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ON A SUBSURFACE OF TÔKI SURFACE

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One of the most illuminating examples in the classification theory of Riemann surfaces is the surface T constructed by Tôki [5] which has the following properties:

α) *T is an infinite regular covering surface (T, U, π) with the unit disk U: $|z| < 1$ as its base surface and π the projection of T onto U;*

β) *There exists a radial slit disk $V = U - \bigcup_\nu \sigma_\nu$ with σ_ν the radial slits in U accumulating only to the circumference of U such that $T - \pi^{-1}(\bigcup_\nu \sigma_\nu) = \sum_{n=1}^{\infty} V_n$ (disjoint union) where V_n $(n = 1, 2, \cdots)$ are copies of V;*

γ) *There exists a bounded harmonic function $\hat{h}$ on U for any given bounded harmonic function h on T such that $h = \hat{h}\circ\pi$.*

Including the one constructed by Tôki any Riemann surface T possessing the above three properties will be referred to as a *Tôki surface.* A subsurface S of a Tôki surface T will be referred to as an *admissible subsurface* if it has the form

$$S = T - \pi^{-1}(K)$$

where K is a compact subset of U such that $K \cap (\bigcup_\nu \sigma_\nu) = \varnothing$ and $U - K$ is a subregion of U whose relative boundary $\partial(U - K)$ relative to U consists of regular points with respect to the Dirichlet problem. The main purpose of this paper is to prove the following

THEOREM. *On any admissible subsurface of a Tôki surface there exists a unique (up to multiplicative constants) $HD^{\sim}$-minimal function but no HB- and HD-minimal functions.*

Here we denote by $HX(R)$ the subclass of the class $H(R)$ of harmonic functions on a Riemann surface R with a property X. As for X we consider B meaning the boundedness, D the finiteness of the Dirichlet integral, and BD both B and D. Moreover a function u is said to have the property $D^{\sim}$ if $u \geqq 0$ and there exists a decreasing sequence $\{u^i\}$ $(i = 1, 2, \cdots)$ in the class $HD(R)$ such that $\{u^i\}$ converges to u. The property $BD^{\sim}$ has the obvious meaning, i.e. B and $D^{\sim}$. An *HX-minimal function u* ($X = B, D, D^{\sim}, BD$ and $BD^{\sim}$) on R is a strictly positive function u in the class $HX(R)$ with *hyperbolic* R such that any nonnegative function v in $HX(R)$ dominated by u is a constant multiple of u. It is known that u is an HY-minimal function if and only if u is an HBY-minimal function for $Y = D$ and $D^{\sim}$ (cf. e.g. Sario-Nakai [4]).

In nos. 1–3 we will discuss the localization of the property γ) for a class of surfaces which contains Tôki surfaces as its subclass. The proof of the above main theorem will then be given in nos. 4–10 divided into a series of lemmas. We will then append in nos. 11 and 12 examples of Riemann surfaces with all possible combinations of existence or non-existence of HX-minimal functions for $X = B$, D and $D^{\sim}$.

1. In the definition of Tôki surfaces T, the condition β) is rather technical, although we need to assume it for the proof of the above theorem, and the essence of Tôki surfaces seems to lie only in conditions α) and γ). For this reason we consider Riemann surfaces R with only two properties α) and γ), and such surfaces will be referred to as surfaces *of Tôki class*. We denote by O_{HX} the class of open Riemann surfaces R such that the class $HX(R)$ consists of only constans. Observe that the condition γ) can be reformulated as

$$HB(R) = HB(U)\circ\pi \tag{1}$$

for surfaces R of Tôki class where (R, U, π) satisfies α). Suppose $u \in HBD(R)$. Then $u = \hat{u}\circ\pi$ with $\hat{u} \in HB(U)$, and the Dirichlet integral

$$D_R(u) = \int_R du \wedge *du = D_U(\hat{u})\cdot\infty\ ,$$

which is finite only if u is a constant. This shows with the Virtanen identity $O_{HD} = O_{HBD}$ (cf. e.g. [4]) that

PROPOSITION. *Any Riemann surface of Tôki class belongs to* $O_{HD} - O_{HB}$.

2. Consider a nonempty open subset F of an open Riemann surface R such that each point of ∂F is regular with respect to the Dirichlet problem for F. We denote by $HB(F; \partial F)$ the relative class consisting of $u \in HB(F) \cap C(R)$ with $u|(R - F) = 0$. The *inextremization* $\lambda = \lambda_F$ is a linear operator: $HB(R) \rightarrow HB(F; \partial F)$ given by

$$\lambda u = \sup s$$

first for $u \geqq 0$ in $HB(R)$, where s runs over all subharmonic functions on R vanishing on $R - F$ and dominated by u, and then for general u in $HB(R)$ by linearity. The *extremization* $\mu = \mu_F$ is a linear operator: $HB(F; \partial F) \rightarrow HB(R)$ given by

$$\mu u = \inf s$$

first for $u \geqq 0$ in $HB(F; \partial F)$, where s runs over all superharmonic functions dominating u, and then for general u in $HB(F; \partial F)$ by linearity. For the details and properties of λ and μ we refer to e.g. Noshiro [3]. The composition $\lambda \circ \mu$ is always an identity but $\mu \circ \lambda$ is not

$$\begin{tikzcd} HB(R) \arrow[dr, "\lambda"'] & HB(F; \partial F) \arrow[l, "\mu"'] \arrow[d, "\lambda \circ \mu = \mathrm{id.}"] \\ & HB(F; \partial F) \end{tikzcd}
\qquad
\begin{tikzcd} HB(R) \arrow[r, "\lambda"] \arrow[d, "\mu \circ \lambda"] & HB(F; \partial F) \arrow[dl, "\mu"] \\ HB(R) & \end{tikzcd}$$

FIGURE 1 FIGURE 2

necessarily so. This comes from the fact that λ is always surjective but μ is not always surjective. This is the place where the following notion comes in: A subset $E \subset R$ is said to be *B-negligible* if there exists an F such that $R - F \supset E$ and μ_F is surjective (cf. Nakai [2]). Thus $R - F$ is B-negligible if and only if μ_F is surjective. Any *compact* subset $E \subset R$ is a trivial example of B-negligible sets. We can then state:

If $R - F$ is B-negligible, then both $\lambda \circ \mu$ and $\mu \circ \lambda$ are identity operators.

3. Let R be of Tôki class and F be a nonempty open subset of U such that each point of ∂F, the relative boundary of F with respect to U, is regular with respect to the Dirichlet problem. Then $F^\infty = \pi^{-1}(F)$ is an open subset of R such that each point of ∂F^∞ is regular with respect to the Dirichlet problem. We are interested in the *localization* of (1), i.e. when is the relation

$$HB(F^\infty; \partial F^\infty) = HB(F; \partial F) \circ \pi \tag{2}$$

valid? We maintain

THEOREM. *If $U - F$ is B-negligible, and if in particular $U - F$ is compact, then the relation* (2) *is valid.*

Since the right hand side of (2) is contained in the left hand side of (2), we only have to show the reversed inclusion. Let $u \in HB(F^\infty; \partial F^\infty)$ with $u \geqq 0$ and $v = \mu_{F^\infty} u$. Then by (1) there exists a $\hat{v} \in HB(U)$ with $v = \hat{v} \circ \pi \geqq 0$. Since μ_F is surjective, there exists a $\hat{u} \in HB(F; \partial F)$ such that $\hat{v} = \mu_F \hat{u}$. Let $h = u - \hat{u} \circ \pi$. Observe that $v - u \geqq 0$, $\hat{v} - \hat{u} \geqq 0$, and $v = \hat{v} \circ \pi$. Therefore

$$|h| \leqq (v - u) + (\hat{v} - \hat{u}) \circ \pi .$$

By the definition of μ, $v - u$ is a *potential* on R, i.e. the greatest harmonic minorant of $v - u$ is zero. Let k be a harmonic minorant of $(\hat{v} - \hat{u})\circ\pi$ on R. In view of (1) there exists a $\hat{k} \in HB(U)$ with $k = \hat{k}\circ\pi$, and a fortiori $\hat{v} - \hat{u} \geqq \hat{k}$ on U. Since $\hat{v} - \hat{u}$ is a potential on U, $\hat{k}$ and then k is nonpositive. Therefore the greatest harmonic minorant of $(\hat{v} - \hat{u})\circ\pi$ is zero, i.e. $(\hat{v} - \hat{u})\circ\pi$ is a potential. Observe that a sum of potentials is again a potential. Thus the subharmonic function $|h|$ is dominated by a potential on R and a fortiori, $|h| = 0$, i.e. $u = \hat{u}\circ\pi$. Since nonnegative members in $HB(F^\infty; \partial F^\infty)$ generates $HB(F^\infty; \partial F^\infty)$, we can deduce the desired inclusion.

4. We proceed to the *proof of the main theorem* stated in the introduction. Let S be an admissible subsurface of a Tôki surface T so that $S = T - \pi^{-1}(K)$ with K as stated in the introduction. We denote by $K_n = \pi^{-1}(K) \cap V_n$ and $S_n = V_n - K_n$ $(n = 1, 2, \cdots)$. Let $\hat{w}$ be the *harmonic measure* of Γ: $|z| = 1$ with respect to $U - K$, i.e. $\hat{w} \in H(U - K) \cap C(U \cup \Gamma)$ with $\hat{w}|\Gamma = 1$ and $\hat{w}|K = 0$. Then $\{\hat{w} = \rho\}$ $(\rho \in (0, 1))$ consists of a finite number of piecewise analytic Jordan curves in $U - K$ separating Γ from K. We set $K_\rho = \{\hat{w} \leqq \rho\} \cup K$ and $K_0 = K$. Observe that there exists an $\eta \in (0, 1)$ such that K_ρ has the same property as K for every $\rho \in [0, \eta]$. We also denote by $(K_\rho)_n = \pi^{-1}(K_\rho) \cap V_n$ $(n = 1, 2, \cdots)$. Since K is compact in U and thus $K = U - (U - K)$ is B-negligible, Theorem 3[*] implies that

$$HB(S; \partial S) = HB(U - K; \partial(U - K))\circ\pi \tag{3}$$

where, as before, $\partial(U - K)$ is the relative boundary of $U - K$ with respect to U. This relation will be used repeatedly.

We will discuss the boundary behavior of a $u > 0$ in $HBD(S)$. Let $u_n = u|S_n$. Since $S_n = V_n - K_n$ may be identified with $V - K$, $\{u_n\}$ can also be considered as a sequence of functions on $V - K$. We fix a point a in $V - K$ and set

$$c_n = c_n(u, a) = u_n(a) .$$

We maintain the following

LEMMA. *The sequence $\{u_n - c_n(u, a)\}$ $(n \geqq 1)$ converges to zero uniformly on each compact subset of $V - K$. In particular* $\lim_{n\to\infty}(c_n(u, a) - c_n(u, b)) = 0$ *for every pair of points a and b in $V - K$.*

The key observation to our proof is:

$$\sum_{n=1}^{\infty} D_{V-K}(u_n - c_n) = \sum_{n=1}^{\infty} D_{V-K}(u_n) = \sum_{n=1}^{\infty} D_{S_n}(u) = D_S(u) < \infty ,$$

[*] We mean by Theorem 3 the theorem in no. 3. Similar conventions are used for lemmas and propositions.

where $c_n = c_n(u, a)$. Thus in particular

$$\lim_{n\to\infty} D_{V-K}(u_n - c_n) = 0 .$$

From $u_n(a) - c_n = 0$ $(n = 1, 2, \cdots)$, the desired conclusion follows.

5. The following, although only a direct consequence of the preceding lemma, will play the central role in proving the main theorem;

LEMMA. *Suppose that* $u > 0$ *in* $HBD(S)$ *dominates an* $\hat{h}\circ\pi$ *with* $\hat{h}$ *in* $HB(U-K; \partial(U-K))$ *on* S. *Then*

$$\liminf_{n\to\infty} c_n(u, a) \geqq \sup_{U-K} \hat{h} .$$

Since $V-K$ is dense in $U-K$, there exists a point $b \in V-K$ for any given positive number ε such that $\hat{h}(b) \geqq \sup_{U-K} h - \varepsilon$. Observe that $u_n \geqq \hat{h}$ on $V-K$ and in particular $u_n(b) \geqq \hat{h}(b)$. From $c_n(u, a) \geqq \hat{h}(b) + (c_n(u, a) - c_n(u, b))$ and Lemma 4, it follows that

$$\liminf_{n\to\infty} c_n(u, a) \geqq \hat{h}(b) \geqq \sup_{V-K} \hat{h} - \varepsilon .$$

On letting $\varepsilon \to 0$, we arrive at the desired conclusion.

6. Recall that $\hat{w}$ is the harmonic measure of Γ: $|z| = 1$ with respect to $U-K$. Let $w = \hat{w}\circ\pi$ which is in $HB(S; \partial S)$.

LEMMA. *The function* w *belongs to* $HD^{\sim}(S) \cap HB(S; \partial S)$.

It is convenient to use the theory of Royden compactification (cf. e.g. Chapter III in [4]). By Proposition 1, $T \in O_{HD} - O_{HB}$ and therefore the Royden harmonic boundary of T consists of a single point p, say. Since $\{p\}$ and $\cup_{j=1}^{n} K_j$ are disjoint compact sets in the Royden compactification T^* of T, there exists a $w_n \in HBD(T - \cup_{j=1}^{n} K_j) \cap C(T^*)$ such that $w_n(p) = 1$ and $w_n | (\cup_{j=1}^{n} K_j) = 0$ for each $n = 1, 2, \cdots$. Since $\{w_n\}$ $(n \geqq 1)$ is decreasing on T, we can easily see that $w^{\sim} = \lim_{n\to\infty} w_n$ belongs to $HD^{\sim}(S) \cap HB(S; \partial S)$. Since

$$\liminf_{z\to z^*} (w_n(z) - w(z)) \geqq 0$$

for every $z^* \in (\partial S) \cup \{p\}$, the maximum principle yields $w_n \geqq w$ $(n \geqq 1)$ and a fortiori $w^{\sim} \geqq w$. On the other hand, (3) implies that $w^{\sim} = \hat{w}^{\sim}\circ\pi$ with a $\hat{w}^{\sim} \in HB(U-K; \partial(U-K))$. Here in view of $0 \leqq w^{\sim} \leqq 1$, we also have $0 \leqq \hat{w}^{\sim} \leqq 1$, and thus $\hat{w}^{\sim} \leqq \hat{w}$, i.e. $w^{\sim} = \hat{w}^{\sim}\circ\pi \leqq \hat{w}\circ\pi = w$. Therefore $w = w^{\sim}$, i.e.

$$w = \lim_{n\to\infty} w_n . \tag{4}$$

7. We insert here a well-known general result on HX-minimal functions (cf. e.g. [4]):

Let F be a subsurface of a Riemann surface R and a a regular point in ∂F with respect to the Dirichlet problem. Then any HX-minimal function on F has a vanishing boundary value at a for $X = B, D$, and $D^{\sim}$.

Applying this to our present S as F, we obtain the following

LEMMA. *Every HX-minimal function belongs to $HB(S; \partial S)$. In particular there exist no HB- and HD-minimal functions on S.*

The first part is clear since every point in ∂S is regular. Let $u \in HD(S) \cap HB(S; \partial S)$. Then $u = \hat{u} \circ \pi$ with $\hat{u} \in HB(U-K; \partial(U-K))$. Thus $u_n = \hat{u}$ on $V-K$ and

$$D_S(u) = \sum_{n=1}^{\infty} D_{V-K}(u_n) = D_{V-K}(\hat{u}) \cdot \infty .$$

This shows that $D_S(u) = 0$ and hence u is constant and actually $u = 0$ on S, i.e. there exist no HD-minimal functions on S. Next suppose that there exists an HB-minimal function u on S. Then since $u \in HB(S; \partial S)$, $u = \hat{u} \circ \pi$ with a $\hat{u} \in HB(U-K; \partial(U-K))$. It is easy to see that $\hat{u}$ is also HB-minimal on $U-K$. Hence $\hat{u}$ has zero boundary values on the whole boundary of $U-K$, considered as a subregion of the complex plane, and thus $\hat{u} = 0$ on $U-K$, i.e. $u = 0$ on S, a contradiction.

8. We are in the stage to prove

LEMMA. *The function w is $HD^{\sim}$-minimal on S.*

Suppose that $0 < u \leqq w$ on S with $u \in HD^{\sim}(S)$. Let $\alpha = \sup_S u$ which is in $(0, 1]$ since $0 < w < 1$ on S. We shall show that $u = \alpha w$ on S. That $w \in HB(S; \partial S)$ implies that $u \in HB(S; \partial S)$, and a fortiori $u = \hat{u} \circ \pi$ with $\hat{u} \in HB(U-K; \partial(U-K))$. In view of $\sup_S u = \alpha$, we clearly have $\hat{u} \leqq \alpha \hat{w}$ on $V-K$. Therefore $u \leqq \alpha w$ on S. We thus have to show that $u \geqq \alpha w$ on S.

Let $\{u^i\}$ $(i = 1, 2, \cdots)$ be a decreasing sequence in $HD(S)$ converging to u on S. Replacing u^i by $u^i \wedge \alpha$ (the greatest harmonic minorant of u^i and α), if necessary, we may assume that $\alpha \geqq u^i \geqq u = \hat{u} \circ \pi$ on S. We fix an arbitrary $\rho \in (0, \eta]$ and $a \in \partial K_\rho$ (cf. no. 4). Clearly $c_n(u^i, a) \leqq \alpha$. With Lemma 5 we deduce

$$\alpha = \sup_{U-K} \hat{u} \leqq \liminf_{n \to \infty} c_n(u^i, a) \leqq \limsup_{n \to \infty} c_n(u^i, a) \leqq \alpha ,$$

i.e. we have

$$\lim_{n\to\infty} c_n(u^i, a) = \alpha .$$

This with Lemma 4 yields

$$\lim_{n\to\infty} \sup_{\partial(K_\rho)_n} |u^i - \alpha| = 0 .$$

Hence for every fixed positive number ε there exists an m such that $u^i + \varepsilon > \alpha$ on $\partial(K_\rho)_n$ for every $n > m$. Let $\bar{u}^i = u^i$ on $S - \cup_{n=1}^m (K_\rho)_n$ and $\bar{u}^i$ be in $H((K_\rho)_n - \partial(K_\rho)_n) \cap C((K_\rho)_n)$, with $\bar{u}^i = u^i$ on $\partial(K_\rho)_n$, on $(K_\rho)_n$ for $n = 1, \cdots, m$. Then $\bar{u}^i$ is a piecewise smooth continuous function on $T - \cup_{n>m} K_n = S \cup (\cup_{n=1}^m K_n)$ and has the finite Dirichlet integral over there. Next we consider a function $v^i \in C(T)$ defined as follows:

$$v^i = \begin{cases} \min(\bar{u}^i + \varepsilon, \alpha) & \text{on} \quad T - \bigcup\limits_{n>m} (K_\rho)_n ; \\ \alpha & \text{on} \quad \bigcup\limits_{n>m} (K_\rho)_n . \end{cases}$$

Then v^i is again a piecewise smooth continuous function on T and has the finite Dirichlet integral over T. Thus v^i belongs to the Royden algebra of T and a fortiori $v^i \in C(T^*)$.

Observe that the closure of $\cup_{n=1}^\infty (K_\rho)_n$ in T^* contains the single Royden harmonic boundary point p of T. If this were not the case, then the closure of $\pi^{-1}(K) = \cup_{n=1}^\infty K_n$ would also be disjoint from p and there would exist an $s \in HD(S) \cap C(T^*)$ such that $s(p) = 1$ and $s(\pi^{-1}(K)) = 0$. Then $s \in HB(S; \partial S)$ and as in no. 7, $D_S(s)$ could not be finite unless $s = 0$ on S, a contradiction. Since $\cup_{n=1}^m (K_\rho)_n$ is compact in T, we deduce that the closure of $\cup_{n>m} (K_\rho)_n$ in T^* contains p. In view of $v^i = \alpha$ on $\cup_{n>m} (K_\rho)_n$, we see that $v^i = \alpha$ on the closure of $\cup_{n>m} (K_\rho)_n$ in T^* and in particular $v^i(p) = \alpha$. Here

$$\liminf_{z\to z^*} \{(v^i(z) + \rho) - \alpha w(z)\} \geqq 0$$

for every $z^* \in (\partial(\pi^{-1}(K_\rho))) \cup \{p\}$, and the maximum principle yields

$$(u^i + \varepsilon) + \rho \geqq \alpha w$$

on $T - \pi^{-1}(K_\rho)$. Here ε can be as small as we wish and therefore

$$u^i + \rho \geqq \alpha w$$

on $T - \pi^{-1}(K_\rho)$. On letting $\rho \to 0$, we finally conclude that $u^i \geqq \alpha w$ on $T - \pi^{-1}(K) = S$ for every $i = 1, 2, \cdots$. Again on letting $i \to \infty$, we obtain the desired inequality $u \geqq \alpha w$.

9. We come to the final step for the proof of our main theorem;

LEMMA. *Any $HD^\sim$-minimal function u on S is a constant multiple* *of w. In particular $\{u \in HD^{\sim}(S) \cap C(S \cup \partial S); u \mid \partial S = 0\}$ is generated by w.*

Without loss of generality we can assume that $0 < u < 1$ on S. We see by Lemma 7 that $u \in HB(S; \partial S)$ and hence $u = \hat{u} \circ \pi$ with a $\hat{u} \in HB(U - K; \partial(U - K))$. Then $0 < \hat{u} < 1$ on $U - K$ and a fortiori $0 < \hat{u} \leqq \hat{w}$ on $U - K$. Therefore $0 < u \leqq w$ on S and the $HD^{\sim}$-minimality of w yields the existence of a constant c with $u = cw$. The second assertion is clear if we observe that w generates $\{u \in HBD^{\sim}(S) \cap C(S \cup \partial S); u \mid \partial S = 0\}$ which in turn generates the class in question in the sense of vector lattice (cf. e.g. [4]).

10. The main theorem in the introduction now follows from Lemmas 7-9. We denote by U_{HX} the class of hyperbolic Riemann surfaces R on which there exists an HX-minimal function ($X = B, D$, and $D^{\sim}$). It is known that both of U_{HX} ($X = B, D$) are contained in $U_{HD^{\sim}}$ (cf. e.g. [4]). Moreover

$$U_{HB} < U_{HD^{\sim}}\ , \quad U_{HD} < U_{HD^{\sim}}\ . \tag{5}$$

The strictness of these two inclusions can be seen by any admissible subsurface S of any Tôki surface T. Kwon [1] has constructed a surface R_K showing the strictness of the second inclusion of (5). It is also seen that there exist unique (up to multiplicative constant) HB- and $HD^{\sim}$-minimal functions but no HD-minimal functions on R_K. In this connection we are interested in finding surfaces with one or zero HX-minimal function ($X = B, D$, and $D^{\sim}$).

11. We denote by x ($x = b, d$, and $d^{\sim}$) the numbers of HX-minimal functions ($X = B, D$, and $D^{\sim}$) and by $R_{bdd^{\sim}}$ a Riemann surface on which there exist x HX-minimal functions ($x = b, d$, and $d^{\sim}$; $X = B, D$, and $D^{\sim}$). We consider the basic case of $x = 0, 1$. We obtain the following table:

Numbers \ Surfaces	R_{000}	R_{001}	R_{010}	R_{011}	R_{100}	R_{101}	R_{110}	R_{111}
b	0	0	0	0	1	1	1	1
d	0	0	1	1	0	0	1	1
$d^{\sim}$	0	1	0	1	0	1	0	1
Existence or Nonexistence	E	E	N	E	N	E	N	E

FIGURE 3

Here E (N, resp.) means that an (any. resp.) $R_{bdd^{\sim}}$ in the column containing E (N, resp.) exists (does not exist, resp.).

The relation (5) means that, for $x=0, 1$ ($x=b, d$, and $d^{\sim}$),

$$b \leqq d^{\sim}\ , \quad d \leqq d^{\sim}\ . \tag{6}$$

Therefore nonexistence of any R_{010}, R_{100}, and R_{110} can trivially be seen. Any hyperbolic plane region is an example of R_{000}. Every admissible subsurface of any Tôki surface is an example of R_{001}. Any Tôki surface itself is an example of R_{011} (cf. no. 7). The Kwon example R_K mentioned above qualifies to be an R_{101}. Finally, any surface R in $O_{HB}-O_G$ (cf. e.g. [4]) is an example of R_{111}.

12. Based on the above observation we deduce the following

THEOREM. *For any triple* $(b, d, d^{\sim})$ *of nonnegative integers with* $\max(b, d) \leqq d^{\sim}$, *there exists a Riemann surface on which there exist* b $(d, d^{\sim}$, *resp.*) *HB-(HD-, HD~-, resp.) minimal functions.*

Let R_{lmn} and $R_{l'm'n'}$ be as described in no. 11, and let β and β' a slit in R_{lmn} and $R_{l'm'n'}$, respectively. Connect $R_{lmn}-\beta$ and $R_{l'm'n'}-\beta'$ crosswise along β and β'. Then it can be seen easily that the resulting surface is of type $R_{l+l',m+m',n+n'}$ (cf. e.g. [4]). In this sense, we can define the operation

$$R_{lmn} \oplus R_{l'm'n'} = R_{l+l',m+m',n+n'}\ . \tag{7}$$

On the other hand, on setting $e_1=(0, 0, 1)$, $e_2=(0, 1, 1)$, $e_3=(1, 0, 1)$, and $e_4=(1, 1, 1)$, it is elementary to see that there exist nonnegative integers n_i $(1 \leqq i \leqq 4)$ such that

$$(b, d, d^{\sim}) = \sum_{i=1}^{4} n_i e_i\ . \tag{8}$$

From (7) and (8) the desired conclusion follows.

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