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STABILITY AND EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF SOME FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction. Assuming the uniqueness of solutions for the initial value problem, several authors discussed that the existence of a bounded solution with some stability property implies the existence of almost periodic solutions (cf. [4], [8], [10], [11], [12]).

Recently, Coppel [3], Yoshizawa [13, 14, 15] and Kato [7] have shown the existence of an almost periodic solution for ordinary differential equations and for functional differential equations without the assumption that the solution is unique. All of them required the existence of a bounded solution which is uniformly stable. Clearly, uniform stability implies the uniqueness of the bounded solution for initial value problem.

In this paper, more generally, we shall discuss functional differential equations with infinite retardation and the existence theorem for almost periodic solutions by assuming the existence of a bounded solution with some stability property which does not imply the uniqueness for initial value problem.

2. Hale's space and some lemmas. First, we shall give a class of Banach spaces considered by Hale [5]. Let $x = (x^1, x^2, \cdots, x^n)$ be any vector in R^n and let $\|x\|_{R^n} = \max_{1 \leqq i \leqq n} |x^i|$ be norm of x. Let $B = B((-\infty, 0], R^n)$ be a Banach space of functions mapping $(-\infty, 0]$ into R^n with norm $\|\cdot\|_B$. For any ϕ in B and any σ in $[0, \infty)$, let ϕ^σ be the restriction of ϕ to the interval $(-\infty, -\sigma]$. This is a function mapping $(-\infty, -\sigma]$ into R^n. We shall denote by B^σ the space of such functions ϕ^σ. For any $\eta \in B^\sigma$, we define the semi-norm $\|\eta\|_{B^\sigma}$ of η by

$$\|\eta\|_{B^\sigma} = \inf_{\phi} \{\|\phi\|_B : \phi^\sigma = \eta\} .$$

Then we can regard the space B^σ as a Banach space with norm $\|\cdot\|_{B^\sigma}$. If x is a function defined on $(-\infty, a)$, $a > 0$, then for each t in $[0, a)$ we define the function x_t by the relation $x_t(s) = x(t + s)$, $-\infty < s \leqq 0$. For numbers a and τ, $a > \tau$, we denote by $A_\tau{}^a$ the class of function x mapping $(-\infty, a)$ into R^n such that x is a continuous function on $[\tau, a)$ and $x_\tau \in B$. The space B is assumed to have the following properties:

(I) If x is in A_τ^a, then x_t is in B for all t in $[\tau, a)$ and x_t is a continuous function of t, where a and τ are constants such that $\tau < a \leqq \infty$.

(II) All bounded continuous functions mapping $(-\infty, 0]$ into R^n are in B.

(III) If a sequence $\{\phi_k\}$, $\phi_k \in B$, is uniformly bounded on $(-\infty, 0]$ with respect to norm $\|\cdot\|_{R^n}$ and converges to ϕ uniformly on any compact subset of $(-\infty, 0]$, then $\phi \in B$ and $\|\phi_k - \phi\|_B \to 0$ as $k \to \infty$.

(IV) There are continuous, increasing and nonnegative functions $b(r)$, $c(r)$ defined on $[0, \infty)$, $b(0) = c(0) = 0$, such that

$$\|\phi\|_B \leqq b(\sup_{-\sigma \leqq s \leqq 0} \|\phi(s)\|_{R^n}) + c(\|\phi^\sigma\|_{B^\sigma})$$

for any ϕ in B and any $\sigma \geqq 0$.

(V) If σ is a nonnegative number and ϕ is an element in B, then $T_\sigma\phi$ defined by $T_\sigma\phi(s) = \phi(s + \sigma)$, $s \in (-\infty, -\sigma]$, is an element in B^σ and $\|T_\sigma\phi\|_{B^\sigma} \to 0$ as $\sigma \to \infty$.

(VI) $\|\phi(0)\|_{R^n} \leqq M_1\|\phi\|_B$ for $M_1 > 0$.

(VII) B is separable.

(VIII) If ϕ and ψ are in B and $\|\phi(\theta)\|_{R^n} \leqq \|\psi(\theta)\|_{R^n}$ for all $\theta \in (-\infty, 0]$, then $\|T_s\phi\|_{B^s} \leqq \|T_s\psi\|_{B^s}$ for all $s \geqq 0$.

REMARK 1. We can easily show that the class of phase spaces considered by Coleman and Mizel [2] has the properties (I)-(VIII).

For an element η in B and for positive numbers N and L, define $S^*(\eta, N, L)$ by

$$S^*(\eta, N, L) = \{\phi \in A_0^\infty;\ \phi_0 = \eta,\ \|\phi(t)\|_{R^n} \leqq N \quad \text{for all} \quad t \geqq 0 \quad \text{and}$$
$$\|\phi(\theta) - \phi(\theta')\|_{R^n} \leqq L|\theta - \theta'| \quad \text{for any} \quad \theta, \theta' \geqq 0\}.$$

LEMMA 1. *Let $\{t_m\}$, $t_m \to \infty$ as $m \to \infty$, and $\{\phi^m\}$, $\phi^m \in S^*(\eta, N, L)$, be sequences. Then $\{\phi^m(t + t_m)\}$ has a subsequence $\{\phi^{m_k}(t + t_{m_k})\}$ such that $\phi^{m_k}(t + t_{m_k})$ converges to a function $y(t)$ uniformly on any compact interval in R^1. Furthermore, $\|\phi^{m_k}{}_{t_{m_k}+t} - y_t\|_B \to 0$ as $k \to \infty$ uniformly on compact subset of R^1.*

PROOF. Take any compact interval $K_n = [-n, n]$. We can assume that $n < t_1$ and $t_m < t_{m+1}$, for $m = 1, 2, \cdots$. Since $\{\phi^m(t + t_m), t \in K_n\}$ is uniformly bounded and equicontinuous, there exists a subsequence of $\{\phi^m(t + t_m)\}$ which converges uniformly on K_n by Ascoli-Arzelà's Theorem. Letting $n = 1, 2, \cdots$ and using the familiar diagonalization procedure, we can get a subsequence $\{\phi^{m_k}(t + t_{m_k})\}$ of $\{\phi^m(t+t_m)\}$ that will be uniformly convergent to a function y on any compact subset of R^1. The limit function y is in B by (II). Define $\phi^{m_k,t}(\theta)$ and $\eta^{m_k,t}(\theta)$ by

$$\phi^{m_k,t}(\theta) = \begin{cases} \phi^{m_k}(t + t_{m_k} + \theta) & \text{for} \quad -(t + t_{m_k}) \leqq \theta \leqq 0 , \\ \eta(0) & \text{for} \quad -\infty < \theta < -(t + t_{m_k}) , \end{cases}$$

and

$$\eta^{m_k,t}(\theta) = \begin{cases} 0 & \text{for} \quad -(t + t_{m_k}) \leqq \theta \leqq 0 , \\ T_{t_{m_k}+t}\eta(\theta) - \eta(0) & \text{for} \quad -\infty < \theta < -(t + t_{m_k}) , \end{cases}$$

respectively. Clearly, $\phi^{m_k,t}$ and $\eta^{m_k,t}$ are in B by (I) and (II). Hypothesis (IV) implies

$$\begin{aligned} ||\phi^{m_k}{}_{t_{m_k}+t} - y_t||_B &= ||\phi^{m_k,t} + \eta^{m_k,t} - y_t||_B \\ &\leqq ||\phi^{m_k,t} - y_t||_B + ||\eta^{m_k,t}||_B \\ &\leqq ||\phi^{m_k,t} - y_t||_B + b\Big(\sup_{-(t+t_{m_k})\leqq\theta\leqq 0} ||\eta^{m_k,t}(\theta)||_{R^n}\Big) \\ &\quad + c(||(\eta^{m_k,t})^{t+t_{m_k}}||_{B^{t+t_{m_k}}}) . \end{aligned}$$

Hence we have

$$(1) \qquad ||\phi^{m_k}{}_{t+t_{m_k}} - y_t||_B \leqq ||\phi^{m_k,t} - y_t||_B + c(||T_{t+t_{m_k}}(\eta - \langle\eta(0)\rangle)||_{B^{t+t_{m_k}}}) ,$$

where $\langle\eta(0)\rangle$ is the constant function $\beta \in B$ such that $\beta(s) = \eta(0)$ for all $s \in (-\infty, 0]$. Since $\phi^{m_k,t}(\theta) \to y(\theta)$ uniformly on any compact set, the right hand side of (1) tends to zero as $m \to \infty$ by (III) and (V). This proves Lemma 1.

LEMMA 2. *The set*

$$S(\eta, N, L) = \{\phi_t;\ t \geqq 0,\ \phi \in S^*(\eta, N, L)\}$$

is relatively compact in B.

PROOF. For any sequence $\{\psi^m\}$, $\psi^m \in S(\eta, N, L)$, there are sequences $\{\phi^m\}$, $\phi^m \in S^*(\eta, N, L)$ and $\{t_m\}$, $t_m \geqq 0$, such that $\psi^m = \phi^m{}_{t_m}$, where we can assume that $t_m \to \infty$ or $t_m \to \tau$ for some constant $\tau \geqq 0$, taking subsequences, if necessary. When $t_m \to \infty$ as $m \to \infty$, the sequence $\{\psi^m\}$ contains a convergent subsequence by Lemma 1, and hence we consider the case where $t_m \to \tau$ as $m \to \infty$.

We can assume that $t_m \leqq \tau_1$ for all m and for some positive constant τ_1, $\tau_1 > \tau$. Since $\{\phi^m(t)\}$ is uniformly bounded and equicontinuous on $[0, \infty)$, there exists a subsequence of $\{\phi^m(t)\}$ which converges to a function y^* uniformly on any compact set of $[0, \infty)$. We shall denote it by $\{\phi^m(t)\}$ again. The limit function y^* is continuous and bounded on $[0, \infty)$. Define $y(t)$ by

$$y(t) = \begin{cases} y^*(t) & \text{for} \quad t \in (0, \infty) , \\ \eta(t) & \text{for} \quad t \in (-\infty, 0] . \end{cases}$$

Then $y \in A_0^\infty$ and $y_0 = \eta$. We have

$$\begin{aligned}||\phi^m{}_{t_m} - y_\tau||_B &\leqq ||\phi^m{}_{t_m} - y_{t_m}||_B + ||y_{t_m} - y_\tau||_B \\ &\leqq b\Big(\sup_{-t_m \leqq s \leqq 0} ||\phi^m(s + t_m) - y(s + t_m)||_{R^n}\Big) \\ &\quad + c(||(\phi^m{}_{t_m})^{t_m} - y^{t_m}{}_{t_m}||_{B^{t_m}}) + ||y_{t_m} - y_\tau||_B \\ &\leqq b\Big(\sup_{-\tau_1 \leqq s \leqq 0} ||\phi^m(s + \tau_1) - y(s + \tau_1)||_{R^n}\Big) \\ &\quad + c(||T_{t_m}\eta - T_{t_m}\eta||_{B^{t_m}}) + ||y_{t_m} - y_\tau||_B .\end{aligned}$$

Hence we have

$$(2) \qquad ||\phi^m{}_{t_m} - y_\tau||_B \leqq b\Big(\sup_{-\tau_1 \leqq s \leqq 0} ||\phi^m(s + \tau_1) - y(s + \tau_1)||_{R^n}\Big) + ||y_{t_m} - y_\tau||_B .$$

Since $\phi^m(t) \to y(t)$ as $m \to \infty$ uniformly on $[0, \tau_1]$ and $y \in A_0^\infty$, the right hand side of (2) tends to zero as $m \to \infty$.

Thus we can see that any sequence $\{\psi^m\}$, $\psi^m \in S(\eta, N, L)$, contains a convergent subsequence, and hence $S(\eta, N, L)$ is relatively compact in B.

3. Asymptotically almost periodic function and definitions of stabilities and separations. Let $f(t)$ be a continuous function defined on $a \leqq t < \infty$. $f(t)$ is said to be asymptotically almost periodic if it is a sum of a continuous almost periodic function $p(t)$ and a continuous function $q(t)$ defined on $a \leqq t < \infty$ which tends to zero as $t \to \infty$, that is

$$(3) \qquad f(t) = p(t) + q(t) .$$

It is well known that $f(t)$ is asymptotic almost periodic if and only if for any sequence $\{\tau_k\}$ such that $\tau_k \to \infty$ as $k \to \infty$ there exists a subsequence $\{\tau_{k_j}\}$ for which $f(t + \tau_{k_j})$ converges uniformly on $a \leqq t < \infty$.

Consider an almost periodic system

$$(4) \qquad \dot{x}(t) = F(t, x_t) ;$$

where $F(t, \phi)$ is continuous on $R^1 \times \bar{B}_M$, $\bar{B}_M = \{\phi \in B; ||\phi||_B \leqq M\}$, and almost periodic in t uniformly for $\phi \in \bar{B}_M$. We assume that there exists an $L > 0$ such that $||F(t, \phi)||_{R^n} \leqq L$ on $R^1 \times \bar{B}_M$. A function $\xi(t)$ is said to be a solution of (4) defined on $[\sigma, \sigma + \tau)$, where $\sigma \in R^1$ and $\tau > 0$, if $\xi \in A_\sigma^{\sigma+\tau}$, $\xi_t \in \bar{B}_M$ for $t \in [\sigma, \sigma + \tau)$ and $\xi(t)$ satisfies (4) for $t \in [\sigma, \sigma + \tau)$. In particular, if $\xi(t)$ is continuous on R^1 and $\xi_t \in \bar{B}_M$ for all $t \in R^1$ and if $\xi(t)$ satisfies (4) for $t \in R^1$, we say $\xi(t)$ is a solution of (4) defined on R^1.

Let $\xi(t)$ be a solution of (4) defined on I, $I = [0, \infty)$, which satisfies $||\xi_t||_B \leqq \beta$, $0 < \beta < M$, for all $t \in I$. Then, clearly, $\xi \in S^*(\xi_0, M_1\beta, L)$. Let $H(\xi)$, $H(F)$ and $H(\xi, F)$ be the hulls of $\xi(t)$, $F(t, \phi)$ and $(\xi(t), F(t, \phi))$, respectively. For the definitions of hulls, see [7]. Let $H^+(\xi)$, $H^+(F)$ and $H^+(\xi, F)$ be the subsets of $H(\xi)$, $H(F)$ and $H(\xi, F)$ whose elements are $x(t)$, $G(t, \phi)$ and $(x(t), G(t, \phi))$ such that there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that $\xi(t + t_k) \to x(t)$ as $k \to \infty$ uniformly on any compact interval in R^1 and $F(t + t_k, \phi) \to G(t, \phi)$ as $k \to \infty$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$, respectively, where $\overline{S(\xi_0, M_1\beta, L)}$ is the closure of $S(\xi_0, M_1\beta, L)$. We shall define $H_G(\xi)$ and $H_G^+(\xi)$ by

$$H_G(\xi) = \{x(t);\ (x(t), G(t, \phi)) \in H(\xi, F)\}$$

and

$$H_G^+(\xi) = \{x(t);\ (x(t), G(t, \phi)) \in H^+(\xi, F)\}\ ,$$

respectively.

The following lemma is one of the conclusions of Theorem 1 in [6].

LEMMA 3. *For any* $x \in H_G(\xi)$, $x(t)$ *is a solution of*

$$\dot{x}(t) = G(t, x_t) \tag{5}$$

defined on I and $G(t, \phi)$ *is almost periodic in t uniformly for* $\phi \in \bar{B}_M$. *In particular,* $x(t)$, $x(t) \in H_G^+(\xi)$, *is a solution of* (5) *defined on* R^1.

REMARK 2. We can easily show that if $F(t, \phi)$ is periodic in t, then Lemma 3 holds without separability of the space B.

THEOREM 1. *If the solution* $\xi(t)$ *of* (4) *is asymptotically almost periodic, then for any* $G \in H^+(F)$, *there exists a sequence* $\{\tau_k\}$ *such that* $\xi(t + \tau_k)$ *tends to an almost periodic solution of the system* (5) *uniformly on* R^1 *as* $k \to \infty$.

PROOF. Since $\xi(t)$ is asymptotically almost periodic, it has the decomposition $\xi(t) = p(t) + q(t)$, where $p(t)$ is almost periodic and $q(t) \to 0$ as $t \to \infty$. Since $G \in H^+(F)$, there exist a sequence $\{\tau_k\}$ and a function $p^*(t)$ such that $\tau_k \to \infty$ as $k \to \infty$ and $F(t + \tau_k, \phi) \to G(t, \phi)$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$ as $k \to \infty$ and that $p(t + \tau_k) \to p^*(t)$ as $k \to \infty$ uniformly on R^1. Then $p^*(t)$ is almost periodic and $(p^*(t), G(t, \phi)) \in H^+(\xi, F)$. By Lemma 3, $p^*(t)$ is an almost periodic solution of (5).

Now we shall give definitions of stabilities and separations.

DEFINITION 1. The solution $\xi(t)$ is uniformly stable with respect to $H_F^+(\xi)$ (in short, u.s. $H_F^+(\xi)$), if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|\xi_t - x_t\|_B < \varepsilon$ for all $t \geqq t_0$, whenever $t_0 \in I$, $x \in H_F^+(\xi)$ and $\|\xi_{t_0} - x_{t_0}\|_B < \delta(\varepsilon)$.

DEFINITION 2. The solution $\xi(t)$ is quasi uniformly asymptotically stable with respect to $H_F^+(\xi)$ (in short, q.u.a.s. $H_F^+(\xi)$), if there exists a $\delta_0 > 0$ and for any $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that $\|\xi_t - x_t\|_B < \varepsilon$ for $t \geqq t_0 + T(\varepsilon)$, whenever $x \in H_F^+(\xi)$ and $\|\xi_{t_0} - x_{t_0}\|_B < \delta_0$ for some $t_0 \in I$.

DEFINITION 3. The solution $\xi(t)$ is uniformly asymptotically stable with respect to $H_F^+(\xi)$ (in short, u.a.s. $H_F^+(\xi)$), if it is u.s. $H_F^+(\xi)$ and q.u.a.s. $H_F^+(\xi)$.

DEFINITION 4. The solution $\xi(t)$ is uniformly asymptotically stable in the large with respect to $H_F^+(\xi)$ (in short, u.a.s.l. $H_F^+(\xi)$), if it is u.s. $H_F^+(\xi)$ and for any $\alpha > 0$ and $\varepsilon > 0$, there exists a $T(\alpha, \varepsilon) > 0$ such that $\|\xi_t - x_t\|_B < \varepsilon$ for $t \geqq t_0 + T(\alpha, \varepsilon)$, whenever $t_0 \in I$, $x \in H_F^+(\xi)$ and $\|\xi_{t_0} - x_{t_0}\|_B < \alpha$.

DEFINITION 5. The solution $\xi(t)$ is stable under disturbances from $H^+(\xi, F)$ (in short, s.d. $H^+(\xi, F)$), if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\|\xi_{t+\tau} - x_t\|_B < \varepsilon$ for $t \geqq 0$, whenever $(x, G) \in H^+(\xi, F)$, $\|\xi_\tau - x_0\|_B < \delta(\varepsilon)$ and $\rho(F^\tau, G) = \sup\{\|F(t+\tau, \phi) - G(t, \phi)\|_{R^n}, t \in R^1, \phi \in \overline{S(\xi_0, M_1\beta, L)}\} < \delta(\varepsilon)$ for some $\tau \geqq 0$.

DEFINITION 6. The solutions in $H^+(\xi)$ are quasi uniformly asymptotically stable with respect to $H^+(\xi)$ with a common pair $(\delta_0, T(\cdot))$ (in short, q.u.a.s. $H^+(\xi)\cdot(\delta_0, T(\cdot))$), if for any $\varepsilon > 0$, any $t_0 \in R^1$ and $G \in H^+(F)$, $\|x_{t_0} - y_{t_0}\|_B < \delta_0$ implies $\|x_t - y_t\|_B < \varepsilon$ for $t \geqq t_0 + T(\varepsilon)$, whenever $x(t)$, $y(t) \in H_G^+(\xi)$.

DEFINITION 7. The solutions in $H^+(\xi)$ are uniformly asymptotically stable with respect to $H^+(\xi)$ with a common triple $(\delta_0, \delta(\cdot), T(\cdot))$ (in short, u.a.s. $H^+(\xi)\cdot(\delta_0, \delta(\cdot), T(\cdot))$), if the solutions in $H^+(\xi)$ are q.u.a.s. $H^+(\xi)\cdot(\delta_0, T(\cdot))$ and for any $\varepsilon > 0$, any $t_0 \geqq 0$ and $G \in H^+(F)$, $\|x_{t_0} - y_{t_0}\|_B < \delta(\varepsilon)$ implies $\|x_t - y_t\|_B < \varepsilon$ for all $t \geqq t_0$, whenever $x(t)$, $y(t) \in H_G^+(\xi)$.

DEFINITION 8. The solutions in $H^+(\xi)$ are uniformly asymptotically stable with a common triple $(\delta_0, \delta(\cdot), T(\cdot))$ (in short, u.a.s. $(\delta_0, \delta(\cdot), T(\cdot))$), if for any $\varepsilon > 0$, any $t_0 \geqq 0$ and any $G \in H^+(F)$, $\|x_{t_0} - y_{t_0}\|_B < \delta(\varepsilon)$ implies $\|x_t - y_t\|_B < \varepsilon$ for all $t \geqq t_0$ and $\|x_{t_0} - y_{t_0}\|_B < \delta_0$ implies $\|x_t - y_t\|_B < \varepsilon$ for $t \geqq t_0 + T(\varepsilon)$, whenever $y(t)$ is a solution of (5) and $x(t) \in H_G^+(\xi)$.

REMARK 3. The definitions of stabilities with respect to hull are weaker definitions than the usual ones, respectively, because $\xi(t)$ is not necessarily unique for initial value problem. For example, the solution $x(t) = 0$ of $\dot{x}(t) = x^{1/3}$ is not uniformly stable for $t \geqq 0$, but u.a.s.

$H^+{}_{x^{1/3}}(x(t)=0)$, s.d. $H^+(x(t)=0, x^{1/3})$ and u.a.s. $H^+(x(t)=0)\cdot(\delta_0, \delta(\cdot), T(\cdot))$.

DEFINITION 9. $H^+(\xi, F)$ is said to satisfy a separation condition if for any $G \in H^+(F)$, $H_G^+(\xi)$ is a finite set and if ϕ and ψ, $\phi, \psi \in H_G^+(\xi)$, are distinct solutions of (5), then there exists a $\lambda(G, \phi, \psi) > 0$ such that $\|\phi_t - \psi_t\|_B \geqq \lambda(G, \phi, \psi)$ for all $t \in R^1$.

If the solution $\xi(t)$ is defined on R^1, then the following separation condition is stronger than the separation condition given in Definition 9.

DEFINITION 10. $H(\xi, F)$ is said to satisfy a separation condition if for any $G \in H(F)$, $H_G(\xi)$ is a finite set and if ϕ and ψ, $\phi, \psi \in H_G(\xi)$, are distinct solutions of (5) defined on R^1, then there exists a $\lambda(G, \phi, \psi) > 0$ such that $\|\phi_t - \psi_t\|_B \geqq \lambda(G, \phi, \psi)$ for all $t \in R^1$.

REMARK 4. The separation conditions on $H^+(\xi, F)$ and $H(\xi, F)$ are weaker conditions than Amerio's condition [1].

4. **Periodic system.** In this section, we assume that the space B has the properties (I)~(VI) and (VIII). Consider the system

$$\dot{x}(t) = F(t, x_t)\,, \tag{6}$$

where $F(t, \phi)$ is continuous on $R^1 \times \bar{B}_M$ and $F(t, \phi) = F(t+\omega, \phi)$, $\omega > 0$. We assume that there exists an $L > 0$ such that $\|F(t, \phi)\|_{R^n} \leqq L$ on $R^1 \times \bar{B}_M$. Moreover, we assume that the system (6) has a solution $\xi(t)$ defined on I such that $\|\xi_t\|_B \leqq \beta$, $0 < \beta < M$, for $t \geqq 0$.

THEOREM 2. *If the solution $\xi(t)$ is u.s. $H_F^+(\xi)$, then $\xi(t)$ is an asymptotically almost periodic solution of* (6). *Consequently, the system* (6) *has an almost periodic solution.*

PROOF. Let $\{\tau_k\}$ be a sequence such that $\tau_k \geqq \omega$ and $\tau_k \to \infty$ as $k \to \infty$. For each τ_k, there exists a positive integer N_k such that $N_k\omega \leqq \tau_k < (N_k+1)\omega$. If we set $\tau_k = N_k\omega + \sigma_k$, then $0 \leqq \sigma_k < \omega$. There exists a σ, $0 \leqq \sigma \leqq \omega$, and a function $\eta(t)$ such that $\sigma_k \to \sigma$ as $k \to \infty$, $\xi(t+\tau_k) \to \eta(t)$ as $k \to \infty$ uniformly on any compact interval in R^1 and $F(t+\tau_k, \phi) \to F(t+\sigma, \phi)$ as $k \to \infty$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$, taking a subsequence, if necessarily. Therefore for a given $\varepsilon > 0$, there is a positive integer $n_0(\varepsilon)$ such that if $k \geqq n_0(\varepsilon)$, then

$$\|\xi_{\tau_k+\omega} - \eta_\omega\|_B < \delta(\varepsilon)/2 \tag{7}$$

by Lemma 1, where $\delta(\cdot)$ is the one for u.s. $H_F^+(\xi)$ of $\xi(t)$. Since $\|\eta(t_1) - \eta(t_2)\|_{R^n} \leqq L|t_1 - t_2|$ for $t_1, t_2 \in R^1$ and $\|\eta(t)\|_{R^n} \leqq M_1\beta$ for all $t \in R^1$, we have for any $\nu \geqq 0$

$$\begin{aligned}\|\eta_{t_1} - \eta_{t_2}\|_B &\leqq b\Big(\sup_{-\nu \leqq s \leqq 0} \|\eta(t_1 + s) - \eta(t_2 + s)\|_{R^n}\Big) + c(\|(\eta_{t_1} - \eta_{t_2})^\nu\|_{B^\nu}) \\ &\leqq b(L|t_1 - t_2|) + c(\|T_\nu\langle 2M_1\beta\rangle\|_{B^\nu})\end{aligned}$$

by (IV) and (VIII), and hence η_t is uniformly continuous by (V). Thus we may assume that for $k \geqq n_0(\varepsilon)$ and for all $t \in R^1$

$$\|\eta_t - \eta_{\sigma_k - \sigma + t}\|_B < \delta(\varepsilon)/2 \,. \tag{8}$$

By Lemma 3, $\eta(t)$ is a solution of $\dot{x}(t) = F(t + \sigma, x_t)$ defined on R^1, and hence $\eta^k(t) = \eta(t - N_k\omega - \sigma)$ is a solution of (6) defined on R^1. Clearly, we have $\eta^k(t) \in H_F^+(\xi)$. Since

$$\begin{aligned}\|\xi_{\tau_k+\omega} - \eta^k{}_{\tau_k+\omega}\|_B &= \|\xi_{\tau_k+\omega} - \eta_{\omega+\sigma_k-\sigma}\|_B \\ &\leqq \|\xi_{\tau_k+\omega} - \eta_\omega\|_B + \|\eta_\omega - \eta_{\omega+\sigma_k-\sigma}\|_B \,,\end{aligned}$$

we have by (7) and (8)

$$\|\xi_{\tau_k+\omega} - \eta^k{}_{\tau_k+\omega}\|_B < \delta(\varepsilon) \,,$$

if $k \geqq n_0(\varepsilon)$, which implies

$$\|\xi_{\tau_k+t} - \eta^k{}_{\tau_k+t}\|_B < \varepsilon \,, \tag{9}$$

because $\xi(t)$ is u.s. $H_F^+(\xi)$. Furthermore, since we have

$$\begin{aligned}\|\xi_{\tau_k+t} - \xi_{\tau_m+t}\|_B &\leqq \|\xi_{\tau_k+t} - \eta^k{}_{\tau_k+t}\|_B + \|\eta^k{}_{\tau_k+t} - \eta^m{}_{\tau_m+t}\|_B \\ &\quad + \|\eta^k{}_{\tau_m+t} - \xi_{\tau_m+t}\|_B \\ &\leqq \|\xi_{\tau_k+t} - \eta^k{}_{\tau_k+t}\|_B + \|\eta_{\sigma_k-\sigma+t} - \eta_{\sigma_m-\sigma+t}\|_B \\ &\quad + \|\eta^m{}_{\tau_m+t} - \xi_{\tau_m+t}\|_B \\ &\leqq \|\xi_{\tau_k+t} - \eta^k{}_{\tau_k+t}\|_B + \|\eta_{\sigma_k-\sigma+t} - \eta_t\|_B \\ &\quad + \|\eta_t - \eta_{\sigma_m-\sigma+t}\|_B + \|\eta^m{}_{\tau_m+t} - \xi_{\tau_m+t}\|_B \,,\end{aligned}$$

(8) and (9) imply

$$\|\xi_{\tau_k+t} - \xi_{\tau_m+t}\|_B < 3\varepsilon \tag{10}$$

for all $t \geqq \omega$ and for $m \geqq k \geqq n_0(\varepsilon)$. It follows from (10) and (VI) that

$$\|\xi(\tau_k + t) - \xi(\tau_m + t)\|_{R^n} < 3M_1\varepsilon \text{ for all } t \geqq \omega \text{ and for } m \geqq k \geqq n_0(\varepsilon) \,.$$

Thus we see that for any sequence $\{\tau_k\}$ such that $\tau_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{\tau_{k_j}\}$ for which $\xi(t + \tau_{k_j})$ converges uniformly on $[\omega, \infty)$ as $j \to \infty$. This shows that $\xi(t)$ is asymptotically almost periodic. The existence of an almost periodic solution follows immediately from Theorem 1.

The following example shows that the converse of Theorem 2 is not true even for a periodic solution.

EXAMPLE. Consider the differential difference equation

$$\dot{x}(t) = x(t - 3\pi/2) . \tag{11}$$

This equation is the special case of the equation (6) (see [5]). Clearly, $\xi(t) = \sin t$ is a bounded periodic solution of (11), that is, an asymptotically almost periodic solution of (11). Set $t_n = (2n + 1)\pi$ and $\eta(t) = \sin(t + \pi)$, then $\eta(t)$ is a solution of (11), $\xi(t + t_n) = \eta(t)$, $|\xi(0) - \eta(0)| = |\sin 0 - \sin \pi| = 0$ and $|\xi(\pi/2) - \eta(\pi/2)| = |\sin \pi/2 - \sin 3\pi/2| = 2$, and hence $\xi(t)$ is not u.s. $H^+_{x(t-3\pi/2)}(\xi)$.

LEMMA 4. *If $\xi(t)$ is u.a.s. $H^+_F(\xi)$, then any $\eta(t)$, $\eta(t) \in H^+_G(\xi)$, is u.a.s. $H^+_G(\xi)$.*

PROOF. For any $\eta(t)$, $\eta(t) \in H^+_G(\xi)$, there exists a sequence $\{\tau_k\}$, $\tau_k \to \infty$ as $k \to \infty$, such that $\xi(t + \tau_k) \to \eta(t)$ as $k \to \infty$ uniformly on any compact interval in R^1 and $F(t + \tau_k, \phi) \to G(t, \phi)$ as $k \to \infty$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$. Set $\tau_k = N_k\omega + \sigma_k$, where $0 \leqq \sigma_k < \omega$. Let $\{k_j\}$ be a subsequence of $\{k\}$ such that $\sigma_{k_j} \to \sigma$ as $j \to \infty$. Then we have $0 \leqq \sigma \leqq \omega$ and $G(t, \phi) = F(t + \sigma, \phi)$.

First, we shall see that $\eta(t)$ is u.s. $H^+_{F^\sigma}(\xi)$. For any $\varepsilon > 0$, let $\delta(\varepsilon)$ be the one for u.s. $H^+_F(\xi)$ of $\xi(t)$, where we can assume that $\varepsilon < (M - \beta)/2$. Let $\zeta(t)$ be in $H^+_{F^\sigma}(\xi)$ and $t_0 \in R^1$. Assume $\|\eta_{t_0} - \zeta_{t_0}\|_B = r < \delta(\varepsilon)$ and put $\xi^{k_j}(t) = \xi(t + \tau_{k_j})$. If j is sufficiently large, we have

$$\|\xi^{k_j}{}_{t_0} - \eta_{t_0}\|_B < (\delta(\varepsilon) - r)/2$$

and

$$\|\xi_{t_0+\sigma+N_{k_j}\omega} - \xi_{t_0+\sigma_{k_j}+N_{k_j}\omega}\|_B = \|\xi^{k_j}{}_{t_0+\sigma-\sigma_{k_j}} - \xi^{k_j}{}_{t_0}\|_B < (\delta(\varepsilon) - r)/2$$

by Lemma 1, and hence

$$\begin{aligned}
&\|\xi_{t_0+\sigma+N_{k_j}\omega} - \zeta_{t_0}\|_B \\
&\quad \leqq \|\xi_{t_0+\sigma+N_{k_j}\omega} - \xi_{t_0+\sigma_{k_j}+N_{k_j}\omega}\|_B + \|\xi_{t_0+\sigma_{k_j}+N_{k_j}\omega} - \eta_{t_0}\|_B \\
&\qquad + \|\eta_{t_0} - \zeta_{t_0}\|_B \\
&\quad \leqq \|\xi^{k_j}{}_{t_0+\sigma-\sigma_{k_j}} - \xi^{k_j}{}_{t_0}\|_B + \|\xi^{k_j}{}_{t_0} - \eta_{t_0}\|_B + \|\eta_{t_0} - \zeta_{t_0}\|_B \\
&\quad < \delta(\varepsilon) .
\end{aligned}$$

Since the solution $\xi(t + \sigma + N_{k_j}\omega)$ is u.s. $H^+_{F^\sigma}(\xi)$, we have

$$\|\xi_{t+\sigma+N_{k_j}\omega} - \zeta_t\|_B < \varepsilon \quad \text{for all} \quad t \geqq t_0 .$$

On the other hand, for an arbitrary $\gamma > 0$, if j is sufficiently large,

$$\begin{aligned}\|\eta_{t_0}-\xi_{t_0+\sigma+N_{k_j}\omega}\|_B &\leqq \|\eta_{t_0}-\xi_{t_0+\sigma_{k_j}+N_{k_j}\omega}\|_B+\|\xi_{t_0+\sigma_{k_j}+N_{k_j}}\\ &\quad -\xi_{t_0+\sigma+N_{k_j}\omega}\|_B\\ &\leqq \|\eta_{t_0}-\xi^{k_j}{}_{t_0}\|_B+\|\xi^{k_j}{}_{t_0}-\xi^{k_j}{}_{t_0+\sigma-\sigma_{k_j}}\|_B\\ &<\delta(\gamma)\end{aligned}$$

by Lemma 1, and hence $\|\eta_t-\xi_{t+\sigma+N_{k_j}\omega}\|_B<\gamma$ for all $t\geqq t_0$. Thus we have $\|\eta_t-\zeta_t\|_B<\varepsilon+\gamma$ for all $t\geqq t_0$. Since γ is arbitrary, we have

$$\|\eta_t-\zeta_t\|_B\leqq\varepsilon \quad \text{for all} \quad t\geqq t_0\,, \quad \text{if} \quad \|\eta_{t_0}-\zeta_{t_0}\|_B<\delta(\varepsilon)\,.$$

This proves that $\eta(t)$ is u.s. $H^+_G(\xi)$.

Next, we show that $\eta(t)$ is u.a.s. $H^+_G(\xi)$. Let $\zeta(t)$ be in $H^+_{F^\sigma}(\xi)$ and $t_0\in R^1$. Assume that $\|\eta_{t_0}-\zeta_{t_0}\|_B=r<\delta_0$, where δ_0 is the one for u.a.s. $H^+_F(\xi)$ of $\xi(t)$. Clearly $\xi(t+\sigma+N_{k_j}\omega)$ is u.a.s. $H^+_{F^\sigma}(\xi)$ with the same δ_0 as the one for $\xi(t)$. We have

$$\|\xi_{t_0+\sigma+N_{k_j}\omega}-\zeta_{t_0}\|_B<\delta_0$$

and

$$\|\xi_{t_0+\sigma+N_{k_j}\omega}-\eta_{t_0}\|_B<\delta_0\,,$$

if j is sufficiently large, by using the same arguments as in the proof of the first part in this lemma. Hence, for sufficiently large j,

$$\|\xi_{t+\sigma+N_{k_j}\omega}-\zeta_t\|_B<\varepsilon \qquad \text{for} \quad t\geqq t_0+T(\varepsilon)$$

and

$$\|\xi_{t+\sigma+N_{k_j}\omega}-\eta_t\|_B<\varepsilon \qquad \text{for} \quad t\geqq t_0+T(\varepsilon)\,.$$

Thus

$$\|\eta_t-\zeta_t\|_B<2\varepsilon \qquad \text{for} \quad t\geqq t_0+T(\varepsilon)\,,$$

if $\|\eta_{t_0}-\zeta_{t_0}\|_B<\delta_0$. This shows that $\eta(t)$ is u.a.s. $H^+_G(\xi)$.

COROLLARY 1. *If the solution $\xi(t)$ is u.s. $H^+_F(\xi)$, then the system* (6) *has an almost periodic solution which is u.s. $H^+_F(\xi)$.*

PROOF. By Theorem 2, $\xi(t)$ is asymptotically almost periodic, and hence $\xi(t)=p(t)+q(t)$, where $p(t)$ is almost periodic and $q(t)\to 0$ as $t\to\infty$. Let $\tau_{k_j}=k_j\omega$ such that $p(t+\tau_{k_j})\to p^*(t)$ uniformly on R^1, where k_j is a positive integer. Then $p^*(t)$ is almost periodic and $\xi(t+\tau_{k_j})\to p^*(t)$ as $j\to\infty$ uniformly on any compact interval in R^1. By Lemmas 3 and 4, $p^*(t)$ is an almost periodic solution of (6) which is u.s. $H^+_F(\xi)$.

THEOREM 3. *If the solution $\xi(t)$ is u.a.s. $H^+_F(\xi)$, then the system* (6) *has a periodic solution of period $m\omega$ for some integer $m\geqq 1$.*

PROOF. There exist a sequence $\{k_j\}$, where k_j is a positive integer and $k_j \to \infty$ as $j \to \infty$, and an $\eta \in H_F^+(\xi)$ such that $\xi(t + k_j\omega) \to \eta(t)$ uniformly on any compact interval in R^1 as $j \to \infty$, because $F(t, \phi)$ is periodic of period ω. There are integers k_p and k_{p+1} such that $\|\xi_{k_p\omega} - \eta_0\|_B < \delta_0$ and $\|\xi_{k_{p+1}\omega} - \eta_0\|_B < \delta_0$ by Lemma 1, where δ_0 is the one for u.a.s. $H_F^+(\xi)$ of $\xi(t)$. Set $m = k_{p+1} - k_p$, $\xi^m(t) = \xi(t + m\omega)$ and $\eta^{k_p}(t) = \eta(t - k_p\omega)$. Clearly, $\eta^{k_p}(t) \in H_F^+(\xi)$ and $\xi^m(t)$ is the solution of (6). Thus we have

$$\|\xi^m{}_{k_p\omega} - \eta^{k_p}{}_{k_p\omega}\|_B = \|\xi_{k_{p+1}\omega} - \eta_0\|_B < \delta_0$$

and

$$\|\xi_{k_p\omega} - \eta^{k_p}{}_{k_p\omega}\|_B = \|\xi_{k_p\omega} - \eta_0\|_B < \delta_0 ,$$

and hence

$$\|\xi^m{}_{k_p\omega+t} - \eta_t\|_B \to 0 \qquad \text{as} \quad t \to \infty$$

and

$$\|\xi_{k_p\omega+t} - \eta_t\|_B \to 0 \qquad \text{as} \quad t \to \infty .$$

Thus we have

$$\|\xi^m{}_t - \xi_t\|_B \to 0 \qquad \text{as} \quad t \to \infty . \tag{12}$$

On the other hand, by Theorem 2, $\xi(t)$ is asymptotically almost periodic, and hence

$$\xi(t) = p(t) + q(t) , \tag{13}$$

where $p(t)$ is almost periodic and $q(t) \to 0$ as $t \to \infty$. From (12) and (13), it follows that

$$\begin{aligned} \|p(t) - p(t + m\omega)\|_{R^n} &\leqq \|\xi(t) - q(t) - \xi(t + m\omega) + q(t + m\omega)\|_{R^n} \\ &\leqq \|\xi(t) - \xi(t + m\omega)\|_{R^n} + \|q(t) - q(t + m\omega)\|_{R^n} \\ &\leqq M_1\|\xi_t - \xi^m{}_t\|_B + \|q(t) - q(t + m\omega)\|_{R^n} \to 0 \end{aligned}$$

as $t \to \infty$. Therefore $p(t) = p(t + m\omega)$ for all $t \in R^1$, because $p(t)$ is almost periodic. If we consider a sequence $\{km\omega\}$, we have

$$\xi(t + km\omega) = p(t) + q(t + km\omega) ,$$

and hence $p(t) \in H_F^+(\xi)$. This shows that the system (6) has a periodic solution $p(t)$ of period $m\omega$, because $p(t)$ also is a solution of (6) by Lemma 3.

REMARK 5. In Theorem 3, if $\xi(t)$ is u.a.s.l. $H_F^+(\xi)$, we have $\|\xi_{t+\omega} - \xi_t\|_B \to 0$ as $t \to \infty$, and hence, clearly $p(t) = p(t + \omega)$.

COROLLARY 2. *If $\xi(t)$ is u.a.s. $H_F^+(\xi)$, then the system* (6) *has a per-*

iodic solution of period $m\omega$ for some integer $m \geqq 1$ which is u.a.s. $H_F^+(\xi)$.

PROOF. By Theorem 3, the system (6) has a periodic solution $p(t)$ of period $m\omega$ for integer $m \geqq 1$ which is in $H_F^+(\xi)$. And $p(t)$ is u.a.s. $H_F^+(\xi)$ by Lemma 4.

5. Almost periodic system. In this section, we assume that the space B has the properties (I)$\sim$(VII). We shall discuss the existence of an asymptotically almost periodic solution of an almost periodic system

$$\dot{x}(t) = F(t, x_t)\ , \tag{14}$$

where $F(t, \phi)$ is continuous on $R^1 \times \bar{B}_M$ and almost periodic in t uniformly for $\phi \in \bar{B}_M$. We assume that there exists an $L > 0$ such that $\|F(t, \phi)\|_{R^n} \leqq L$ on $R^1 \times \bar{B}_M$. Moreover, we assume that the system (14) has a bounded solution $\xi(t)$ defined on I such that $\|\xi_t\|_B \leqq \beta$, $0 < \beta < M$, for $t \geqq 0$.

THEOREM 4. *If the solution $\xi(t)$ is s.d. $H^+(\xi, F)$, then $\xi(t)$ is an asymptotically almost periodic solution. Consequently,* (14) *has an almost periodic solution.*

PROOF. Let $\{\tau_k\}$ be any sequence such that $\tau_k > 0$ and $\tau_k \to \infty$ as $k \to \infty$. Since $F(t, \phi)$ is almost periodic in t uniformly for $\phi \in \bar{B}_M$, we may assume the existence of an $(x(t), G(t, \phi)) \in H^+(\xi, F)$ such that $\xi(t + \tau_k) \to x(t)$ as $k \to \infty$ uniformly on any compact interval in R^1 and $F(t + \tau_k, \phi) \to G(t, \phi)$ as $k \to \infty$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$. For any $\varepsilon > 0$ there exists a $k_0(\varepsilon) > 0$ such that if $k \geqq k_0(\varepsilon)$, then $\|\xi_{\tau_k} - x_0\|_B < \delta(\varepsilon)$ and $\rho(F^{\tau_k}, G) < \delta(\varepsilon)$, where $\delta(\varepsilon)$ is the one for s.d. $H^+(\xi, F)$ of $\xi(t)$, which implies that

$$\|\xi_{\tau_k+t} - x_t\|_B < \varepsilon \qquad \text{for} \quad t \geqq 0\ ,$$

because $x_t \in \overline{S(\xi_0, M_1\beta, L)}$ for all $t \in R^1$ by Lemmas 1 and 2. Therefore $\|\xi_{\tau_k+t} - \xi_{\tau_m+t}\|_B < 2\varepsilon$ for all $t \geqq 0$, if $m \geqq k \geqq k_0(\varepsilon)$. Thus we see that $\xi(t)$ is asymptotically almost periodic. The existence of an almost periodic solution follows immediately from Theorem 1.

LEMMA 5. *Let $\{x^k(t), G^k(t, \phi)\}$, $(x^k, G^k) \in H^+(\xi, F)$, and $\{s_k\}$, $s_k \in R^1$, be any sequences. Then $\{x^k(t + s_k), G^k(t + s_k, \phi)\}$ has a subsequence $\{x^{k_j}(t+s_{k_j}), G^{k_j}(t + s_{k_j}, \phi)\}$ such that for some $(y, G) \in H^+(\xi, F)$, $x^{k_j}(t + s_{k_j}) \to y(t)$ uniformly on any compact interval in R^1 as $j \to \infty$ and $G^{k_j}(t + s_{k_j}, \phi) \to G(t, \phi)$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$ as $j \to \infty$.*

PROOF. Put $y^k(t) = x^k(t + s_k)$, then $y^k(t)$ is uniformly bounded and equicontinuous on R^1, and hence $y^k(t)$ has a subsequence which tends to a function $y(t)$ uniformly on any compact interval in R^1. Since $H(F)$ is compact (cf. see [7]), $G^k(t+s_k, \phi)$ has a subsequence which tends to an almost periodic function $G(t, \phi)$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$. By Lemma 3, $y(t)$ is a solution of (5). Hence we have the conclusion if we can show that $y(t)$ is in $H^+(\xi)$.

For a positive integer m, let K_m be a compact interval $[-m, m]$. There exists an $N(m) > 0$ such that $\|x^k(t+s_k) - y(t)\|_{R^n} < 1/2m$ for $t \in K_m$, if $k \geqq N(m)$. Since $x^{N(m)} \in H^+(\xi)$, there exists a sequence $\{\tau_m\}$, $\tau_m \geqq m$, such that $\|\xi(t+\tau_m) - x^{N(m)}(t+s_{N(m)})\|_{R^n} < 1/2m$ for $t \in K_m$. Thus $\|\xi(t+\tau_m) - y(t)\|_{R^n} < 1/m$ for $t \in K_m$. This implies that $y(t)$ is in $H^+(\xi)$, because $\tau_m \longrightarrow \infty$ as $m \longrightarrow \infty$.

To make expressions simple, we shall use the following notations. For a sequence $\{\alpha_k\}$, we shall denote it by α and $\beta \subset \alpha$ means that β is a subsequence of α. For $\alpha = \{\alpha_k\}$ and $\beta = \{\beta_k\}$, $\alpha + \beta$ will denote the sequence $\{\alpha_k + \beta_k\}$. Moreover $L_\alpha x$ will denote $\lim_{k\to\infty} x(t+\alpha_k)$, where $\alpha = \{\alpha_k\}$ and the limit exists for each t.

LEMMA 6. *Suppose that $H^+(\xi, F)$ satisfies the separation condition. Then $\lambda(G, \phi, \psi)$ does not depend on G, ϕ and ψ.*

PROOF. It is clear that $\lambda(G, \phi, \psi)$ does not depend on ϕ and ψ. Let G_1 and G_2 be in $H^+(F)$. Then there exists a sequence $\{r'_k\}$ such that

$$G_2(t, \phi) = \lim_{k\to\infty} G_1(t + r'_k, \phi)$$

uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$, that is, $L_{r'}G_1 = G_2$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$. Let $x^1(t)$ and $x^2(t)$ be solutions in $H^+_{G_1}(\xi)$. Then there exists a subsequence $r \subset r'$ for which $L_r x^1 = y^1$, $L_r x^2 = y^2$ uniformly on any compact interval in R^1. By Lemma 3, $y^1(t)$ and $y^2(t)$ are solutions of

$$\dot{x}(t) = G_2(t, x_t) \tag{15}$$

defined on R^1 and by Lemm 5, $y^1(t)$ and $y^2(t)$ are in $H^+(\xi)$. If $x^1(t)$ and $x^2(t)$ are distinct solutions, we have

$$\inf_{t\in R} \|x^1{}_{t+r_k} - x^2{}_{t+r_k}\|_B = \inf_{t\in R} \|x^1{}_t - x^2{}_t\|_B = \alpha_{12} > 0 ,$$

and hence

$$\inf_{t\in R} \|y^1{}_t - y^2{}_t\|_B = \beta_{12} \geqq \alpha_{12} > 0 , \tag{16}$$

which means that $y^1(t)$ and $y^2(t)$ are distinct solutions of (15). Let $p_1 \geqq 1$ and $p_2 \geqq 1$ be the numbers of distinct solutions of $\dot{x}(t) = G_1(t, x_t)$ and (15), respectively. Clearly, $p_1 \leqq p_2$. In the same way, we have $p_2 \leqq p_1$.

Therefore $p_1 = p_2 = p$.

Now, let $\alpha = \min\{\alpha_{ik}, i, k = 1, 2, \cdots, p, i \neq k\}$ and $\beta = \min\{\beta_{jm}, j, m = 1, 2, \cdots, p, j \neq m\}$. Then by (16), we have $\alpha \leqq \beta$. In the same way, we have $\alpha \geqq \beta$. Therefore $\alpha = \beta = \lambda_0$.

By Lemma 6, if $H^+(\xi, F)$ satisfies the separation condition, we can choose a positive constant λ_0 independent of G, ϕ and ψ for which $\|\phi_t - \psi_t\|_B \geqq \lambda_0$ for all $t \in R^1$. We shall call λ_0 the separation constant for $H^+(\xi, F)$.

THEOREM 5. *Suppose that $H^+(\xi, F)$ satisfies the separation condition. Then $\xi(t)$ is an asymptotically almost periodic solution. Consequently,* (14) *has an almost periodic solution.*

PROOF. For any sequence $\{\tau'_k\}$ such that $\tau'_k \to \infty$ as $k \to \infty$, there is a subsequence $\{\tau_k\}$ of $\{\tau'_k\}$ and an $(\eta, G) \in H^+(\xi, F)$ such that $\xi(t + \tau_k) \to \eta(t)$ as $k \to \infty$ uniformly on any compact interval in R^1 and $F(t + \tau_k, \phi) \to G(t, \phi)$ as $k \to \infty$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$.

Suppose that $\xi(t + \tau_k)$ is not convergent uniformly on I. Then for some $\varepsilon > 0$ such that $\varepsilon < \lambda_0/2$, where λ_0 is the separation constant, there are sequence $\{t'_j\}$, $\{k_j\}$ and $\{m_j\}$ such that $k_j \to \infty$, $m_j \to \infty$ as $j \to \infty$ and

$$\|\xi(t'_j + \tau_{k_j}) - \xi(t'_j + \tau_{m_j})\|_{R^n} \geqq M_1\varepsilon ,$$

that is,

$$\|\xi_{t'_j+\tau_{k_j}} - \xi_{t'_j+\tau_{m_j}}\|_B \geqq \varepsilon .$$

Since ξ_{τ_k} is convergent by Lemma 1, we have $\|\xi_{\tau_{k_j}} - \xi_{\tau_{m_j}}\|_B < \lambda_0/2$, if j is sufficiently large. Set $\psi^j(t) = \xi(t + \tau_{k_j}) - \xi(t + \tau_{m_j})$. Then $\|\psi^j_0\|_B < \lambda_0/2$ and $\|\psi^j_{t'_j}\|_B \geqq \varepsilon$ for all large j. Since $\varepsilon < \lambda_0/2$, there exists a t_j such that $\varepsilon \leqq \|\psi^j_{t_j}\|_B < \lambda_0/2$. Thus we have sequences $\{t_j\}$, $\{\tau_{k_j}\}$ and $\{\tau_{m_j}\}$ for which

$$\varepsilon \leqq \|\xi_{t_j+\tau_{k_j}} - \xi_{t_j+\tau_{m_j}}\|_B < \lambda_0/2 . \tag{17}$$

Now we shall denote by r the sequence $\{\tau_k\}$. Then $r' = \{\tau_{k_j}\} \subset r$ and $r'' = \{\tau_{m_j}\} \subset r$. Let $\alpha = \{t_j\}$. For the sequences α, r' and r'', there exists $\alpha' \subset \alpha$, $\beta \subset r'$ and $\beta' \subset r''$ such that

$$L_{\alpha'+\beta}F = L_\alpha L_\beta F,\ L_{\alpha'+\beta'}F = L_{\alpha'} L_{\beta'} F \text{ exist uniformly on } R^1 \times \overline{S(\xi_0, M_1\beta, L)}$$

and

$$L_{\alpha'+\beta}\xi = x,\ L_{\alpha'+\beta'}\xi = y \text{ exist uniformly on any compact interval in } R^1 .$$

Since $L_\beta F = L_{\beta'}F = G$, we have $L_{\alpha'+\beta}F = L_{\alpha'+\beta'}F = L_{\alpha'}G = H$. By Lemma 5, $x(t)$ and $y(t)$ are in $H^+_H(\xi)$.

On the other hand, we have by (17) and Lemma 1

$$\varepsilon \leqq ||x_0 - y_0||_B \leqq \lambda_0/2 ,$$

which shows that $x(t)$ and $y(t)$ are distinct solutions of $\dot{x}(t) = H(t, x_t)$, and hence

$$||x_0 - y_0||_B \geqq \lambda_0 .$$

Thus there arises a contradiction. Therefore $\xi(t + \tau_k)$ converges uniformly on I, consequently $\xi(t)$ is an asymptotically almost periodic solution. By Theorem 1, system (14) has an almost periodic solution.

REMARK 6. For ordinary differential equations, Nakajima [9] has shown that the separation condition on $H^+(\xi, F)$ implies the existence of an almost periodic solution.

6. Stability properties and separation conditions. In this section, we are not required the property (VIII) of the space B except for the last theorem. We shall discuss separation conditions and stability properties in almost periodic systems. Let $F(t, \phi)$ and $\xi(t)$ be the ones given in Section 5, respectively.

We shall say that the solution $\xi(t)$ is unique for initial value problem with respect to $H^+_F(\xi)$ when $\xi_t = x_t$ for all $t \geqq t_0$, whenever $x \in H^+_F(\xi)$, if $\xi_{t_0} = x_{t_0}$ for some $t_0 \geqq 0$.

THEOREM 6. *Suppose that $\xi(t)$ is unique for initial value problem with respect to $H^+_F(\xi)$ and $H^+(\xi, F)$ satisfies the separation condition. Then $\xi(t)$ is s.d. $H^+(\xi, F)$.*

PROOF. Suppose that $\xi(t)$ is not s.d. $H^+(\xi, F)$. Then there exist an $\varepsilon > 0$ and sequences $(x^k, G_k) \in H^+(\xi, F)$, $\tau_k \geqq 0$ and $t_k > 0$ such that

$$||\xi_{t_k+\tau_k} - x^k{}_{t_k}||_B = \varepsilon(<\lambda_0/2) , \tag{18}$$

$$||\xi_{\tau_k} - x^k{}_0||_B < 1/k \tag{19}$$

and

$$\rho(F^{\tau_k}, G_k) < 1/k . \tag{20}$$

First, we shall show that $t_k + \tau_k \to \infty$ as $k \to \infty$. Suppose not. Then there exists a subsequence of $\{\tau_k\}$, which we shall denote by $\{\tau_k\}$ again, and a constant $\tau \geqq 0$ such that $\tau_k \to \tau$ as $k \to \infty$, because $0 \leqq \tau_k < \tau_k + t_k < \infty$. Since

$$\rho(F^\tau, G_k) \leqq \rho(F^\tau, F^{\tau_k}) + \rho(F^{\tau_k}, G_k) ,$$

we have by (20)

$$\rho(F^{\tau}, G_k) \to 0 \quad \text{as} \quad k \to \infty . \tag{21}$$

By (21) and Lemmas 3 and 5, $F^{\tau} \in H^+(F)$ and $x^k(t)$ can be assumed to tend to a solution $y(t)$ of

$$\dot{x}(t) = F(t + \tau, x_t)$$

uniformly on any compact interval of R^1 as $k \to \infty$. It follows from (I), (19) and Lemma 1 that

$$\|\xi_{\tau} - y_0\|_B \leqq \|\xi_{\tau} - \xi_{\tau_k}\|_B + \|\xi_{\tau_k} - x^k{}_0\|_B + \|x^k{}_0 - y_0\|_B \to 0 \quad \text{as} \quad k \to \infty .$$

Since

$$\|\xi_{t_k+\tau_k} - y_{t_k}\|_B \geqq \|\xi_{t_k+\tau_k} - x^k{}_{t_k}\|_B - \|x^k{}_{t_k} - y_{t_k}\|_B$$

and t_k is bounded, (18) and Lemma 1 imply that $\|\xi_{t_k+\tau_k} - y_{t_k}\|_B \geqq \varepsilon/2$ for a sufficiently large k, which contradicts the uniqueness of $\xi(t)$ with respect to $H_F^+(\xi)$. Thus $t_k + \tau_k \to \infty$ as $k \to \infty$.

Set $q_k = t_k + \tau_k$ and $x^k(t + t_k) = v^k(t)$. Then $\xi(t + q_k)$ and $v^k(t)$ are solutions of $\dot{x}(t) = F(t + q_k, x_t)$ and $\dot{x}(t) = G_k(t + t_k, x_t)$, respectively. There exists an $(\eta(t), P(t, \phi)) \in H^+(\xi, F)$, such that $\xi(t + q_k) \to \eta(t)$ uniformly on any compact interval in R^1 and $F(t + q_k, \phi) \to P(t, \phi)$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$ as $k \to \infty$, taking a subsequence of $\{q_k\}$, if necessarily, because $q_k \to \infty$ as $k \to \infty$. By (20), we have

$$\begin{aligned}\rho(P, G_k{}^{t_k}) &\leqq \rho(P, F^{q_k}) + \rho(F^{q_k}, G_k{}^{t_k}) \\ &\leqq \rho(P, F^{q_k}) + \rho(F^{\tau_k}, G_k) \to 0 \quad \text{as} \quad k \to \infty ,\end{aligned}$$

and hence, by Lemma 5, there exist a subsequence $\{v^{k_j}(t)\}$ of $\{v^k(t)\}$ and a $z(t) \in H_P^+(\xi)$ such that $v^{k_j}(t) \to z(t)$ uniformly on any compact interval of R^1 as $j \to \infty$. Since we have

$$\begin{aligned}&\lim_{j\to\infty} \{\|\xi_{t_{k_j}+\tau_{k_j}} - x^{k_j}{}_{t_{k_j}}\|_B - \|v^{k_j}{}_0 - z_0\|_B - \|\eta_0 - \xi_{q_{k_j}}\|_B\} \\ &\qquad \leqq \|\eta_0 - z_0\|_B \\ &\qquad \leqq \lim_{j\to\infty} \{\|\xi_{t_{k_j}+\tau_{k_j}} - x^{k_j}{}_{t_{k_j}}\|_B + \|v^{k_j}{}_0 - z_0\|_B + \|\eta_0 - \xi_{q_{k_j}}\|_B\} ,\end{aligned}$$

it follows from (18) that $\|\eta_0 - z_0\|_B = \varepsilon$, which contradicts the separation condition of $H^+(\xi, F)$.

THEOREM 7. *The following three propositions are equivalent:*

(i) *$H^+(\xi, F)$ satisfies the separation condition.*

(ii) *The solutions in $H^+(\xi)$ are q. u. a. s. $H^+(\xi)\cdot(\delta_0, T(\cdot))$.*

(iii) *The solutions in $H^+(\xi)$ are u.a.s. $H^+(\xi)\cdot(\delta_0, \delta(\cdot), T(\cdot))$.*

PROOF. It is clear that (iii) implies (ii). It is easily shown that (i) implies (iii), because for any $G \in H^+(F)$, the number of elements of $H_G^+(\xi)$ is finite. In fact, we can set $\delta_0 = \lambda_0$ and for any $\varepsilon > 0$ we can set $\delta(\varepsilon) = \lambda_0/2$. Thus we shall show that (ii) implies (i).

Assume that the condition (ii) holds. First of all, we shall see that any distinct solutions $x(t)$, $y(t)$ in $H_G^+(\xi)$ satisfy

$$\varliminf_{t\to-\infty} \|x_t - y_t\|_B \geqq \delta_0 . \tag{22}$$

Suppose not. Then for some $G \in H^+(F)$, there exist two distinct solutions $x(t)$ and $y(t)$ in $H_G^+(\xi)$ which satisfy

$$\varliminf_{t\to-\infty} \|x_t - y_t\|_B < \delta_0 . \tag{23}$$

Since $x(t)$ and $y(t)$ are distinct solutions, we have $\|x_{t_0} - y_{t_0}\|_B = \varepsilon$ at some t_0 and for some $\varepsilon > 0$. Then there is a t_1 such that $t_1 < t_0 - T(\varepsilon/2)$ and

$$\|x_{t_1} - y_{t_1}\|_B < \delta_0$$

by (23). Since $x(t)$ is q.u.a.s. $H_G^+(\xi)$, we have

$$\|x_{t_0} - y_{t_0}\|_B < \varepsilon/2 ,$$

which contradicts $\|x_{t_0} - y_{t_0}\|_B = \varepsilon$. Thus we have (22).

Since $\overline{S(\xi_0, M_1\beta, L)}$ is a compact set, there are a finite number of coverings which consist of m_0 balls with diameter $\delta_0/4$. We shall show that the number of solutions in $H_G^+(\xi)$ is at most m_0. Suppose not. Then there are $m_0 + 1$ solutions in $H_G^+(\xi)$, $x^j(t)$, $j = 1, 2, \cdots, m_0 + 1$, and a t_2 such that

$$\|x^j_{t_2} - x^i_{t_2}\|_B \geqq \delta_0/2 \quad \text{for} \quad i \neq j , \tag{24}$$

by (22). Since $x^j_{t_2}$, $j = 1, 2, \cdots, m_0 + 1$, are in $\overline{S(\xi_0, M_1\beta, L)}$ by Lemmas 1 and 2, some of these solutions, say $x^i(t)$, $x^j(t)$ $(i \neq j)$, are in one ball at time t_2, and hence

$$\|x^j_{t_2} - x^i_{t_2}\|_B < \delta_0/4 ,$$

which contradicts (24). Therefore the number of solutions in $H_G^+(\xi)$ is $m \leqq m_0$. Thus

$$H_G^+(\xi) = \{x^1(t), x^2(t), \cdots, x^m(t)\} \tag{25}$$

and

$$\varliminf_{t\to-\infty} \|x^i_t - x^j_t\|_B \geqq \delta_0 , \qquad i \neq j . \tag{26}$$

Consider a sequence $\{\tau_k\}$ such that $\tau_k \to -\infty$ as $k \to -\infty$ and $G(t+\tau_k, \phi) \to$

$G(t, \phi)$ uniformly on $R^1 \times \overline{S(\xi_0, M_1\beta, L)}$ as $k \to \infty$. Set $v^{j,k}(t) = x^j(t + \tau_k)$, $j = 1, 2, \cdots, m$. Then $v^{j,k}(t)$ is equicontinuous and uniformly bounded on R^1, and hence $v^{j,k}(t)$ can be assumed to tend to $y^j(t)$ uniformly on any compact interval in R^1 as $k \to \infty$ for $j = 1, 2, \cdots, m$. By Lemma 5, $y^j \in H_G^+(\xi)$ and by Lemmas 1 and 2, $y^j{}_t \in \overline{S(\xi_0, M_1\beta, L)}$ for all $t \in R^1$. Since

$$||y^j{}_t - y^i{}_t||_B = \lim_{k \to \infty} ||v^{j,k}{}_t - v^{i,k}{}_t||_B$$

for $t \in R^1$, it follows from (26) that

$$||y^j{}_t - y^i{}_t||_B \geqq \delta_0 \quad \text{for all} \quad t \in R^1 \quad \text{and} \quad i \neq j\,. \tag{27}$$

Since the number of solutions in $H_G^+(\xi)$ is m, $H_G^+(\xi)$ consists of $y^1(t)$, $y^2(t)$, $\cdots$, $y^m(t)$ and we have (27), which shows that $H^+(\xi, F)$ satisfies the separation condition.

By Theorems 5 and 7, we have the following corollary.

COROLLARY 3. *Suppose that the solutions in* $H^+(\xi)$ *are q.u.a.s.* $H^+(\xi)\cdot(\delta_0, T(\cdot))$. *Then* $\xi(t)$ *is an asymptotically almost periodic solution. Consequently,* (14) *has an almost periodic solution.*

REMARK 7. For functional differential equations, it is known by Kato [7] that if the solutions in $H^+(\xi)$ are u.a.s. $(\delta_0, \delta(\cdot), T(\cdot))$, then (14) has an almost periodic solution, and for ordinary differential equations, Nakajima [9] has shown that if the solutions in $H^+(\xi)$ are u.a.s. $(\delta_0, \delta(\cdot), T(\cdot))$, then $H^+(\xi, F)$ satisfies the separation condition.

If $\xi(t)$ is u.s. $H_F^+(\xi)$, then $\xi(t)$ is unique for initial value problem with respect to $H_F^+(\xi)$. Therefore we have the following corollary by Theorems 6 and 7.

COROLLARY 4. *Suppose that* $\xi(t)$ *is u.s.* $H_F^+(\xi)$ *and the solutions in* $H^+(\xi)$ *are q.u.a.s.* $H^+(\xi)\cdot(\delta_0, T(\cdot))$. *Then* $\xi(t)$ *is s.d.* $H^+(\xi, F)$.

In the following lemmas and theorem, we assume that the space B has the property (VIII).

LEMMA 7. *For any* $\varepsilon > 0$, *there exists a* $\pi(\varepsilon) > N$ *such that for any* $\phi \in S^*(\eta, N, L)$, $||T_s\phi_t||_{B^s} < \varepsilon$ *for all* $s \geqq \pi$ *and for all* $t \geqq N/L$.

PROOF. Define $\operatorname{sgn}(\eta^i(0))$, $i = 1, 2, \cdots, n$, and $\tilde{N} \in R^n$ by

$$\operatorname{sgn}(\eta^i(0)) = \begin{cases} 1 & \text{if} \quad \eta^i(0) \geqq 0\,, \\ -1 & \text{if} \quad \eta^i(0) < 0\,, \end{cases}$$

and

$$\tilde{N} = (\operatorname{sgn}(\eta^1(0))N, \operatorname{sgn}(\eta^2(0))N, \cdots, \operatorname{sgn}(\eta^n(0))N),$$

respectively. There exists a function $\zeta(t)$, $\zeta \in A_0^\infty$ such that $\zeta(t) = \tilde{N}$ for all $t \geqq N/L$ and $\|\phi(t)\|_{R^n} \leqq \|\zeta(t)\|_{R^n}$ for any $\phi \in S^*(\eta, N, L)$ and $t \in R^1$. We have by (VIII) $\|T_s\phi_t\|_{B^s} \leqq \|T_s\zeta_t\|_{B^s}$ for all $t \geqq 0$, $s \geqq 0$ and $\phi \in S^*(\eta, N, L)$. Put $\tilde{\zeta}(t) = \zeta(t) - \tilde{N}$ and

$$\tilde{N}^*(t) = (\operatorname{sgn}(\eta^1(0))N^*(t), \operatorname{sgn}(\eta^2(0))N^*(t), \cdots, \operatorname{sgn}(\eta^n(0))N^*(t)),$$

where

$$N^*(t) = \begin{cases} 0 & \text{for } 0 < t < \infty, \\ -t & \text{for } -N \leqq t \leqq 0, \\ N & \text{for } -\infty < t < -N. \end{cases}$$

Then we have

$$\begin{aligned} \|T_s\zeta_t\|_{B^s} &= \|T_s(\langle\tilde{N}\rangle_t + \tilde{\zeta}_t)\|_{B^s} \\ &\leqq \|T_s\langle\tilde{N}\rangle_t\|_{B^s} + \|T_s\tilde{\zeta}_t\|_{B^s} \\ &\leqq \inf_{\psi}\{\|\psi\|_B;\ \psi^s = T_s\langle\tilde{N}\rangle_t\} + \inf_{\psi}\{\|\psi\|_B;\ \psi^s = T_s\tilde{\zeta}_t\} \\ &\leqq \|\tilde{N}^*_{s-N}\|_B + \|\tilde{\zeta}_{s+t}\|_B \\ &\leqq b\Big(\sup_{-(s-N)\leqq\theta\leqq 0}\|\tilde{N}^*(s-N+\theta)\|_{R^n}\Big) + c(\|T_{s-N}\tilde{N}^*_0\|_{B^{s-N}}) \\ &\quad + b\Big(\sup_{-(s+t-N/L)\leqq\theta\leqq 0}\|\tilde{\zeta}(s+t+\theta)\|_{R^n}\Big) \\ &\quad + c(\|T_{s+t-N/L}\tilde{\zeta}_{N/L}\|_{B^{s+t-N/L}}) \\ &\leqq c(\|T_{s-N}\tilde{N}^*_0\|_{B^{s-N}}) + c(\|T_{s+t-N/L}\tilde{\zeta}_{N/L}\|_{B^{s+t-N/L}}) \end{aligned}$$

for $t \geqq N/L$ and $s \geqq N$. By (V), for a given $\varepsilon > 0$ there exists a $\pi(\varepsilon) > N$ such that

$$c(\|T_{s-N}\tilde{N}^*_0\|_{B^{s-N}}) < \varepsilon/2$$

and

$$c(\|T_{s+t-N/L}\tilde{\zeta}_{N/L}\|_{B^{s+t-N/L}}) < \varepsilon/2$$

for all $s \geqq \pi$ and $t \geqq N/L$, and hence we have

$$\|T_s\phi_t\|_{B^s} \leqq \|T_s\zeta_t\|_{B^s} < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $s \geqq \pi$ and $t \geqq N/L$.

LEMMA 8. *If $f \in S^*(\eta, N, L)$ and $f(t)$ is an asymptotically almost periodic function whose decomposition is given by* (3), *then*

$$||f_t - p_t||_B \to 0 \quad as \quad t \to \infty .$$

PROOF. Define $q^*(t)$ by

$$q^*(t) = \begin{cases} q(t) & \text{for} \quad t \in (0, \infty) , \\ \eta(t) - p(t) & \text{for} \quad t \in (-\infty, 0] . \end{cases}$$

Then we can easily show that $q^*{}_t = f_t - p_t$ for all $t \geqq 0$. Since $p(t)$ is almost periodic, there exists a sequence $\{\tau_k\}$ such that $\tau_k \to \infty$ as $k \to \infty$ and $p(t + \tau_k) \to p(t)$ as $k \to \infty$. Then we have $f(t + \tau_k) = p(t + \tau_k) + q(t + \tau_k)$ for $t + \tau_k \geqq 0$ by (3). Thus we have $p_t \in \overline{S(\eta, N, L)}$ for all $t \in R^1$ by Lemmas 1 and 2. Consequently, we have $q^* \in S^*(\eta_0 - p_0, 2N, 2L)$. By Lemma 7, for any $\varepsilon > 0$, there exists a $\pi(\varepsilon) > 2N$ such that $||T_\pi q^*_{t_1}||_{B^\pi} < c^{-1}(\varepsilon/2)$ for all $t \geqq N/L$. We may assume that $||q(t)||_{R^n} < b^{-1}(\varepsilon/2)$ for $t \geqq \pi$, because $q(t) \to 0$ as $t \to \infty$. Hence we have

$$\begin{aligned} ||q^*{}_t||_B &\leqq b\Big(\sup_{-\pi \leqq \theta \leqq 0} ||q^*(t + \theta)||_{R^n}\Big) + c(||(q^*_t)^\pi||_{B^\pi}) \\ &\leqq b\Big(\sup_{-\pi \leqq \theta \leqq 0} ||q(t + \theta)||_{R^n}\Big) + c(||T_\pi q^*{}_{t-\pi}||_{B^\pi}) \\ &< b(b^{-1}(\varepsilon/2)) + c(c^{-1}(\varepsilon/2)) \\ &< \varepsilon \end{aligned}$$

for $t \geqq 2\pi + N/L$. Thus $||q^*{}_t||_B = ||f_t - p_t||_B \to 0$ as $t \to \infty$.

THEOREM 8. *Suppose that $\xi(t)$ is a solution of* (14) *defined on R^1 and $||\xi_t||_B \leqq \beta$ for all $t \in R^1$. If $H(\xi, F)$ satisfies the separation condition, then $\xi(t)$ is an almost periodic solution of* (14) *which is s.d. $H^+(\xi, F)$.*

PROOF. By the same argument as in the proof of Lemma 6, we can choose a separation constant λ_0 for $H(\xi, F)$, and hence, by Theorem 5, $\xi(t)$ is asymptotic almost periodic. Thus $\xi(t)$ has the decomposition $\xi(t) = p(t) + q(t)$, where $p(t)$ is almost periodic and $q(t) \to 0$ as $t \to \infty$.

First, we shall see that $p(t)$ is a solution of (14). Since $\xi_t \in \overline{S(\xi_0, M_1\beta, L)}$ for $t \geqq 0$ and $p_t \in \overline{S(\xi_0, M_1\beta, L)}$ for $t \in R^1$, we can show that $F(t, p_t)$ is almost periodic in t by the same argument as in the proof of Theorem 2.7 in [15]. Since $\xi(t)$ is the solution of (14), we have

$$\dot{\xi}(t) = F(t, p_t) + F(t, \xi_t) - F(t, p_t) \tag{28}$$

for $t \geqq 0$. By Lemma 8, $||\xi_t - p_t||_B \to 0$ as $t \to \infty$. Consequently, uniform continuity of $F(t, \phi)$ implies that $F(t, \xi_t) - F(t, p_t) \to 0$ as $t \to \infty$. Thus (28) shows that $\dot{\xi}(t)$ also is asymptotically almost periodic, and hence it follows from Theorem 3.3 in [15] that

$$\dot{p}(t) = F(t, p_t) \quad \text{for} \quad t \in R^1 .$$

Thus $p(t)$ is a solution of (14) defined on R^1.

The property of separation implies $\|\xi_t - p_t\|_B = 0$ for all $t \in R^1$. Since $\xi(t)$ and $p(t)$ are continuous on R^1, we have $\xi(t) = p(t)$ for all $t \in R^1$.

Let $x \in H_F^+(\xi)$ and $x_{t_0} = \xi_{t_0}$ for some $t_0 \geqq 0$. Then $x(t) = \xi(t)$ for all $t \in (-\infty, t_0]$, because $\xi(t)$ is continuous on R^1. Since $x(t)$ also is a solution of (14) defined on R^1 by Lemma 3, $\|x_t - \xi_t\|_B = 0$ or $\|x_t - \xi_t\|_B \geqq \lambda_0$ for all $t \in R^1$. However, $\|x_{t_0} - \xi_{t_0}\|_B = 0$ implies $\|x_t - \xi_t\|_B = 0$ for all $t \geqq 0$. Hence $\xi(t)$ is unique for initial value problem with respect to $H_F^+(\xi)$. Thus Theorem 6 implies that $\xi(t)$ is s.d. $H^+(\xi, F)$.

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