Tôhoku Math. Journ.
28 (1976), 583-590.

COMMUTATIVE NORMAL *-DERIVATIONS III*),**)

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(Received July 14, 1975)

1. **Introduction.** In the present paper, we shall continue the study of commutative derivations. We shall show general methods to obtain all *KMS*-states. Also we shall generalize the theorem in [6] to commutative derivations with infinite range interaction, and we shall examine the relation between this theorem and Dyson conjecture [1] and Kac-Thompson conjecture [1] concerning one-dimensional Ising ferromagnet.

2. **Theorems.** Let $\mathfrak{A}$ be a uniformly hyperfinite C^*-algebra, and let δ be a normal *-derivation in $\mathfrak{A}$-i.e., there is an increasing sequence of finite type I subfactors $\{\mathfrak{A}_n\}$ in $\mathfrak{A}$ such that $\bigcup_{n=1}^{\infty}\mathfrak{A}_n$ is dense in $\mathfrak{A}$ and the domain $\mathfrak{D}(\delta)$ of δ is $\bigcup_{n=1}^{\infty}\mathfrak{A}_n$. Then there is a sequence of self-adjoint elements $\{h_n\}$ in $\mathfrak{A}$ such that $\delta(a) = i[h_n, a]$ $(a \in \mathfrak{A}_n)$ $(n = 1, 2, \cdots)$. Suppose that δ is commutative -i.e., we can choose (h_n) as a commutative family.

Let $\mathfrak{L}_n$ be the C^*-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_n$ and h_n. Suppose that $h_n \in \bigcup_{m=1}^{\infty}\mathfrak{A}_m (n = 1, 2, \cdots)$; then $\mathfrak{L}_n$ is finitedimensional. Let $(p_{n,j})_{j=1}^{m(n)}$ be the family of all minimal projections in the center Z_n of $\mathfrak{L}_n$. Then $\mathfrak{L}_n = \sum_{j=1}^{m(n)} \mathfrak{L}_n p_{n,j}$. Let $\{\rho(t)\}$ be the strongly continuous one-parameter subgroup of *-automorphisms on $\mathfrak{A}$ corresponding to δ (cf. [5]). Then $\rho(t)(a) = e^{+tih_n}\, a e^{-tih_n}$ $(a \in \mathfrak{L}_n)$ and so $\rho(t)\mathfrak{L}_n p_{n,j} = \mathfrak{L}_n p_{n,j}$.

Now we shall show the following theorem.

THEOREM 1. *Suppose* $h_n \in \bigcup_{m=1}^{\infty}\mathfrak{A}_m (n = 1, 2, \cdots)$ *and let* $\psi_{n,j,\beta}(x) = \tau(xe^{-\beta h_n}p_{n,j})/\tau(e^{-\beta h_n}p_{n,j})$ $(x \in \mathfrak{A})$, *where* $-\infty < \beta < +\infty$ *and* τ *is the unique tracial state on* $\mathfrak{A}$. *Let*

$$G_\beta = \{\psi_{n,j,\beta} \mid 1 \leqq j \leqq m(n) \text{ and } n = 1, 2, 3, \cdots\} .$$

Then the set of all accumulation points in the $\sigma(\mathfrak{A}^*, \mathfrak{A})$-*closure* $\bar{G}_\beta$ *of* G_β *in the state space of* $\mathfrak{A}$ *contains all extreme KMS states on* $\mathfrak{A}$ *for* $\{\rho(t)\}$ *at* β. *Moreover every accumulation point of* $\bar{G}_\beta$ *in the* $\sigma(\mathfrak{A}^*, \mathfrak{A})$-*topology is a KMS state for* $\{\rho(t)\}$ *at* β, *where* $\mathfrak{A}^*$ *is the dual Banach space of* $\mathfrak{A}$.

*) This research is supported by National Science Foundation.
**) The content of this paper was announced in the international conference of K theory and operator algebras at University of Georgia on April 19, 1975.

PROOF. Let ϕ_β be an accumulation point of $\bar{G}_\beta$; then there is a subsequence (n_k) of (n) such that $\psi_{n_k,j_k,\beta} \to \phi_\beta$ in $\sigma(\mathfrak{A}^*, \mathfrak{A})$.

For n with $n \leqq n_k$, we can easily see that $\psi_{n_k,j_k,\beta}$ is a *KMS* state on $\mathfrak{L}_n$ for $\{\rho(t)\}$ at β and so ϕ_β is a *KMS* state on $\bigcup_{n=1}^\infty \mathfrak{A}_n$ for $\{\rho(t)\}$ at β; hence ϕ_β is a *KMS* state on $\mathfrak{A}$ for $\{\rho(t)\}$ at β (cf. Theorem 3.2 in [4]).

Conversely let ψ_β be a *KMS* state on $\mathfrak{A}$ for $\{\rho(t)\}$ at β. Since each $\mathfrak{L}_n p_{n,j}$ is invariant under $\rho(t)$ and it is a full matrix algebra, $\psi_\beta = \sum_{j=1}^{m(n)} \lambda_{n,j,\beta}\psi_{n,j,\beta}$ on $\mathfrak{L}_n$, where $\lambda_{n,j,\beta} > 0$ and $\sum_{j=1}^{m(n)} \lambda_{n,j,\beta} = 1$. The strict positivity of $\lambda_{n,j,\beta}$ comes from the fact that a *KMS* state on $\mathfrak{A}$ is faithful, since $\mathfrak{A}$ is simple.

Hence ψ_β is contained in the $\sigma(\mathfrak{A}^*, \mathfrak{A})$-closure of the convex subset of $\mathfrak{S}$ generated by G_β, where $\mathfrak{S}$ is the state space of $\mathfrak{A}$.

Now suppose that ψ_β is an extreme *KMS* state on $\mathfrak{A}$ for $\{\rho(t)\}$ at β; then by the well known theorem about convex set, all extreme points of the $\sigma(\mathfrak{A}^*, \mathfrak{A})$-closed convex subset generated by G_β belong to the $\sigma(\mathfrak{A}^*, \mathfrak{A})$-closure $\bar{G}_\beta$ of G_β. This completes the proof.

Now suppose that $h_n \in \bigcup_{m=1}^\infty \mathfrak{A}_m (n = 1, 2, \cdots)$. By replacing (n) by a suitable subsequence, we may assume that $h_n \in \mathfrak{A}_{n+1} (n = 1, 2, \cdots)$. Let $\{\mu_{n,j} \mid j = 1, 2, \cdots, m(n)\}$ be a family of real numbers and let $l_n = h_n + \sum_{j=1}^{m(n)} \mu_{n,j} p_{n,j}$; then $[h_n - l_n, a] = 0$ $(a \in \mathfrak{A}_n)$. It is clear that (l_n) is a commutative family corresponding to δ. Moreover

$$\frac{\tau(xe^{-\beta l_n})}{\tau(e^{-\beta l_n})} = \sum_{j=1}^{m(n)} \frac{\tau(p_{n,j}e^{-\beta l_n})}{\tau(e^{-\beta l_n})} \cdot \frac{\tau(xe^{-\beta l_n}p_{n,j})}{\tau(e^{-\beta l_n}p_{n,j})}$$
$$= \sum_{j=1}^{m(n)} \frac{\tau(p_{n,j}e^{-\beta l_n})}{\tau(e^{-\beta l_n})} \frac{\tau(xe^{-\beta h_n}p_{n,j})}{\tau(e^{-\beta h_n}p_{n,j})} \quad (x \in \mathfrak{L}_n) .$$

Since $\tau(p_{n,j}e^{-\beta l_n}) = \tau(p_{n,j}e^{-\beta h_n})e^{-\beta\mu_{n,j}}$, for an arbitrary family of $\{\lambda_1, \lambda_2, \cdots, \lambda_{m(n)}\}$ of positive numbers with $\sum_{j=1}^{m(n)} \lambda_j = 1$, there is a family of $\{\mu_{n,1}, \mu_{n,2}, \cdots, \mu_{n,m(n)}\}$ of real numbers such that

$$\tau(p_{n,j}e^{-\beta h_n})e^{-\beta\mu_{n,j}} = \lambda_j \quad \text{for} \quad j = 1, 2, \cdots, m(n);$$

then

$$\frac{\tau(p_{n,j}e^{-\beta l_n})}{\tau(e^{-\beta l_n})} = \frac{\lambda_j}{\sum_{j=1}^{m(n)} \tau(p_{n,j}e^{-\beta l_n})} = \lambda_j \quad \text{for} \quad j = 1, 2, \cdots, m(n) .$$

From these considerations we can easily conclude the following theorem.

THEOREM 2. *Suppose that δ is a commutative normal *-derivation with $\delta(\mathfrak{D}(\delta)) \subset \mathfrak{D}(\delta)$, and let $\{\rho(t)\}$ be the strongly continuous one-parameter group of *-automorphisms on $\mathfrak{A}$ corresponding to δ.*

Let ϕ_β *be an arbitrary KMS state on* $\mathfrak{A}$ *for* $\{\rho(t)\}$ *at* β. *Then there exists a commutative family of self-adjoint elements* $\{h_n\}$ *(depending on* β*) corresponding to* δ *such that*

$$h_n \in \mathfrak{D}(\delta) \quad and \quad \phi_\beta(a) = \frac{\tau(ae^{-\beta h_n})}{\tau(e^{-\beta h_n})} \quad (a \in \mathfrak{L}_n)\ (n = 1, 2, \cdots)\ ,$$

where $\mathfrak{L}_n$ *is the* C^**-subalgebra of* $\mathfrak{A}$ *generated by* $\mathfrak{A}_n$ *and* h_n.

Conversely, if for a state ψ_β *on* $\mathfrak{A}$, *there is a commutative family* $\{h_n\}$ *of self-adjoint elements in* $\mathfrak{A}$ *corresponding to* δ *such that*

$$\psi_\beta(a) = \frac{\tau(ae^{-\beta h_n})}{\tau(e^{-\beta h_n})} \quad (a \in \mathfrak{L}_n)\ (n = 1, 2, \cdots)\ ,$$

where $\mathfrak{L}_n$ *is the* C^**-subalgebra generated by* $\mathfrak{A}_n$ *and* h_n, *then* ψ_β *is a KMS state on* $\mathfrak{A}$ *for* $\{\rho(t)\}$ *at* β.

PROOF. Take a commutative family (r_n) of self-adjoint elements in $\mathfrak{D}(\delta)$ corresponding to δ (the existence of (r_n) with $r_n \in \mathfrak{D}(\delta)$ is equivalent to $\delta(\mathfrak{D}(\delta)) \subset \mathfrak{D}(\delta)$ (cf. [4])); then $\delta(a) = i[r_n, a]$ $(a \in \mathfrak{A}_n)$ $(n = 1, 2, \cdots)$.

Take a subsequence (r_{n_k}) of (r_n) such that $r_{n_k} \in \mathfrak{A}_{n_k+1}$.

Let $\mathfrak{M}_{n_k}$ be the C^*-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_{n_k}$ and r_{n_k}; then

$$\phi_\beta = \sum_{j=1}^{p} \lambda_{n_k,j,\beta}\psi_{n_k,j,\beta} \quad \text{on} \quad \mathfrak{M}_{n_k}\ ,$$

where $p =$ the dimension of the center of $\mathfrak{M}_{n_k}$, $\psi_{n_k,j,\beta}$ is the unique *KMS* state on $\mathfrak{M}_{n_k}p_{n_k,j}$ for $\{\rho(t)\}$ at β, $\lambda_{n_k,j,\beta} > 0$ and $\sum_{j=1}^{p} \lambda_{n_k,j,\beta} = 1$. Then by the previous discussions, we can choose h_{n_k} in $\mathfrak{M}_{n_k}$ such that

$$[h_{n_k} - r_{n_k}, a] = 0 \quad (a \in \mathfrak{M}_{n_k})$$

and

$$\frac{\tau(p_{n_k,j}e^{-\beta h_{n_k}})}{\tau(e^{-\beta h_{n_k}})} = \lambda_{n_k,j,\beta} \quad \text{for} \quad j = 1, 2, \cdots, p\ ,$$

where $\{p_{n_k,j}\}_{j=1}^{p}$ is the family of all minimal central projections in the center of $\mathfrak{M}_{n_k}$. Then for $a \in \mathfrak{M}_{n_k}$,

$$\frac{\tau(ae^{-\beta h_{n_k}})}{\tau(e^{-\beta h_{n_k}})} = \sum_{j=1}^{p} \frac{\tau(p_{n_k,j}e^{-\beta h_{n_k}})}{\tau(e^{-\beta h_{n_k}})} \cdot \frac{\tau(ae^{-\beta h_{n_k}}p_{n_k,j})}{\tau(e^{-\beta h_{n_k}}p_{n_k,j})}\ .$$

By the uniqueness of *KMS* states on $\mathfrak{M}_{n_k}p_{n_k,j}$,

$$\frac{\tau(ae^{-\beta h_{n_k}}p_{n_k,j})}{\tau(e^{-\beta h_{n_k}}p_{n_k,j})} = \psi_{n_k,j,\beta}(ap_{n_k,j}) \quad (a \in \mathfrak{M}_{n_k})\ .$$

Hence

$$\frac{\tau(ae^{-\beta h_{n_k}})}{\tau(e^{-\beta h_{n_k}})} = \sum_{j=1}^{p} \lambda_{n_k,j,\beta}\psi_{n_k,j,\beta}(ap_{n_k,j}) = \phi_\beta(a) \quad (a \in \mathfrak{M}_{n_k}) .$$

Let $\mathfrak{L}_{n_k}$ be the C^*-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_{n_k}$ and h_{n_k}; then $\mathfrak{L}_{n_k} \subset \mathfrak{M}_{n_k}$; hence we have

$$\frac{\tau(ae^{-\beta h_{n_k}})}{\tau(e^{-\beta h_{n_k}})} = \phi_\beta(a) \quad (a \in \mathfrak{L}_{n_k}) .$$

Now for n with $n_{k-1} < n \leqq n_k$, we define $h_n = h_{n_k}$. Then clearly

$$\phi_\beta(a) = \frac{\tau(ae^{-\beta h_n})}{\tau(e^{-\beta h_n})} \quad (a \in \mathfrak{L}_n) ,$$

where $\mathfrak{L}_n$ is the C^*-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_n$ and h_n, since $\mathfrak{A}_n \subset \mathfrak{A}_{n_k}$ and so $\mathfrak{L}_n \subset \mathfrak{L}_{n_k}$. The second part of the theorem is clear. This completes the proof.

REMARK 1. For $a \in \mathfrak{A}_n$ and $n \leqq m$, consider two analytic functions $\tau(ae^{-\gamma h_n})/\tau(e^{-\gamma h_n})$, $\tau(ae^{-\gamma h_m})/\tau(e^{-\gamma h_m})$ on $(-\infty, +\infty)$ $(\gamma \in (-\infty, +\infty))$. They coincide with $\phi_\beta(a)$ at β. It would be an interesting problem how these two functions are related each other in a neighborhood of β.

Let ϕ_γ be the *KMS*-state on $\mathfrak{A}$ for $\{\rho(t)\}$ at γ defined by an accumulation point of

$$\left\{\xi_{n,\gamma} \mid \xi_{n,\gamma}(x) = \frac{\tau(xe^{-\gamma h_n})}{\tau(e^{-\gamma h_n})} \quad (x \in \mathfrak{A})\right\}$$

in the state space of $\mathfrak{A}$.

Is there a nice relation between ϕ_β and ϕ_γ where γ is in a neighborhood of β ?

Under what conditions can we choose (h_n) such that

$$(a\phi_\gamma) = \frac{\tau(ae^{-\gamma h_n})}{\tau(e^{-\gamma h_n})} \quad (a \in \mathfrak{A}_n) \ (n = 1, 2, \cdots)$$

for γ in a neighborhood of β ? There problems are quite important from the stand point of the phase transition theory.

Next we shall extend Theorem 1 to commutative derivations with infinite range interaction.

Suppose that $\mathfrak{A} = \bigotimes_{n=1}^{\infty} \mathfrak{B}_n$, where $\mathfrak{B}_n$ are finite type I factors and $\bigotimes_{n=1}^{\infty} \mathfrak{B}_n$ is the infinite C^*-tensor product of $\{\mathfrak{B}_n\}$. Let C_n be a maximal commutative C^*-subalgebra of $\mathfrak{B}_n$ and let $C = \bigotimes_{n=1}^{\infty} C_n$. Then C is considered as a commutative C^*-subalgebra of $\mathfrak{A}$. Put $\mathfrak{A}_n = \bigotimes_{m=1}^{n} \mathfrak{B}_m \subset \mathfrak{A}$ and let δ be a *-derivation of $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ into $\mathfrak{A}$ such that $\delta(a) = i[h_n, a]$ $(a \in \mathfrak{A}_n)$ with $h_n \in C$ $(n = 1, 2, \cdots)$. Then δ is a commutative normal

*-derivation. All derivations arising from classical lattice systems and Ising models satisfy these properties. Let

$D_n = \bigotimes_{m=n+1}^{\infty} C_n \subset \bigotimes_{m=n+1}^{\infty} \mathfrak{B}_m \subset \bigotimes_{m=1}^{\infty} \mathfrak{B}_m = \mathfrak{A}$: then $\mathfrak{A}_n \otimes D_n$ is invariant under $\rho(t)$, and

$$\rho(t)(a) = e^{+ith_n} a e^{-ith_n} \quad (a \in \mathfrak{A}_n \otimes D_n) .$$

Put $D_n = C(K_n)$, where $C(K_n)$ is the C^*-algebra of all complex valued continuous functions on a compact space K_n. Let Φ be the unique $C(K_n)$-valued tracial state on $\mathfrak{A}_n \otimes D_n$. For $s \in K_n$, let

$$\mathfrak{J}_{n,s} = \{x \mid \Phi(x^*x)(s) = 0,\ x \in \mathfrak{A}_n \otimes D_n\};$$

then $\mathfrak{J}_{n,s}$ is a maximal ideal of $\mathfrak{A}_n \otimes D_n$ and it is invariant under $\rho(t)$.

Since an extreme KMS state defines a factor representation,

$$\left\{\phi_{n,\beta,s} \mid \phi_{n,\beta,s}(x) = \frac{\Phi(xe^{-\beta h_n})(s)}{\Phi(e^{-\beta h_n})(s)} \quad (x \in \mathfrak{A}_n \otimes D_n); \quad s \in K_n\right\}$$

is the set of all extreme KMS states on $\mathfrak{A}_n \otimes D_n$ for $\{\rho(t)\}$ at β. Let

$$E_{n,\beta} = \left\{\psi_{n,\beta,s} \mid \psi_{n,\beta,s}(x) = \frac{\Phi(xe^{-\beta h_n})(s)}{\Phi(e^{-\beta h_n})(s)} \quad (x \in \mathfrak{A})\right\} .$$

Now let ϕ be a KMS state on $\mathfrak{A}$ for $\{\rho(t)\}$ at β; then for each n there is a unique probability measure μ_n on K_n such that

$$\phi(x) = \int_{K_n} \frac{\Phi(xe^{-\beta h_n})(s)}{\Phi(e^{-\beta h_n})(s)} d\mu_n(s) \quad (x \in \mathfrak{A}_n \otimes D_n) .$$

Hence we can easily show the following theorem.

THEOREM 3. *Let* $H_\beta = \bigcup_{n=1}^{\infty} E_{n,\beta}$; *then the* $\sigma(\mathfrak{A}^*, \mathfrak{A})$-*closure* $\bar{H}_\beta$ *of* H_β *in the state space of* $\mathfrak{A}$ *contains all extreme KMS state on* $\mathfrak{A}$ *for* $\{\rho(t)\}$ *at* β.

Since K_n is totally disconnected, we can easily conclude the following theorem from Theorem 3.

THEOREM 4. *Let*

$$\psi_{n,p_n,\beta}(x) = \frac{\tau(xe^{-\beta h_n} p_n)}{\tau(e^{-\beta h_n} p_n)} \quad (x \in \mathfrak{A}) ,$$

where $-\infty < \beta < +\infty$ *and for a projection* p_n *in* D_n *and let*

$$U_\beta = \{\psi_{n,p_n,\beta} \mid n = 1, 2, \cdots;\ p_n \in D_n^p\} ,$$

where D_n^p *is the set of all non-zero projections in* D_n. *Then the* $\sigma(\mathfrak{A}^*, \mathfrak{A})$-*closure of* U_β *in the state space of* $\mathfrak{A}$ *contains all extreme KMS states on* $\mathfrak{A}$ *for* $\{\rho(t)\}$ *at* β.

Now we shall extend the theorem in [6] to derivations with infinite interaction.

THEOREM 5. *Suppose that* $h_n \in \bigotimes_{m=1}^{\infty} C_m$ $(n = 1, 2, \cdots)$ *and let* P_n *be the canonical conditional expectation of* $\mathfrak{A}$ *onto* $\mathfrak{A}_n = \bigotimes_{m=1}^{n} \mathfrak{B}_m$.

If a sequence $\{\|h_n - P_n(h_n)\|\}$ *is bounded, then the strongly continuous one-parameter group* $\{\rho(t)\}$ *of *-automorphisms on* $\mathfrak{A}$ *corresponding to* δ *has no phase transition at arbitrary inverse temperature.*

PROOF. The proof is quite similar with the proof of the theorem in [6].

Since $P_n(h_n) \in \bigotimes_{m=1}^{n} C_m$, by the same method with the proof of Lemma 1 in [6] we can show that

$$\frac{1}{K}\frac{\tau(ye^{-P_n(h_n)})}{\tau(e^{-P_n(h_n)})} \leqq \frac{\tau(ye^{-h_n}p)}{\tau(e^{-h_n}p)} \leqq K\frac{\tau(ye^{-P_n(h_n)})}{\tau(e^{-P_n(h_n)})}$$

for $y(\geqq 0) \in \bigotimes_{m=1}^{n} C_m$, where p is an arbitrary non-zero projection p in $\bigotimes_{m=n+1}^{\infty} C_m$ and K is a fixed number. Hence

$$\frac{1}{K^2}\frac{\tau(ye^{-h_n})}{\tau(e^{-h_n})} \leqq \frac{\tau(ye^{-h_n}p)}{\tau(e^{-h_n}p)} \leqq K^2\frac{\tau(ye^{-h_n})}{\tau(e^{-h_n})}\,.$$

Without loss of generality, we may assume that $\{\tau(ye^{-h_n})/\tau(e^{-h_n})\}$ converges to a $\phi(x)$ in $\sigma(\mathfrak{A}^*, \mathfrak{A})$; then by Theorem 4, we have

$$\frac{1}{K^2}\phi(y) \leqq \psi(y) \leqq K^2\phi(y) \quad (y(\geqq 0) \in \bigotimes_{m=1}^{\infty} C_m)$$

for all extreme KMS states ψ on $\mathfrak{A}$ for $\{\rho(t)\}$ at $\beta = 1$. On the other hand, for $x \in \mathfrak{A}_n \otimes D_n$, there is a unique probability measure μ_n on K_n such that

$$\psi(x) = \int_{K_m} \frac{\Phi(xe^{-h_n})(s)}{\Phi(e^{-h_n})(s)} d\mu_n(s)$$

and similarly there is a unique probability measure ν_n on K_n such that

$$\phi(x) = \int_{K_m} \frac{\Phi(xe^{-h_n})(s)}{\Phi(e^{-h_n})(s)} d\nu_n(s)\,.$$

Since

$$\frac{1}{K^2}\phi(y) \leqq \psi(y) \leqq K^2\phi(y) \quad (y(\geqq 0) \in \bigotimes_{m=n}^{\infty} C_m)\,,$$

$$\frac{1}{K^2}\nu_n \leqq \mu_n \leqq K^2\nu_n\,.$$

Hence we have

$$\frac{1}{K^2}\phi(x) \leqq \psi(x) \leqq K^2\phi(x) \quad (x(\geqq 0) \in \mathfrak{A}_n \otimes D_n) \quad (n = 1, 2, \cdots) .$$

Therefore

$$\frac{1}{K^2}\phi(x) \leqq \psi(x) \leqq K^2\phi(x) \quad (x(\geqq 0) \in \mathfrak{A}) .$$

Since ψ is extreme, $\phi = \psi$ and so there is no phase transition at $\beta = 1$. Since $\|\beta h_n - P_n(\beta h_n)\| = |\beta| \|h_n - P(h_n)\|$, by the same discussions we can conclude that there is no phase transition at arbitrary temperature. This completes the proof. Finally we shall state some remarks on the problem whether we can relax the boundedness condition of the sequence $\{\|h_n - P_n(h_n)\|\}$ to incur "no phase transition at arbitrary inverse temperature".

Consider one-dimensional Ising ferromagnet. Let Z be the group of integers and let B be the full matrix algebra of 2×2. For each $p \in Z$, we shall consider a copy B_p of B. Let $\mathfrak{A} = \bigotimes_{p \in Z} B_p$ and let $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in B$ and let σ_p be the element in B_p corresponding to σ.

Consider the total energy:

$$H = -\sum_{p>q} J(p - q)\sigma_p\sigma_q$$

with $J(n) \geqq 0$, $n = 1, 2, \cdots$. Let $\Lambda_n = \{p \mid -n \leqq p \leqq n,\ p \in Z\}$ and let $\mathfrak{A}_n = \bigotimes_{p \in \Lambda_n} B_p \subset \mathfrak{A}$. Put $M_0 = \sum_{n=1}^{\infty} J(n)$ and $M_1 = \sum_{n=1}^{\infty} nJ(n)$. Suppose that $M_0 < +\infty$; then for $a \in \mathfrak{A}_n$,

$$\begin{aligned} \delta(a) &= i[H, a] \\ &= -i\Big[\sum_{k=-n}^{n}\sum_{j=1}^{\infty} J(j)\sigma_{k+j}\sigma_k + \sum_{k=-n}^{n}\sum_{j=1}^{\infty} J(k + n + j)\sigma_k\sigma_{-n-j},\ a\Big] \quad (a \in \mathfrak{A}_n) . \end{aligned}$$

Since

$$\begin{aligned} &\sum_{k=-n}^{n}\sum_{j=1}^{\infty} \| J(j)\sigma_{k+j}\sigma_k \| + \sum_{k=-n}^{n}\sum_{j=1}^{\infty} \| J(k + n + j)\sigma_k\sigma_{-n-j} \| \\ &\qquad \leqq (2n + 1)\sum_{j=1}^{\infty} J(j) + (2n + 1)\sum_{j=1}^{\infty} J(j) = 2(2n + 1)M_0 < +\infty . \end{aligned}$$

Therefore, put

$$h_n = -\sum_{k=-n}^{n}\sum_{j=1}^{\infty} J(j)\sigma_{k+j}\sigma_k - \sum_{k=-n}^{n}\sum_{j=1}^{\infty} J(k + n + j)\sigma_k\sigma_{-n-j} .$$

Then $h_n \in \mathfrak{A}$ and $\delta(a) = i[h_n, a]$ $(a \in \mathfrak{A}_n)$. Clearly (h_n) is a commutative family, so that δ is a commutative normal *-derivation with $\mathfrak{D}(\delta) = \bigcup_{n=1}^{\infty} \mathfrak{A}_n$. Therefore by the theorem in [5], there is a strongly continuous one-parameter group $\{\rho(t)\}$ of *-automorphisms on $\mathfrak{A}$ such that $\rho(t)(a) = \exp it\delta_{h_n}(a)$ $(a \in \mathfrak{A}_n)$ and $t \in (-\infty, \infty)$ $(n = 1, 2, \cdots)$, where $\delta_{h_n}(a) = [h_n, a]$ $(a \in \mathfrak{A})$. Let $r_n = -\sum_{k=-n}^{n} \sum_{j=1}^{n-k} J(j)\sigma_{k+j}\sigma_k$; then $r_n \in \mathfrak{A}_n$ and so $\|h_n - P_n(h_n)\| \leqq \|h_n - r_n\|$. Moreover, in this case $P_n(h_n) = r_n$ since $P_n(\sigma_p) = 0$ $(|p| > n)$ and $P_n(\sigma_p\sigma_q) = P_n(\sigma_p)\sigma_q$ if $|q| \leqq n$. Hence

$$h_n - P_n(h_n) = h_n - r_n = -\sum_{j=1}^{\infty} J(j)\sigma_{n+j}\sigma_n$$
$$-\sum_{j=2}^{\infty} J(j)\sigma_{n-1+j}\sigma_{n-1} - \cdots - \sum_{j=2n+1}^{\infty} J(j)\sigma_{-n+j}\sigma_{-n}$$
$$-\sum_{k=-n}^{n} \sum_{j=1}^{\infty} J(k+n+j)\sigma_k\sigma_{-n-j} .$$

Hence

$$\|h_n - P_n(h_n)\|$$
$$\leqq 2\{J(1) + 2J(2) + \cdots + (2n+1)J(2n+1) + (2n+1) \sum_{j=2n+2}^{\infty} J(j)\}$$
$$\leqq 2\{\sum_{j=1}^{2n+1} jJ(j) + (2n+1) \sum_{j=2n+2}^{\infty} J(j)\} .$$

If $M_1 < +\infty$, then

$$\|h_n - P_n(h_n)\| \leqq 2 \sum_{j=1}^{\infty} jJ(j) = 2M_1 < +\infty .$$

Let C_p be the C^*-subalgebra of B_p generated by $\{\sigma_p, 1\}$; then $h_n \in \bigotimes_{p \in Z} C_p$ $(n = 1, 2, \cdots)$. Therefore $\{h_n\}$ satisfies all the conditions in Theorem 6. Therefore if $M_1 < +\infty$, there is no phase transition at arbitrary temperature (cf. Ruelle's theorem in [3]).

It is an outstanding open question whether or not "no phase transition at arbitrary inverse temperature" implies $M_1 < +\infty$ (cf. [1, 2]).

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