umbilical immersion. Let $\hat{M}^n = E^n$ be the simply connected Riemannian covering of M^n and denote the covering map by $\hat{\pi}$. Then the immersion $f\circ\hat{\pi}\colon \hat{M}^n \to \bar{M}^{n+p}(\bar{c})$ is also an umbilical immersion which is congruent to (E)(i) or (H)(ii)(e). Therefore $f\circ\hat{\pi}$ is one to one and thus $\hat{\pi}$ is one to one. It follows that M^n is an Euclidean space.

q.e.d.

REMARK 3.1. Combining (3.7) and (3.11) with (3.13), we have

$$\sum_\beta A_\beta^2 = \frac{||H||^2}{n} I\,. \tag{3.17}$$

Thus we see from (1.10) that M^n is an Einstein manifold.

The following equation is useful in the process of reducing planar geodesic immersions to essential ones.

LEMMA 3.6. *Define* $T = (T_{\alpha\beta})$ *by* $T_{\alpha\beta} = \mathrm{trace}\,(A_\alpha A_\beta)$ *which is a symmetric linear transformation on the normal space. Then we have*

$$\sum_\beta T_{\alpha\beta}A_\beta = \frac{n}{2}(2||\eta||^2 + \bar{c} - \lambda^2)A_\alpha - \frac{1}{2}(\bar{c} - \lambda^2)(\mathrm{trace}\,A_\alpha)I \tag{3.18}$$

for any α.

PROOF. Substituting (3.17) into (3.16), we get

$$2\sum_\beta A_\beta A_\alpha A_\beta = \frac{1}{2n}[\{4||H||^2 - n^2(\bar{c} + \lambda^2)\}A_\alpha + n(\bar{c} + \lambda^2)(\mathrm{trace}\,A_\alpha)I]\,.$$

Combining this equation and (3.7) with (3.14), we obtain (3.18). q.e.d.

4. Reduction of planar geodesic immersions to their essential ones. In the preceding section, we have proved that the second fundamental form is parallel if $f: M^n \to \bar{M}^{n+p}(\bar{c})$ is a planar geodesic immersion. First, using this fact, we reduce the codimension to the dimension of the first normal space, where we recall that the first normal space $N_1(x)$ at $x \in M^n$ is the subspace spanned by the set $\{H(X, Y): X, Y \in T_xM^n\}$ in the normal space N_xM^n. It is easy to prove

LEMMA 4.1. *The symmetric linear transformation T on N_xM defined in §3 is positive semidefinite. Let $Pos(T)$ and $Sym(T_xM)$ denote the maximal subspace on which T is positive definite and the vector space consisting of symmetric linear transformations on T_xM^n respectively. Then we have*

$$N_1(x) = \{\xi: A_\xi = 0\}^\perp = Pos(T) \approx \{A_\xi: \xi \in N_xM\} \subset Sym(T_xM) ,$$

$\{\ \}^\perp$ being the orthogonal complement of $\{\ \}$ in N_xM.

The following lemma guarantees the reduction of the codimension.

LEMMA 4.2. *The dimension of the first normal space is constant and N_1 is invariant by the parallel displacement with respect to the normal connection.*

PROOF. Let x and y be arbitrary two points of M^n. Let σ be a curve from x to y in M^n. Take an orthonormal basis $\{X_1, \cdots, X_n\}$ for T_xM^n and parallel translate this frame to y along σ with respect to the Riemannian connection ∇ of M^n. Thus we have orthonormal frame field parallel along σ, which is denoted by $\{Y_1, \cdots, Y_n\}$. Then $H(Y_i, Y_j)$ is parallel along σ with respect to the normal connection $\nabla^\perp$, because

$$\nabla^\perp_{\dot{\sigma}}(H(Y_i, Y_j)) = (\nabla'_{\dot{\sigma}}H)(Y_i, Y_j) + H(\nabla_{\dot{\sigma}}Y_i, Y_j) + H(Y_i, \nabla_{\dot{\sigma}}Y_j) = 0 .$$

Noting that the set $\{H(Y_i(y), Y_j(y)): i, j = 1, \cdots, n\}$ spans $N_1(y)$, we see that the parallel displacement along any σ from x to y with respect to $\nabla^\perp$ gives a linear isomorphism of $N_1(x)$ to $N_1(y)$. Therefore the dimension of N_1 is constant and N_1 is invariant by the parallel displacement with respect to $\nabla^\perp$. q.e.d.

If $\lambda = 0$, then f is a totally geodesic immersion which is exhibited in §1. Thus, in the sequel, we assume $\lambda \neq 0$. Let $r = \dim N_1$. Then there exists a totally geodesic submanifold $\bar{M}^{n+r}(\bar{c})$ of dimension $n + r$ in $\bar{M}^{n+p}(\bar{c})$ such that $f(M^n) \subset \bar{M}^{n+r}(\bar{c})$. This is immediate consequence from Lemma 4.2 and a theorem of Erbacher [5]. Since $\dim Sym(TM) = n(n+1)/2$, we have an inequality $r \leqq n(n+1)/2$ by Lemma 4.1. Therefore

it suffices to consider a planar geodesic immersion $f: M^n \to \bar{M}^{n+r}(\bar{c})$ under the assumption that f is full, $r \leqq n(n+1)/2$ and the normal space coincides with the first normal space. Moreover all equations obtained in §3 are valid. Indices α, β, γ run over the range $\{n+1, \cdots, n+r\}$.

We shall now consider the symmetric linear transformation T.

LEMMA 4.3. *Let x be arbitrary fixed point and U be the normal coordinate neighborhood around x in M^n. Then, with respect to a suitable orthonormal normal frame field $\{\xi_{n+1}, \cdots, \xi_{n+r}\}$ on U, T can be diagonalized as the following types:*

$$\text{(I)} \qquad T = \begin{pmatrix} a^2 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & a^2 \end{pmatrix}, \quad a^2 = \frac{n}{2}(\bar{c} - \lambda^2), \ \operatorname{trace} A_\alpha = 0 \quad \text{for any } \alpha\ ;$$

$$\text{(II)} \qquad T = \begin{pmatrix} b^2 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & b^2 \\ & & & & n\|\eta\|^2 \end{pmatrix}, \quad b^2 = \frac{n}{2}(2\|\eta\|^2 + \bar{c} - \lambda^2), \ \operatorname{trace} A_\alpha = 0$$

$$\text{for } \alpha \neq n+r, \ A_{n+r} = \frac{1}{n}(\operatorname{trace} A_{n+r})I\ .$$

In the case (I), *we have* $\bar{c} > \lambda^2 > 0$, $r \leqq (n-1)(n+2)/2$ *and f is minimal. In the case* (II), *we have* $2\|\eta\|^2 > \lambda^2 - \bar{c}$ *if* $r \neq 1$ *and we see that f is totally umbilical if* $r = 1$.

PROOF. Let $\{\xi_{n+1}, \cdots, \xi_{n+r}\}$ be an orthonormal basis for $N_x M$ such that $T\xi_\alpha = \mu_\alpha \xi_\alpha$. Parallel translate this normal frame to each point in U along the unique geodesic in M^n issuing from x with respect to the normal connection. Since T is parallel with respect to $\nabla^\perp$, T can be diagonalized by this orthonormal normal frame field locally defined over U. Then (3.18) becomes

$$\mu_\alpha A_\alpha = \frac{n}{2}(2\|\eta\|^2 + \bar{c} - \lambda^2)A_\alpha - \frac{1}{2}(\bar{c} - \lambda^2)(\operatorname{trace} A_\alpha)I\ . \tag{4.1}$$

Taking trace, we have

$$\mu_\alpha(\operatorname{trace} A_\alpha) = n\|\eta\|^2(\operatorname{trace} A_\alpha)$$

which implies that $\mu_\alpha = n\|\eta\|^2$ or $\operatorname{trace} A_\alpha = 0$. If $\operatorname{trace} A_\beta = 0$ for some β, then

$$\mu_\beta = \frac{n}{2}(2\|\eta\|^2 + \bar{c} - \lambda^2)$$

because $A_\alpha \neq 0$ for any α, that is caused by the circumstance the normal

space is equal to the first normal space. Thus we conclude that T has at most two eigenvalues.

Suppose now $\lambda^2 \neq \bar{c}$. If the immersion f is minimal, then $\operatorname{trace} A_\alpha = 0$ for any α and so $\mu_\alpha = (n/2)(\bar{c} - \lambda^2)$ for any α. This case is of type (I). If the immersion f is not minimal, i.e., $\|\eta\| \neq 0$, then there exist at most two eigenvalues b^2 and $n\|\eta\|^2$. We may assume that $\mu_\alpha = b^2$ for $\alpha = n+1, \cdots, n+r_1$ and $\mu_\alpha = n\|\eta\|^2$ for $\alpha = n+r_1+1, \cdots, n+r$ and moreover we may assume $1 \leqq r_1$, because if $\mu_\alpha = n\|\eta\|^2$ for any α, then (4.1) shows that f is totally umbilical, i.e., $r = 1$. We see from (4.1) that $\operatorname{trace} A_\alpha = 0$ for $n+1 \leqq \alpha \leqq n+r_1$ and $A_\alpha = (1/n)(\operatorname{trace} A_\alpha)I$ for $n+r_1+1 \leqq \alpha \leqq n+r$. Then we have

$$\sum_{\alpha=n+r_1+1}^{n+r} \operatorname{trace} A_\alpha^2 = \frac{1}{n} \sum_{\alpha=n+r_1+1}^{n+r} (\operatorname{trace} A_\alpha)^2 = \frac{1}{n} \sum_{\alpha=n+1}^{n+r} (\operatorname{trace} A_\alpha)^2$$
$$= n\|\eta\|^2 .$$

On the other hand,

$$\sum_{\alpha=n+r_1+1}^{n+r} \operatorname{trace} A_\alpha^2 = (r - r_1)n\|\eta\|^2 ,$$

so that $r_1 = r - 1$. This case is of type (II).

Next suppose $\lambda^2 = \bar{c}$. Equation (4.1) implies that $\mu_\alpha = n\|\eta\|^2$ for all α. Since T is positive definite, $\|\eta\| \neq 0$. Noting that η is parallel with respect to $\nabla^\perp$, we can choose $\eta/\|\eta\|$ as ξ_{n+r}. This case is of type (II).

In the case of type (I), the minimality of f gives a restriction for r (cf. Lemma 4.1), i.e., $r \leqq (n-1)(n+2)/2$. This bound is equal to the dimension of the vector space spanned by the elements of $Sym(TM)$ whose traces are 0. q.e.d.

Let us consider the case of type (II). Even if the ambient manifold is a hyperbolic space, we have

LEMMA 4.4. *In the case of type* (II), $\bar{c} + \|\eta\|^2$ *is positive if* $r \neq 1$.

PROOF. We note that $2\|\eta\|^2 > \lambda^2 - \bar{c}$. The square of the length of the second fundamental form is given as

$$\|H\|^2 = (r-1)b^2 + n\|\eta\|^2 = nr\|\eta\|^2 + \frac{n}{2}(r-1)(\bar{c} - \lambda^2) .$$

Combining this equation with (3.7), it follows that

$$\|\eta\|^2 = \frac{1}{n+2r}\{(n+r+1)\lambda^2 - (r-1)\bar{c}\} .$$

Thus we find $2\{(n+r+1)\lambda^2 - (r-1)\bar{c}\}/(n+2r) > \lambda^2 - \bar{c}$ which implies $\lambda^2 + \bar{c} > 0$.

It follows that $\bar{c} + ||\eta||^2 > (\lambda^2 + \bar{c})/2 > 0$. q.e.d.

Note that $\xi_{n+r} = \eta/||\eta||$ and $A_{n+r} = (1/n)(\text{trace } A_{n+r})I$ in the case of type (II). By a similar method to that used in [11], it can be shown that if $r \neq 1$, M^n must be minimally immersed into a totally umbilical hypersurface of $\bar{M}^{n+r}(\bar{c})$ with curvature $\bar{c} + ||\eta||^2 > 0$ which is orthogonal to the mean curvature normal η. This totally umbilical hypersurface will be denoted by $\tilde{M}^{n+q}(\tilde{c})$ where $q = r - 1 \leqq (n-1)(n+2)/2$ and $\tilde{c} = \bar{c} + ||\eta||^2$. We must note that $\tilde{M}^{n+q}(\tilde{c})$ is congruent to (E) (ii), (S) (ii) or (H) (ii) (s) exhibited in §1 and regarding the immersion f as that of M^n into $\tilde{M}^{n+q}(\tilde{c})$, the second fundamental tensors corresponding to the normal vectors $\xi_{n+1}, \cdots, \xi_{n+q}$ are also $A_{n+1}, \cdots, A_{n+q}$ respectively. Moreover it is easily verified that $f: M^n \to \tilde{M}^{n+q}(\tilde{c})$ is also isotropic, whose isotropy constant μ^2 is equal to $\lambda^2 - ||\eta||^2$.

If f is totally umbilical in the case of type (I), then f must be totally geodesic which contradicts our assumption $\lambda \neq 0$. If $r = 1$ in the case of tpye (II), then f is totally umbilical. Conversely, if f is totally umbilical in the case of type (II), then $r = 1$. Thus, in the sequel, we exclude the totally umbilical case from our consideration. After all, in order to solve our problem introduced in §0, we have only to consider a full, isotropic and minimal immersion $f: M^n \to \tilde{M}^{n+q}(\tilde{c})$ whose isotropy constant μ^2 is equal to $\lambda^2 - ||\eta||^2$ where the case of tpye (I) may be regarded as that $\eta = 0$. Moreover we may assume that $q \leqq (n-1)(n+2)/2$ and f is also planar geodesic; because the intersection of a 2-dimensional great sphere with a small hypersphere in a sphere is empty, a point or a circle.

We need the following lemma to determine the codimension q.

LEMMA 4.5. *We have equations*

$$\mu^2 = \frac{q\tilde{c}}{n+q+2} \tag{4.2}$$

$$\rho = n(n-1)\tilde{c} - \frac{n(n+2)}{2}\mu^2 . \tag{4.3}$$

PROOF. Since $f: M^n \to \tilde{M}^{n+q}(\tilde{c})$ is planar geodesic, equation (3.7) is valid. By virture of Lemma 4.3, we obtain $||H||^2 = nq(\tilde{c} - \mu^2)/2$, H being the second fundamental form of the immersion $f: M^n \to \tilde{M}^{n+q}(\tilde{c})$. Substituting $\eta = 0$ into (3.7), it follows that

$$nq(\tilde{c} - \mu^2) = n(n+2)\mu^2$$

which implies (4.2). From (1.12), we have (4.3). q.e.d.

Finally, we show that the universal covering manifold of M^n is a compact simply connected symmetric space of rank one and determine its maximal sectional curvature.

Let $f: M^n \to \tilde{M}^{n+q}(\tilde{c})$ be the immersion as above. Let ι be the natural imbedding of $\tilde{M}^{n+q}(\tilde{c}) = S^{n+q}(\tilde{c})$ into E^{n+q+1}. Then we have

LEMMA 4.6. *If $\sigma: (-t_1, t_1) \to M^n$ is a unit speed geodesic such that $\sigma(0) = x_0$ and $\dot{\sigma}(0) = X$, then*

$$\iota f(\sigma(t)) = \iota f(x_0) + \frac{1}{\nu}(\sin \nu t)\iota f(X) + \frac{1}{\nu^2}(1 - \cos \nu t)(\iota H(X, X) - \tilde{c}\iota f(x_0)) , \tag{4.4}$$

where $\nu^2 = \tilde{c} + \mu^2$ and $\iota f(X)$ means $\iota_* f_* X$.

PROOF. Since f is planar geodesic, there exists a 2-dimensional great sphere P such that $f(\sigma(-t_1, t_1)) \subset P$ for sufficiently small t_1. P is the intersection of $\tilde{M}^{n+q}(\tilde{c})$ with 3-dimensional plane $\tilde{P}$ spanned by $\iota f(x_0)$, $\iota f(X)$ and $\iota H(X, X)$. Set

$$X_1 = \sqrt{\tilde{c}}\,\iota f(x_0) , \quad X_2 = \iota f(X) \text{ and } X_3 = \frac{1}{\mu}\iota H(X, X) .$$

Then $\{X_1, X_2, X_3\}$ is an orthonormal base for $\tilde{P}$. Using this base, we can write

$$\iota f(\sigma(t)) = w_1(t)X_1 + w_2(t)X_2 + w_3(t)X_3 \tag{4.5}$$

where $w_b(t)$ $(b = 1, 2, 3)$ are certain differentiable functions defined on $(-t_1, t_1)$ and they satisfy $\sum_{b=1}^{3} w_b^2 = 1/\tilde{c}$. The straightforward computation shows

$$\iota \tilde{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})) = \sum_{b=1}^{3} (w_b''' + \tilde{c}w_b')X_b ,$$

$\tilde{\nabla}$ being the covariant differentiation in $\tilde{M}^{n+q}(\tilde{c})$. Taking account of (3.9), we thus have differential equations

$$w_b''' = -(\tilde{c} + \mu^2)w_b' \quad \text{for } b = 1, 2, 3$$

with initial conditions

$$w_1(0) = 1/\sqrt{\tilde{c}},\ w_1'(0) = 0,\ w_1''(0) = -\sqrt{\tilde{c}};\ w_2(0) = 0,\ w_2'(0) = 1,\ w_2''(0) = 0;$$
$$w_3(0) = 0,\ w_3'(0) = 0,\ w_3''(0) = \mu .$$

The solutions are given by

$$w_1(t) = \frac{1}{\sqrt{\tilde{c}}\,\nu^2}(\tilde{c}\cos \nu t + \mu^2),\ w_2(t) = \frac{1}{\nu}\sin \nu t,\ w_3(t) = \frac{\mu}{\nu^2}(1 - \cos \nu t) .$$

Substituting these equations into (4.5), we get (4.4). q.e.d.

REMARK 4.1. Equation (4.4) shows that any geodesic in M^n is mapped to a circle by f.

Let $\tilde{H}$ be the second fundamental form of the immersion $\iota \circ f: M^n \to E^{n+q+1}$. Then it is easy to show that $\tilde{H}$ satisfies

$$\tilde{H}(X, Y) = \iota H(X, Y) - \tilde{c}\langle X, Y\rangle \iota f(X)$$

for any X, Y tangent to M^n. The immersion $\iota \circ f$ is also planar geodesic, whose isotropy constant is equal to ν^2. In terms of $\tilde{H}$, (4.4) can be written as

$$\iota f(\sigma(t)) = \iota f(x_0) + \frac{1}{\nu}(\sin \nu t)\iota f(X) + \frac{1}{\nu^2}(1 - \cos \nu t)\tilde{H}(X, X) \tag{4.6}$$

which is the same equation as S. L. Hong obtained in [7]. By virture of (4.6), we can represent locally the immersion $\iota \circ f$ in terms of normal coordinates centered at x_0. Using this fact, Hong has proved

LEMMA 4.7. *The sectional curvature of M^n is 1/4-pinched, i.e.,*

$$\frac{1}{4}\nu^2 \leqq s(X, Y) \leqq \nu^2 \tag{4.7}$$

for any orthonormal vectors X, Y tangent to M^n. In particular, if M^n is of constant sectional curvature, then it is $\nu^2/4$.

PROOF. In [7], the consideration for the case $n = 2$ is lacking Thus we give a proof only when $n = 2$. Let X, Y be any orthonormal vectors tangent to M^n. Since f is minimal we have $H(X, X) = -H(Y, Y)$. It follows from (3.5) that $\|H(X, Y)\|^2 = \mu^2$. Using (3.6), we obtain $s(X, Y) = \tilde{c} - 2\mu^2$. On the other hand, it is easy to see that the case $q = 1$ does not occur and hence $q = 2$. Therefore (4.2) implies $s(X, Y) = \tilde{c}/3$, from which we have $s(X, Y) = \nu^2/4$. q.e.d.

Let $\hat{M}^n$ be the simply connected Riemannian covering of M^n and denote the covering map by $\hat{\pi}$. Then $f \circ \hat{\pi}: \hat{M}^n \to \tilde{M}^{n+q}(\tilde{c})$ is an immersion with the same property as f. Thus $\hat{M}^n$ is a compact symmetric space of rank one, because Lemmas 3.4 and 4.7 hold for $f \circ \hat{\pi}$. Let $d = 2$ or 4 according as $\hat{M}^n = \boldsymbol{C}P^m$ or $\boldsymbol{Q}P^m$ where $n = md$. We obtain

LEMMA 4.8. *$\hat{M}^n$ is a simply connected compact symmetric space of rank one. Let c denote the maximal sectional curvature, i.e., holomorphic, Q-or C-sectional curvature. If $\hat{M}^n = S^n(c)$, then $c = (n/2(n+1))\tilde{c}$, $q = (n-1)(n+2)/2$ and $\mu^2 = ((n-1)/(n+1))\tilde{c}$. If $\hat{M}^n = \boldsymbol{C}P^m(c)$ or $\boldsymbol{Q}P^m(c)$, then $c = (2m/(m+1))\tilde{c}$, $q = (m-1)(md+2)/2$ and $\mu^2 = ((m-1)/(m+1))\tilde{c}$. If $\hat{M}^n = \boldsymbol{C}ayP^2(c)$, then $c = (4/3)\tilde{c}$, $q = 9$ and $\mu^2 = (1/3)\tilde{c}$.*

PROOF. Lemma 4.7 implies that if $\hat{M}^n = S^n$, then the sectional curvature is $\nu^2/4$ and if $\hat{M}^n = \boldsymbol{CP}^m$, $\boldsymbol{QP}^m$ or $\boldsymbol{CayP}^2$, then the maximal sectional curvature is ν^2. Thus the scalar curvature ρ is given by

$$\rho = \frac{1}{4}n(n-1)\nu^2,\ m(m+1)\nu^2,\ 4m(m+2)\nu^2 \text{ or } 16\times 9\nu^2$$

according as $\hat{M}^n = S^n$, $\boldsymbol{CP}^m$, $\boldsymbol{QP}^m$ or $\boldsymbol{CayP}^2$ (cf. [2], [8]). Combining them with (4.3), we obtain our assertion for μ^2. We also obtain our assertion for q from (4.2). q.e.d.

REMARK 4.2. Since $\boldsymbol{CP}^1(c)$ and $\boldsymbol{QP}^1(c)$ are of constant sectional curvature, they should be regarded as $S^2(c)$ and $S^4(c)$ respectively.

5. Main theorems and corollaries. In the preceding section, we have reduced planar geodesic immersions into space forms to the full, minimal and planar geodesic immersions of compact rank one symmetric spaces into spheres. We consider their rigidity. Note that the values c and q obtained in Lemma 4.8 are equal to those of the immersions explained in §2. Let $f: M^n \to S^{n+q}(\tilde{c})$ be a planar geodesic immersion which is full and minimal. Then the simply connected Riemannian covering manifold $\hat{M}^n$ of M^n is $S^n(c)$, $\boldsymbol{CP}^m(c)$, $\boldsymbol{QP}^m(c)$ or $\boldsymbol{CayP}^2(c)$ by virture of Lemma 4.8. On the other hand, in Theorems 1 and 2 we constructed the immersion $\hat{f}: \hat{M}^n \to S^{n+q}(\tilde{c})$ with same properties as f. Let $f_1\circ\hat{\pi}, \cdots, f_{n+q+1}\circ\hat{\pi}$ be the coordinate functions of $f\circ\hat{\pi}: \hat{M}^n \to S^{n+q}(\tilde{c}) \subset E^{n+q+1}$, where $\hat{\pi}$ is the covering map $\hat{M}^n \to M^n$. Then they are linearly independent because f is full. On the other hand, the coordinate functions $\hat{f}_1, \cdots, \hat{f}_{n+q+1}$ of $\hat{f}$ are orthogonal (cf. Remark 2.3). It follows that there exists a non-singular linear transformation U on E^{n+q+1} such that $f\circ\hat{\pi} = U\circ\hat{f}$. Since f is analytic and of degree 2 in the sense of [15], we can apply [15] Proposition 11.1 to $f\circ\hat{\pi}$, so that U is an orthogonal linear transformation. Noting that $\hat{f}$ is an imbedding for $\boldsymbol{CP}^m$, $\boldsymbol{QP}^m$ and $\boldsymbol{CayP}^2$ (resp. two fold immersion for S^n), $\hat{\pi}$ is one to one (resp. two fold). Thus M^n must be $S^n(c)$, $\boldsymbol{RP}^n(c)$, $\boldsymbol{CP}^m(c)$, $\boldsymbol{QP}^m(c)$ or $\boldsymbol{CayP}^2(c)$.

Let's go back to the original situation. Let $f: M^n \to \bar{M}^{n+p}(\bar{c})$ be a planar geodesic immersion. Then f is a minimal immersion into $S^{n+p}(\bar{c})$ or lies minimally in a totally umbilical submanifold $\tilde{M}^{n+q}(\tilde{c})$ of $\bar{M}^{n+p}(\bar{c})$ with curvature $\tilde{c} = \bar{c} + \|\eta\|^2 > 0$ except for totally geodesic and umbilical case, η being the mean curvature normal of f. Any two $(n+q)$-dimensional totally umbilical submanifolds in $\bar{M}^{n+p}(\bar{c})$ with same curvature are congruent each other and any rotation of a totally umbilical submanifold can be extended to an isometry of the ambient manifold $\bar{M}^{n+p}(\bar{c})$. Therefore we have

THEOREM 3. *Let $f: M^n \to \bar{M}^{n+p}(\bar{c})$ be a planar geodesic immersion. Then f is congruent to a totally geodesic, umbilical immersion exhibited in §1 or one of the model immersions stated in Theorems 1, 2 and their Corollaries.*

COROLLARY. *If $f: M^n \to \bar{M}^{n+p}(\bar{c})$ is an immersion which maps every geodesic in M^n to circles in $\bar{M}^{n+p}(\bar{c})$, then we have the same conclusion as Theorem 3.*

COROLLARY. *If $f: M^n \to \bar{M}^{n+p}(\bar{c})$ is an isotropic immersion with parallel second fundamental tensor, then we have the same conclusion as Theorem 3.*

PROOF. In order to prove that f is planar geodesic, we may assume that f is a minimal and isotropic immersion of M^n into $S^{n+q}(\tilde{c})$ with parallel second fundamental tensor. Let σ be any geodesic in M^n. Then we have a differential equation

$$\tilde{\nabla}_{f\dot{\sigma}} f\dot{\sigma} = H(\dot{\sigma}, \dot{\sigma}) , \quad \tilde{\nabla}_{f\dot{\sigma}}(H(\dot{\sigma}, \dot{\sigma})) = -\mu^2 f\dot{\sigma} ,$$

$\tilde{\nabla}$ being the covariant differentiation of $S^{n+q}(\tilde{c})$, where we have used the fact that equation (3.8) is equivalent to the isotropic condition (3.4) and the second fundamental form H is parallel. The initial conditions are $\sigma(0) = x_0$, $\dot{\sigma}(0) = X$ and $\tilde{\nabla}_{f\dot{\sigma}} f\dot{\sigma}|_{t=0} = H(X, X)$. Then the solution is given as (4.4) which shows that f is planar geodesic. q.e.d.

Finally, we give an interesting property of planar geodesic immersions into spheres which are full and minimal. We also give a different proof for their rigidity when the manifold M^n is a sphere, complex or quaternion projective space. First, let $f_n: S^n(1) \to S^{n+p_n}(k_n)$ be a planar geodesic immersion which is full and minimal, where

$$k_n = \frac{2(n+1)}{n} , \qquad p_n = \frac{(n-1)(n+2)}{2} \tag{5.1}$$

(cf. Lemma 4.8). The isotropy constant μ_n of f_n is given by

$$\mu_n^2 = \frac{2(n-1)}{n} . \tag{5.2}$$

Fix a point x in $S^n(1)$ and consider the unit hypersphere $S^{n-1}(1)$ in the tangent space $T_x S^n(1)$ at x. Then the second fundamental form H_n gives a map $\phi_n: S^{n-1}(1) \to S^{p_n-1}(1/\mu_n^2) \subset N_x S^n(1)$ by $\phi_n(X) = H_n(X, X)$ for any $X \in S^{n-1}(1)$.

PROPOSITION 5.1. *The map $\phi_n/2$ coincides with a full and minimal planar geodesic immersion $f_{n-1}: S^{n-1}(1) \to S^{n-1+p_{n-1}}(k_{n-1})$ whose isotropy constant is μ_{n-1}.*

PROOF. First we note that $p_n - 1 - (n-1) = p_{n-1}$ and $4/\mu_n^2 = k_{n-1}$. Let $X, Y \in S^{n-1}(1)$ be any orthogonal vectors. The great circle $\sigma(t) = (\cos t)X + (\sin t)Y$ in $S^n(1)$ is mapped by $\phi_n/2$ as

$$\frac{1}{2}\phi_n(\sigma(t)) = \frac{1}{4}\{H(X,X) + H(Y,Y) + (\cos 2t)(H(X,X) - H(Y,Y)) + 2(\sin 2t)H(X,Y)\}. \tag{5.3}$$

Thus we have $\|(\phi_n/2)_*(Y)\|^2 = \|H(X,Y)\|^2$. On the other hand, applying (3.6) to f_n, we obtain $\|H(X,Y)\|^2 = 1$. Therefore $\phi_n/2$ is an isometric immersion. This is also planar geodesic. In fact (5.3) is a circle with center $(H(X,X) + H(Y,Y))/4$ and with radius 1/2.

Let $\xi \in N_x S^n(1)$. The height function defined over $S^{n-1}(1)$ in the direction ξ is given by

$$h_\xi(X) = \frac{1}{2}\langle A_\xi X, X\rangle \tag{5.4}$$

where we have used (1.3). Since f_n is minimal and hence $\operatorname{trace} A_\xi = 0$ for any ξ, the height function h_ξ is a spherical harmonic of degree 2 corresponding to the eigenvalue $2n$. Thus $\phi_n/2$ is minimal.

Let $\{\xi_{n+1}, \cdots, \xi_{n+p_n}\}$ be an orthonormal base for $N_x S^n(1)$. If the image $(\phi_n/2)(S^{n-1}(1))$ is contained in a hyperplane of $N_x S^n(1)$, say $\sum_\alpha a^\alpha h_\alpha = 0$ where $h_\alpha = h_{\xi_\alpha}$, then we have from (5.4) $\sum a^\alpha \langle A_\alpha X, X\rangle = 0$ for any $X \in S^{n-1}(1)$ which implies $\sum a^\alpha A_\alpha = 0$. However, from Lemma 4.1, A_α's are linearly independent. Thus we have a contradiction. We have proved that $\phi_n/2$ is full. q.e.d.

PROPOSITION 5.2. *$f_n: S^n(1) \to S^{n+p_n}(k_n)$ is rigid. In particular, $f_2: S^2(1) \to S^4(3)$ is a Veronese surface (cf. [3]). The map f_1 is congruent to the Hopf map: $S^1(1) \to S^1(4)$.*

PROOF. Let $f_n': S^n(1) \to S^{n+p_n}(k_n)$ be another planar geodesic immersion which is full and minimal. Similarly we have ϕ_n' and hence a planar geodesic immersion f_{n-1}' which is full and minimal. Since $S^{n+p_n}(k_n)$ is frame homogeneous, we may assume that $f_n(x) = f_n'(x)$ and $f_{n^*} = f_{n^*}'$ at x. Thus the normal spaces at x with respect to f_n and f_n' coincide. Suppose that there exists an orthogonal linear transformation U_1 of $N_x S^n(1)$ such that $U_1 \circ f_{n-1} = f_{n-1}'$. U_1 can be extended to an orthogonal linear transformation U of $S^{n+p_n}(k_n) \subset E^{n+p_n+1}$ by means of $U(f_n(x)) = f_n(x)$, $U|_{f_{n^*}(T_x S^n)} = \text{Identity}$ and $U|_{N_x S^n} = U_1$. Then the second fundamental form of the composite $U \circ f_n$ is given by $U_1 \circ H_n$ at x. Our assumption shows $U_1 \circ H_n = H_n'$, H_n' being the second fundamental form of f_n'. Since $U \circ f_n$ is also planar geodesic, $U \circ f_n$ is locally determined by $U_1 \circ H_n$ (cf.

Lemma 4.6). Similarly, f'_n is locally determined by H'_n. Noting that f_n and f'_n are analytic, it follows that $U \circ f_n = f'_n$.

By the consideration above, we have only to prove the rigidity of $f_2: S^2(1) \to S^4(3)$. The coordinate functions of $f_1: S^1(1) \to S^1(4)$ are given as follows:

$$h_1(t) = \frac{1}{2}(a_1 \cos 2t + b_1 \sin 2t), \quad h_2(t) = \frac{1}{2}(a_2 \cos 2t + b_2 \sin 2t),$$
$$h_1^2 + h_2^2 = 1/4,$$

because h_1 and h_2 are spherical harmonics of degree 2 on $S^1(1)$. Clearly, the matrix $\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is orthogonal which shows that f_1 is unique up to rotation and reflection of $S^1(4)$. q.e.d.

Next, let $f_m: \boldsymbol{FP}^m(4) \to S^{md+p_m}(k_m)$ be a planar geodesic immersion which is full and minimal where $\boldsymbol{F} = \boldsymbol{C}$ or $\boldsymbol{Q}$ and

$$k_m = \frac{2(m+1)}{m}, \qquad p_m = \frac{(m-1)(dm+2)}{2} \tag{5.5}$$

(cf. Lemma 4.8). The isotropy constant μ_m of f_m is given by

$$\mu_m^2 = \frac{2(m-1)}{m}. \tag{5.6}$$

Fix a point x in $\boldsymbol{FP}^m(4)$ and let $S^{dm-1}(1)$ be the unit hypersphere in the tangent space $T_x\boldsymbol{FP}^m$. Then the second fundamental form H_m gives a map $\phi_m: S^{dm-1}(1) \to S^{p_m-1}(1/\mu_m^2) \subset N_x\boldsymbol{FP}^m(4)$ by $\phi_m(X) = H_m(X, X)$ for any $X \in S^{dm-1}(1)$.

PROPOSITION 5.3. *The map $\phi_m/2$ induces a full and minimal planar geodesic immersion $f_{m-1}: \boldsymbol{FP}^{m-1}(4) \to S^{d(m-1)+p_{m-1}}(k_{m-1})$ such that $\phi_m/2 = f_{m-1} \circ \pi_{m-1}$ where π_{m-1} denotes the Hopf fibering: $S^{dm-1}(1) \to \boldsymbol{FP}^{m-1}(4)$. The isotropy constant of f_{m-1} is μ_{m-1}.*

PROOF. Note that $p_m - 1 - d(m-1) = p_{m-1}$ and $4/\mu_m^2 = k_{m-1}$. Let J_1 denote the complex structure on $\boldsymbol{CP}^m(4)$ and $\{J_1, J_2, J_3\}$ be the quaternion structure on $\boldsymbol{QP}^m(4)$. J_1 is a globally defined (1, 1)-tensor field such that $\langle J_1X, J_1Y\rangle = \langle X, Y\rangle$ and $J_1^2 = -1$, but J_1, J_2, J_3 are locally defined (1, 1)-tensor fields such that $\langle J_iX, J_iY\rangle = \langle X, Y\rangle$, $J_i^2 = -1$ $(i = 1, 2, 3)$ and satisfy $J_1J_2 = -J_2J_1 = J_3$, $J_2J_3 = -J_3J_2 = J_1$, $J_3J_1 = -J_1J_3 = J_2$. The curvature tensor of $\boldsymbol{FP}^m(4)$ is given by (cf. [8])

$$\begin{aligned} \langle R(X, Y)Z, W\rangle &= \langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle \\ &+ \sum_{i=1}^{d-1} \{\langle J_iY, Z\rangle\langle J_iX, W\rangle - \langle J_iX, Z\rangle\langle J_iY, W\rangle \\ &+ 2\langle X, J_iY\rangle\langle J_iZ, W\rangle\}. \end{aligned} \tag{5.7}$$

Taking account of (1.7), from (5.7), we have

$$\begin{aligned}(5.8)\qquad &\langle H(Y, Z), H(X, W)\rangle - \langle H(X, Z), H(Y, W)\rangle \\ &= \sum_{i=1}^{d-1}\{\langle J_iY, Z\rangle\langle J_iX, W\rangle - \langle J_iX, Z\rangle\langle J_iY, W\rangle \\ &\quad + 2\langle X, J_iY\rangle\langle J_iZ, W\rangle\} \\ &\quad - \frac{m+2}{m}\{\langle Y, Z\rangle\langle X, W\rangle - \langle X, Z\rangle\langle Y, W\rangle\}\,.\end{aligned}$$

Let X be any unit vector tangent to $FP^m(4)$. Putting $Y = Z = J_jX$ and $W = X$ in (5.8), we obtain

$$\mu_m^2 = \langle H(X, X), H(J_jX, J_jX)\rangle - \|H(X, J_jX)\|^2 \qquad \text{for each } j\ (1 \leqq j \leqq d-1)\,.$$

On the other hand, using (3.5), we obtain

$$\mu_m^2 = \langle H(X, X), H(J_jX, J_jX)\rangle + 2\|H(X, J_jX)\|^2 \quad \text{for each } j\,.$$

Thus we have for any j

$$(5.9)\qquad H(J_jX, J_jX) = H(X, X)\,, \quad H(X, J_jX) = 0\,.$$

Now, let's consider the map $\phi_m/2: S^{dm-1}(1) \to S^{d(m-1)+p_{m-1}}(k_{m-1})$. The equation (5.9) means that $\phi_m/2$ induces a map $f_{m-1}: FP^{m-1}(4) \to S^{d(m-1)+p_{m-1}}(k_{m-1})$ such that $\phi_m/2 = f_{m-1}\circ\pi_{m-1}$. Let $X \in S^{dm-1}(1)$ and let $Y \in S^{dm-1}(1)$ satisfy $X \perp Y$, $J_iX \perp Y$ for any i. Then every geodesic in $FP^{m-1}(4)$ can be written as $\pi_{m-1}(\cos t\, X + \sin t\, Y)$, which is mapped by f_{m-1} as (5.3). It follows that $\|f_{m-1*}(Y)\|^2 = \|H(X, Y)\|^2$. Putting $Z = Y$ and $W = X$ in (5.8),

$$-\frac{m+2}{m} = \langle H(X, X), H(Y, Y)\rangle - \|H(X, Y)\|^2\,.$$

Combining this equation with (3.5), we obtain $\|H(X, Y)\| = 1$ which shows that f_{m-1} is an isometric immersion. Clearly, f_{m-1} also planar geodesic.

A similar method to that taken in the proof of Lemma 5.1 shows the minimality and fullness of f_{m-1}. q.e.d.

PROPOSITION 5.4. *$f_m: FP^m(4) \to S^{md+p_m}(k_m)$ is rigid. In particular, f_1 is an isometry of $S^d(4)$ and satisfies $\phi_2/2 = f_1\circ\pi_1$ where $\pi_1: S^{2d-1}(1) \to S^d(4)$ is the Hopf map.*

PROOF. A similar argument to Proposition 5.2 implies that the rigidity of f_m can be reduced to that of $f_2: FP^2(4) \to S^{3d+1}(3)$. The map $\phi_2/2$ induces an isometric immersion $f_1: FP^1(4) \to S^d(4)$ such that $\phi_2/2 = f_1\circ\pi_1$. However $FP^1(4)$ and $S^d(4)$ are of the same dimension. Thus f_1 is a Riemannian

covering and so isometry, because $S^d(4)$ is simply connected. q.e.d.

Taking (1.3) and (5.9) into account, we have

THEOREM 4. *Let $f: M^n \to S^{n+p}(\bar{c})$ be a planar geodesic immersion which is full, minimal and non-totally geodesic where $M^n = S^n$, CP^m or QP^m. Then the vector space $\{A_\xi: \xi \in N_x M^n\}$ can be identified with the vector space spanned by Hermitian matrices of trace 0 over each field $F = R, C$ or Q, i.e., $\mathfrak{H}_0(m, F)$.*

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