SOME CONDITIONS ON A REDUCTIVE OPERATOR IMPLYING NORMALITY OR SPECTRALITY

KOH-ICHI KITANO

(Received July 31, 1975)

In this note, we shall show some sufficient conditions that a reductive operator acting in a complex Hilbert space be normal or spectral. It is a generalization of some results on operators with reducing invariant subspaces in [9]. In §2, if a reductive operator T is the sum of a normal operator, whose spectrum lies on a C^2 -Jordan curve, and a compact operator which belongs to the Macaev class $\mathcal{C}_\omega$, then T is normal. In §3, if the spectrum of a reductive operator T lies on a C^2 -Jordan curve J and if the resolvent of T satisfies the growth condition (C) near J , then T is spectral.

Throughout this note $\mathcal{H}$ will be a (not necessarily separable) complex Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators acting in $\mathcal{H}$. We say that the bounded linear operator T is reductive if every invariant subspace of T reduces T . A well-known unsolved problem is "must a reductive operator be normal?". It has recently been shown that this problem is equivalent to the invariant subspace conjecture [5]. The answer is affirmative for compact operators [1] and for polynomially compact operators [11], [10] and in [9] has been shown that if $T - T^*$ or $I - T^*T$ belongs to a von Neumann-Schatten class $\mathcal{C}_p$ ($1 \leq p < \infty$) or if T is similar to the sum of a normal operator and a commuting compact operator, then T is reductive only if T is normal, and if the spectrum of a reductive operator T lies on a C^2 -Jordan curve and if the resolvent of T satisfies an exponential growth condition, then T is spectral.

In this note we always represent J by the canonic form $J = \{g(s): s \in [0, l]\}$, where s denotes the length of arc (if J is a Jordan curve we have $g(0) = g(l)$). The set of all operators $T \in \mathcal{B}(\mathcal{H})$ such that the spectrum of T lies on a C^2 -Jordan curve J will be denoted by $\mathcal{B}(\mathcal{H}, J)$. We denote by $\sigma(T)$ and $\rho(T)$, the spectrum and the resolvent set of T , respectively. For any complex number λ and a subset E of a complex plane C , we put $d(\lambda, E) = \inf \{|\lambda - z|: z \in E\}$. Let $\mathcal{H}$ be an invariant subspace of T , we denote by $T|_{\mathcal{H}}$ the restriction of T to $\mathcal{H}$.

1. Preliminaries. In this section we note some known results and preliminary lemmas for later use.

Let $T \in \mathcal{B}(\mathcal{H}, J)$.

DEFINITION 1.1. ([7]) We say that the resolvent of T has the growth condition (C) near J if

$$\int_0^\varepsilon \log \log M(\delta) d\delta < \infty$$

for some sufficiently small $\varepsilon > 0$, where $M(\delta) = \sup \{ \|(\lambda I - T)^{-1}\| : d(\lambda, J) > \delta \}$.

In [9], the resolvent is said to have an exponential growth condition, if $d(\lambda, J)^p \log \|(\lambda I - T)^{-1}\|$, $\lambda \notin J$ is bounded near J for some $p \geq 1$. If the resolvent has an exponential growth condition near J , then it has the growth condition (C) near J .

LEMMA 1.2. Let $T \in \mathcal{B}(\mathcal{H}, J)$. Suppose that the resolvent of T satisfies the growth condition (C) near J . If J_0 is an open subarc of J and if we define by

$$\mathcal{S} = \{x \in \mathcal{H} : (\lambda I - T)^{-1}x \text{ has an analytic continuation to the complement of } \bar{J}_0\},$$

(bar denote the closure), then $\mathcal{S}$ is a closed hyperinvariant subspace of T .

Moreover, if $\sigma(T) \cap J_0 \neq \emptyset$, then $\mathcal{S} \neq \{0\}$ and $\sigma(T|_{\mathcal{S}}) \subset \bar{J}_0$. If $\sigma(T)$ consists of more than one point, then we can choose an open subarc J_0 of J such that $\mathcal{S}$ is a proper subspace.

PROOF. In Lemma 2.4 of [7], we have shown that $\mathcal{S}$ is a closed invariant subspace of T . Moreover, taking into consideration of Lemmas 2.4 and 2.5 in [7], if $\sigma(T) \cap \bar{J}_0 \neq \emptyset$, then $\mathcal{S} \neq \{0\}$ and if $\sigma(T)$ contains more than one point, we can choose an open subarc J_0 of J such that $\mathcal{S}$ is a proper subspace.

For the proof of this lemma, next we show that $\mathcal{S}$ is a hyperinvariant subspace of T . If $S \in \mathcal{B}(\mathcal{H})$, which commutes with T , then S commutes with the resolvent, $(\lambda I - T)^{-1}$, of T for every λ in $\rho(T)$. Now, let $R(\lambda, T; x)$ denote the analytic continuation of $(\lambda I - T)^{-1}x$, from the equation

$$(\lambda I - T)^{-1}Sx = S(\lambda I - T)^{-1}x$$

for every $x \in \mathcal{S}$, then it is clear that $SR(\lambda, T; x)$ is an analytic continuation of $(\lambda I - T)^{-1}Sx$. Therefore $Sx \in \mathcal{S}$ for every $x \in \mathcal{S}$, i.e., $\mathcal{S}$ is hyperinvariant subspace of T .

To complete the proof we have only to show that $\sigma(T|_{\mathcal{S}}) \subset \bar{J}_0$ for

the case of $\sigma(T) \cap J_0 \neq \emptyset$. In the following, we can use an argument of (p. 201, [3]). We first show that $R(\lambda, T: x) \in \mathcal{S}$ for every $x \in \mathcal{S}$. For a fixed $\lambda \in C \setminus \bar{J}_0$, we define the function $G_\lambda(\mu)$ such that

$$G_\lambda(\mu) = \begin{cases} -\frac{R(\mu, T: x) - R(\lambda, T: x)}{\mu - \lambda} & \text{if } \mu \neq \lambda, \\ -R'(\lambda, T: x) & \text{if } \mu = \lambda, \end{cases}$$

then $G_\lambda(\mu)$ is analytic in $C \setminus \bar{J}_0$ and follows that

$$\begin{aligned} (\mu I - T)G_\lambda(\mu) &= -(\mu I - T) \left\{ \frac{R(\mu, T: x) - R(\lambda, T: x)}{\mu - \lambda} \right\} \\ &= -(\mu - \lambda)^{-1} \{ (\mu I - T)R(\mu, T: x) - (\lambda I - T)R(\lambda, T: x) \\ &\quad - (\mu - \lambda)R(\lambda, T: x) \} \\ &= -(\mu - \lambda)^{-1} \{ x - x - (\mu - \lambda)R(\lambda, T: x) \} = R(\lambda, T: x), \end{aligned}$$

for $\mu \neq \lambda$. If we make $\mu \rightarrow \lambda$, we obtain that the equality between the extreme terms of the above relation of equalities holds also for $\mu = \lambda$. Therefore if $x \in \mathcal{S}$, then $R(\lambda, T: x) \in \mathcal{S}$.

Next, let $\lambda_0 \in C \setminus \bar{J}_0$. We define the operator H acting in $\mathcal{S}$ by the following

$$Hx = R(\lambda_0, T: x).$$

It is evident that H is linear. We will show that H is closed. Let $\{x_n\} \subset \mathcal{S}$ with $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} Hx_n = y$. From the fact that $\mathcal{S}$ is closed it follows that $x \in \mathcal{S}$ and $y = \lim_{n \rightarrow \infty} Hx_n = \lim_{n \rightarrow \infty} R(\lambda_0, T: x_n) \in \mathcal{S}$. We have

$$(\lambda_0 I - T)Hx_n = (\lambda_0 - T)R(\lambda_0, T: x_n) = x_n,$$

from which follows $(\lambda_0 I - T)y = x$. While,

$$(\lambda_0 I - T)R(\lambda_0, T: x) = x,$$

therefore we have

$$(\lambda_0 I - T)\{y - R(\lambda_0, T: x)\} = 0.$$

Let us now put $z = y - R(\lambda_0, T: x)$. Having $y, R(\lambda_0, T: x) \in \mathcal{S}$, it follows that $z \in \mathcal{S}$. Thus $(\lambda I - T)^{-1}z$ has an analytic continuation to the complement of $\bar{J}_0$. Let the function h from $C \setminus \{\lambda_0\}$ into $\mathcal{H}$ be defined by

$$h(\lambda) = (\lambda - \lambda_0)^{-1}z.$$

Then we have

$$\begin{aligned} (\lambda I - T)h(\lambda) &= (\lambda - \lambda_0)^{-1}(\lambda I - T)z \\ &= (\lambda - \lambda_0)^{-1}\{(\lambda_0 I - T)z + (\lambda - \lambda_0)z\} = z. \end{aligned}$$

Consequently it results that $R(\lambda, T; z)$ is an entire function. Since it satisfies

$$(\lambda I - T)R(\lambda, T; z) = z$$

for $|\lambda| > \|T\|$. We have

$$R(\lambda, T; z) = (\lambda I - T)^{-1}z \rightarrow 0 \quad (|\lambda| \rightarrow \infty),$$

therefore by Liouville's theorem $R(\lambda, T; z) \equiv 0$. Thus we have

$$z = \frac{1}{2\pi i} \int_{\Gamma} R(\lambda, T; z) d\lambda = 0,$$

where $\Gamma = \{\lambda: |\lambda| = \|T\| + 1\}$, therefore $y = R(\lambda_0, T; x) = Hx$. It follows that the operator H is closed and, therefore according to Banach's closed graph theorem (see, p. 79 in [15]), $H \in \mathcal{B}(\mathcal{S})$. We observe that

$$(\lambda_0 I - T)Hx = (\lambda_0 I - T)R(\lambda_0, T; x) = x$$

for every $x \in \mathcal{S}$, therefore we have

$$(*) \quad (\lambda_0 I|_{\mathcal{S}} - T|_{\mathcal{S}})H = I|_{\mathcal{S}}.$$

In the next place for every $x \in \mathcal{S}$, we have

$$R(\lambda_0, T; (\lambda_0 I - T)x) = (\lambda_0 I - T)R(\lambda_0, T; x),$$

from which, according to the definition of H , follows that

$$H(\lambda_0 I - T)x = R(\lambda_0, T; (\lambda_0 I - T)x) = x,$$

therefore we have

$$(**) \quad H(\lambda_0 I|_{\mathcal{S}} - T|_{\mathcal{S}}) = I|_{\mathcal{S}}.$$

From (*), (**) follows that $H = (\lambda_0 I|_{\mathcal{S}} - T|_{\mathcal{S}})^{-1}$, therefore we have $\sigma(T|_{\mathcal{S}}) \subset \bar{J}_0$. The proof of Lemma 1.2 is now completed.

The following lemma is found in [7].

LEMMA 1.3. *Let $T \in \mathcal{B}(\mathcal{H})$. Write $T = A + B$, where T is the sum of a normal operator A , whose spectrum lies on J , and a compact operator B which belongs to the Macaev class $\mathcal{E}_\omega$. Then any point λ with $\lambda \notin \sigma(A)$ is either $\lambda \in \rho(T)$ or an eigenvalue with finite multiplicity of T .*

REMARK. The Macaev class $\mathcal{E}_\omega$ is the set of compact operators for which $\|T\|_\omega = \sum_n \mu_n/n < \infty$, where μ_n ($n = 1, 2, 3, \dots$) are the eigenvalues of $(T^*T)^{1/2}$. It is easy to see that $\mathcal{E}_\omega$ contains all von Neumann-Schatten classes $\mathcal{E}_p$ ($1 \leq p < \infty$) of compact operators (cf. §3 in [7] or [8]).

The next lemma was established in the proof of Theorem 3.1 in [7].

LEMMA 1.4. *Let T be as above (in Lemma 1.3). Suppose further that the spectrum of A does not separate the complex plane $\mathbb{C}$ and the point spectrum of T is empty. Then the resolvent of T satisfies the growth condition (C) near J .*

The next lemma is a trivial generalization of Lemma 2.3 in [6], (see, Lemma 2.5 in [9]).

LEMMA 1.5. *If T is a quasi-nilpotent operator and $T^*T - TT^*$ is a compact operator, then T is compact.*

The following lemma was established in [9].

LEMMA 1.6. *Let $T \in \mathcal{B}(\mathcal{H})$. If $\{\mathcal{M}_\alpha\}$ is a family of reducing subspaces for T such that $T|_{\mathcal{M}_\alpha}$ is normal for each α , then $T|_{(\vee \{\mathcal{M}_\alpha\})}$ is normal, where $\vee \{\mathcal{M}_\alpha\}$ denotes the closed linear span of the family $\{\mathcal{M}_\alpha\}$.*

LEMMA 1.7. *Suppose $K = T^*T - TT^*$ is a compact operator, and let $\{\mathcal{M}_\alpha\}$ be a totally ordered (by inclusion relation) family of subspaces each of which reduces T and satisfies $\|K|_{\mathcal{M}_\alpha}\| = \|K\|$. Then $\|K|_{\mathcal{M}_0}\| = \|K\|$, where $\mathcal{M}_0 = \bigcap \mathcal{M}_\alpha$.*

PROOF. The proof is quite similar to one of Lemma 2 in [11]. Let $\varepsilon > 0$ be given. For each α , there exists a unit vector x_α in $\mathcal{H}$ such that

$$\|(P_0 - P_\alpha)Kx_\alpha\| > \|(P_0 - P_\alpha)K\| - \varepsilon/4,$$

where P_α, P_0 are orthogonal projections on $\mathcal{M}_\alpha, \mathcal{M}_0$, respectively. Since $\{x_\alpha\}$ is a bounded set and K is a compact operator, we can choose a subset $\{x_{\alpha_\nu}\}$ such that $\{Kx_{\alpha_\nu}\}$ converges strongly to a certain vector x . Since P_0 is a strong limit of P_α , there exists a ν such that

$$\|(P_{\alpha_\nu} - P_0)x\| < \varepsilon/4, \|Kx_{\alpha_\nu} - x\| < \varepsilon/4.$$

Combining above inequalities we have

$$\|(P_0 - P_{\alpha_\nu})K\| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\|K|_{\mathcal{M}_0}\| = \|P_0K\| = \lim \|P_{\alpha_\nu}K\| = \|K\|.$$

The proof of Lemma 1.7 is now completed.

2. Some sufficiently conditions on a reductive operator implying normality. In this section we shall see the following theorem to be valid.

THEOREM 2.1. *If T is a reductive operator, if $T = A + B$, where T is the sum of a normal operator A , whose spectrum lies on J (the spectrum*

of A does not separate the complex plane) and a compact operator B which belongs to the Macaev class $\mathcal{E}_\omega$, then T is normal.

As a immediate consequence of this theorem, we have the following corollary.

COROLLARY 2.2. *If T is a reductive operator and if the imaginary part of T is in the Macaev class $\mathcal{E}_\omega$, then T is normal.*

PROOF OF THEOREM 2.1. For each point $\lambda \in \sigma_p(T)$ (the point spectrum of T), set $\mathcal{N}(\lambda) = \{x \in \mathcal{H} : Tx = \lambda x\}$. Since $\mathcal{N}(\lambda)$ is an invariant subspace of T , each $\mathcal{N}(\lambda)$, $\lambda \in \sigma_p(T)$ reduces T by assumption, we have $TT^*x = \lambda T^*x$ for $x \in \mathcal{N}(\lambda)$, it follows $\|T^*x - \bar{\lambda}x\|^2 = 0$, therefore $T^*x = \bar{\lambda}x$ for $x \in \mathcal{N}(\lambda)$. Thus the restriction of T onto $\mathcal{N} = \bigvee \{\mathcal{N}(\lambda) : \lambda \in \sigma_p(T)\}$ is a normal operator (by Lemma 1.6). We must show that the restriction of T to the reducing subspace $\mathcal{N}^\perp$ is also normal. Clearly by Lemma 1.3 $\sigma(T|_{\mathcal{N}^\perp}) \subset J$ and Lemma 1.4 implies that the resolvent of $T|_{\mathcal{N}^\perp}$ satisfies the growth condition (C) near J .

We shall break the proof into portions. Hereafter to simplify we suppose that $\sigma(T) \subset J$ and the resolvent of T satisfies the growth condition (C) near J .

(i) Every non-zero reducing subspace of T includes a proper invariant subspace of T or else is one-dimensional.

Let $\mathcal{M}$ be a reducing subspace of T . Since $\sigma(T|_{\mathcal{M}}) \subset \sigma(T)$, the resolvent of $T|_{\mathcal{M}}$ satisfies the same growth condition as the resolvent of T does, consequently the resolvent of $T|_{\mathcal{M}}$ satisfies the growth condition (C) near J . In the first place if $\sigma(T|_{\mathcal{M}})$ consists of more than one-point. From Lemma 1.2, there exists a proper invariant subspace of $T|_{\mathcal{M}}$, thus it also a proper invariant subspace of T . The next, if $\sigma(T|_{\mathcal{M}})$ consists of a single point, say $\{\lambda\}$. Since $\mathcal{M}$ is a reducing subspace of T , $(T|_{\mathcal{M}})^* = T^*|_{\mathcal{M}}$, we have

$$(\lambda I - T)^*(\lambda I - T) - (\lambda I - T)(\lambda I - T)^* = T^*T - TT^*.$$

On the other hand, A is normal and B is compact, it follows that $T^*T - TT^*$ is compact. According to Lemma 1.5, $(\lambda I - T)|_{\mathcal{M}}$ is compact. Hence $\mathcal{M}$ is a one-dimensional or $(\lambda I - T)|_{\mathcal{M}}$ has a proper invariant subspace by Aronszajn-Smith theorem [2]. It follows from above arguments that the only minimal invariant subspace of T are one-dimensional and reducing.

(ii) Let $\mathcal{N}_0$ be the closed linear span of the one-dimensional reducing subspaces of T and let $K = T^*T - TT^*$. Then $\mathcal{N}_0$ reduces T and K . According to Lemma 1.6, $T|_{\mathcal{N}_0}$ is normal, thus to complete the proof

we have to show that $T|_{\mathcal{N}_0^\perp}$ is normal, it suffices to prove that $K|_{\mathcal{N}_0^\perp} = 0$.

(iii) Next we shall show that $K|_{\mathcal{N}_0^\perp} = 0$. Since $\mathcal{N}_0^\perp$ is reducing subspace of T , we have $\sigma(T|_{\mathcal{N}_0^\perp}) \subset \sigma(T) \subset J$. Therefore the resolvent of $T|_{\mathcal{N}_0^\perp}$ satisfies the growth condition (C) near J . From the result in (i), every non-zero reducing subspace of $T|_{\mathcal{N}_0^\perp}$ includes a proper invariant subspace of $T|_{\mathcal{N}_0^\perp}$ or else is one-dimensional. Now we suppose to be $K|_{\mathcal{N}_0^\perp} \neq 0$. Let $\mathcal{F}$ be the family of subspaces $\mathcal{M} \subset \mathcal{N}_0^\perp$ such that reduce $T|_{\mathcal{N}_0^\perp}$ and satisfy the condition $\|P_{\mathcal{M}}K|_{\mathcal{N}_0^\perp}\| = \|K|_{\mathcal{N}_0^\perp}\|$, where $\{P_{\mathcal{M}}\}$ are the orthogonal projections on $\mathcal{M}$. By Lemma 1.7, $\mathcal{F}$ is inductive, therefore Zorn's lemma implies that $\mathcal{F}$ contains a minimal member $\mathcal{M}_0$. On account of $K|_{\mathcal{N}_0^\perp} \neq 0$ (by assumption), $\mathcal{M}_0 \neq \{0\}$, if $\dim \mathcal{M}_0 \geq 2$, then $\mathcal{M}_0$ contains a proper invariant subspace $\mathcal{M}_1$ (by Lemma 1.2). By hypothesis of this theorem $\mathcal{M}_1$ reduces $T|_{\mathcal{N}_0^\perp}$ and we have

$$\begin{aligned} \|K|_{\mathcal{N}_0^\perp}\| &= \|P_{\mathcal{M}_0}K|_{\mathcal{N}_0^\perp}\| \\ &= \max \{ \|P_{\mathcal{M}_1}K|_{\mathcal{N}_0^\perp}\|, \|(P_{\mathcal{M}_0} - P_{\mathcal{M}_1})K|_{\mathcal{N}_0^\perp}\| \}, \end{aligned}$$

where $P_{\mathcal{M}_0}, P_{\mathcal{M}_1}$ are the orthogonal projections on $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively. It follows that $\mathcal{M}_1$ or $\mathcal{M}_0 \cap \mathcal{M}_1^\perp$ is a member of $\mathcal{F}$. This contradicts the minimality of $\mathcal{M}_0$, for both $\mathcal{M}_1$ and $\mathcal{M}_0 \cap \mathcal{M}_1^\perp$ are properly contained in $\mathcal{M}_0$. Thus $\dim \mathcal{M}_0 = 1$, i.e., there exists a one-dimensional reducing subspace of $T|_{\mathcal{N}_0^\perp}$. Again this contradicts the construction of $\mathcal{N}_0$, therefore $K|_{\mathcal{N}_0^\perp} = 0$, i.e., $T|_{\mathcal{N}_0^\perp}$ is normal. The proof of Theorem 2.1 is now completed.

3. A sufficient condition on a reductive operator implying spectrality.

In this section we shall show that if a reductive operator belongs to $\mathcal{B}(\mathcal{H}, J)$ whose resolvent satisfies the growth condition (C) near J is spectral, (in fact, is the sum of a normal operator and a commuting quasinilpotent operator). The same result has been obtained in [9] for an exponential growth condition, and some related results have also been obtained by J. G. Stampfli [13], [14]. The idea of the proof is to construct a spectral measure locally by means of techniques similar to those used by J. T. Schwartz in [12]. The arguments in this section is quite similar to that of [9].

The following lemma was established in [9].

LEMMA 3.1. *Suppose T has closed hyperinvariant reducing subspaces $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ such that, for each i , $T|_{\mathcal{M}_i} = A_i + B_i$ where A_i is normal on $\mathcal{M}_i$, B_i is quasi-nilpotent on $\mathcal{M}_i$ and A_i commutes with B_i . If the $\mathcal{M}_i (i = 1, 2, \dots, k)$ span $\mathcal{H}$, then $T = A + B$, where A is normal*

and B is quasi-nilpotent such that A commutes with B .

LEMMA 3.2. *Let $T \in \mathcal{B}(\mathcal{H}, J)$ be a reductive operator. Suppose that the resolvent T satisfies the growth condition (C) near J . If $\mathcal{S}$ is an invariant subspace in Lemma 1.2, then $T|_{\mathcal{S}} = A + B$, where A is normal, B is quasi-nilpotent and A commutes with B .*

PROOF. Let $J_0 = \{g(s): s \in (a, b)\}$, (take note of $J = \{g(s): s \in [0, l]\}$). Then $0 < a < b < l$. By Lemma 1.2, $\mathcal{S}$ is a closed hyperinvariant subspace for T . Choose c and d such that $0 < c < a < b < d < l$, and for $t \in (c, d)$ now we define $\mathcal{S}_t$ by

$$\mathcal{S}_t = \{x \in \mathcal{H}: (\lambda I - T)^{-1}x \text{ has an analytic continuation} \\ \text{to the complement of } \{g(s): s \in [c, t]\}\}.$$

Then each $\mathcal{S}_t$ is a closed hyperinvariant subspace for T , and $\sigma(T|_{\mathcal{S}_t}) \subset \{g(s): s \in [c, t]\}$, (by Lemma 1.2). For each $t \in (c, d)$, let P_t denote the orthogonal projection onto $\mathcal{S}_t$, since T is reductive, it follows that every P_t commute with T . Now $t_1 < t_2$ implies $\mathcal{S}_{t_1} \subset \mathcal{S}_{t_2}$, or $P_{t_1} \leq P_{t_2}$. Next we define the normal operator A by

$$A = \int_a^b g(t) dP_t.$$

Now $\mathcal{S}$ is invariant under every operator commuting with T , and is thus invariant under each P_t and hence reduces A .

The desired conclusion will therefore be established as soon as it is shown that $(T|_{\mathcal{S}}) - (A|_{\mathcal{S}})$ is quasi-nilpotent. For this we first show that $t_1 < t_2$ implies that

$$\sigma(T|(\mathcal{S}_{t_2} \ominus \mathcal{S}_{t_1})) \subset \{g(s): s \in [t_1, t_2]\}.$$

We know by Lemma 1.2 that

$$\sigma(T|(\mathcal{S}_{t_2})) \subset \{g(s): s \in [c, t_2]\},$$

so we need only show that $s_0 \in [c, t_1)$ implies

$$g(s_0) \notin \sigma(T|(\mathcal{S}_{t_2} \ominus \mathcal{S}_{t_1})).$$

We put $T_1 = T|(\mathcal{S}_{t_2} \ominus \mathcal{S}_{t_1})$. Note that

$$(\lambda I - T_1)^{-1} = (\lambda I - T)^{-1}|(\mathcal{S}_{t_2} \ominus \mathcal{S}_{t_1})$$

for $\lambda \notin \sigma(T)$. If $s_0 \in [c, t_1)$, let J_1 be an open subarc of J contained in $\{g(s): s \in [c, t_1)\}$ and containing $g(s_0)$. Since $(\lambda I - T_1)^{-1}$ satisfies the same growth condition as $(\lambda I - T)^{-1}$ does, i.e., $(\lambda I - T_1)^{-1}$ satisfies the growth condition (C) near J , therefore Lemma 1.2 implies that the space

$\mathcal{M} = \{x \in \mathcal{S}_{i_2} \ominus \mathcal{S}_{i_1} : (\lambda I - T_1)^{-1}x \text{ has an analytic continuation to the complement of } \bar{J}_1\},$

is a closed hyperinvariant subspace of T_1 . But if $x \in \mathcal{M}$, then since $(\lambda I - T)^{-1}x = (\lambda I - T_1)^{-1}x$ for $\lambda \notin \sigma(T)$, implies that $x \in \mathcal{S}_{i_1}$. Hence $\mathcal{M} = \{0\}$, and from Lemma 1.2 implies $J_1 \cap \sigma(T_1) = \emptyset$. Thus $g(s_0) \notin \sigma(T_1)$ and $\sigma(T|(\mathcal{S}_{i_2} \ominus \mathcal{S}_{i_1})) \subset \{g(s) : s \in [t_1, t_2]\}$. We can now use an argument of Dunford's (p. 1940 in Part III of [4]) to show that $(T|_{\mathcal{S}}) - (A|_{\mathcal{S}})$ is a quasi-nilpotent operator.

To prove this, given any $\varepsilon > 0$ form a partition

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b$$

of $[a, b]$ such that every diameter of subarc $\{g(s) : s \in [t_{i-1}, t_i]\}$ ($i = 1, 2, \dots, n$) less than ε . Then the spectrum of operators

$$T_i = T|(\mathcal{S}_{i_i} \ominus \mathcal{S}_{i_{i-1}}), A_i = A|(\mathcal{S}_{i_i} \ominus \mathcal{S}_{i_{i-1}})$$

are each contained in $\{g(s) : s \in [t_{i-1}, t_i]\}$ and, since T_i and A_i commute, the spectrum of $T_i - A_i$ is contained in $\{g(s_1) - g(s_2) : s_1 \text{ and } s_2 \in [t_{i-1}, t_i]\}$. Thus the spectral radius of $T_i - A_i$ is at most 2ε . Now we define the space by

$$\mathcal{S} = [\mathcal{S} \cap \mathcal{S}_{i_0}] \oplus [\mathcal{S} \cap (\mathcal{S}_{i_1} \ominus \mathcal{S}_{i_0})] \oplus [\mathcal{S} \cap (\mathcal{S}_{i_2} \ominus \mathcal{S}_{i_1})] \oplus \cdots \oplus [\mathcal{S} \cap (\mathcal{S}_{i_n} \ominus \mathcal{S}_{i_{n-1}})].$$

Let T_0 and A_0 respectively denote the restriction of T and A to $\mathcal{S} \cap \mathcal{S}_{i_0}$. (Note that $\mathcal{S} \cap \mathcal{S}_{i_0}$ could be $\{0\}$ in which case this term is unnecessary.) Then $\sigma(T_0) = \sigma(A_0) = \{g(t_0)\}$, and $\sigma(T_0 - A_0) = \{0\}$. Now

$$(T - A)|_{\mathcal{S}} = \sum_{i=0}^n \oplus P(T_i - A_i),$$

where P is the orthogonal projection onto $\mathcal{S}$, and since the spectrum of the direct sum of finite operators is the union of their spectrum, it follows that the spectral radius of $(T - A)|_{\mathcal{S}}$ is at most 2ε . Since ε is an arbitrary positive number, it results that $\sigma((T - A)|_{\mathcal{S}}) = \{0\}$. Thus $(T|_{\mathcal{S}}) - (A|_{\mathcal{S}})$ is a quasi-nilpotent operator. The proof of Lemma 3.2 is now completed.

THEOREM 3.3. *Let $T \in \mathcal{B}(\mathcal{H}, J)$ be a reductive operator. Suppose that the resolvent of T satisfies the growth condition (C) near J . Then $T = A + B$, where A is normal, B is quasi-nilpotent and A commutes with B , i.e., T is spectral.*

PROOF. We can choose an open subarc J_0 of J such that the corresponding closed hyperinvariant subspace $\mathcal{S}$ given by Lemma 1.2 is not

zero. According to Lemma 3.2, $T|_{\mathcal{S}}$ is the sum of a normal operator and a commuting quasi-nilpotent operator. Thus by Lemma 3.1, we need only show that the whole space $\mathcal{H}$ is the closed linear span of finitely many such subspaces $\mathcal{S}$. Since $\sigma(T)$ is compact, there exists a finite collection $\mathcal{E}$ of arcs such as J_0 above whose union includes $\sigma(T)$. Let $\mathcal{M}_0$ be the closed linear span of the corresponding subspaces $\mathcal{S}$. Next, we must show that $\mathcal{M}_0^\perp = \{0\}$. If not, then let $\lambda_0 \in \sigma(T|_{\mathcal{M}_0^\perp})$ and consider the arc J_0 in $\mathcal{E}$ such that $\lambda_0 \in J_0$. Then the closed invariant subspace for $T|_{\mathcal{M}_0^\perp}$ corresponding to J_0 given by Lemma 1.2 would contain a non-zero vector x . Since $\mathcal{M}_0^\perp$ reduces T , we have

$$(\lambda I - T|_{\mathcal{M}_0^\perp})^{-1}x = (\lambda I - T)^{-1}x,$$

and thus $(\lambda I - T)^{-1}x$ would have an analytic continuation to the complement of $\bar{J}_0$. Hence $x \in \mathcal{S}$, when $\mathcal{S}$ is the invariant subspace for T corresponding to J_0 . But $\mathcal{S} \subset \mathcal{M}_0$, contradicting the fact that $x \in \mathcal{M}_0^\perp$, it results that $\mathcal{M}_0^\perp = \{0\}$. Thus $\mathcal{M}_0 = \mathcal{H}$ and the conclusion of this theorem follows from Lemma 3.1.

REFERENCES

- [1] T. ANDÔ, Note on invariant subspaces of a compact normal operator, Arch. der Math., 14 (1963), 337-340.
- [2] N. ARONSZAJN AND K. T. SMITH, Invariant subspaces of completely continuous operators, Ann. of Math., II Ser., 60 (1954), 345-350.
- [3] I. COLOJOARA, Elemente de Teorie Spectrala, Editura Academiei Republicii Socialiste Romania, Bucuresti, 1968.
- [4] N. DUNFORD AND J. T. SCHWARTZ, Linear Operators, Part I, Part II, Part III, Interscience, New York, 1958, 1963, 1971.
- [5] J. A. DYER, E. A. PEDERSEN AND P. PORCELLI, An equivalent formulation of the invariant subspace conjecture, Bull. Amer. Math. Soc., 78 (1972), 1020-1023.
- [6] K. KITANO, Invariant subspaces of some non-selfadjoint operators, Tôhoku Math. J., II Ser., 20 (1968), 313-322.
- [7] K. KITANO, The growth of the resolvent and hyperinvariant subspaces, Tôhoku Math. J., II Ser., 25 (1973), 317-331.
- [8] V. I. MACAEV, A class of completely continuous operators, Doklady Akad. Nauk SSSR, 139 (1961), 548-551.
- [9] E. NORDGREN, H. RADJAVI AND P. ROSENTHAL, On operators with reducing invariant subspaces. (to appear)
- [10] P. ROSENTHAL, Completely reducible operators, Proc. Amer. Math. Soc., 19 (1968), 826-830.
- [11] T. SAITÔ, Some remarks on Andô's theorems, Tôhoku Math. J., II Ser., 18 (1966), 404-409.
- [12] J. T. SCHWARTZ, Subdiagonalization of operators in Hilbert space with compact imaginary part, Comm. Pure Appl. Math., 15 (1962), 159-172.
- [13] J. G. STAMPFLI, A local spectral theory for operators, J. Functional Analysis, 4 (1969), 1-10.

- [14] J. G. STAMPFLI, A local spectral theory for operators, III; Resolvents, spectral sets and similarity, Trans. Amer. Math. Soc., 168 (1972), 133-151.
- [15] K. YOSIDA, Functional Analysis, 2nd Edition, Springer, Berlin-Heidelberg-New York, 1968.

DEPARTMENT OF MATHEMATICS
TÔHOKU UNIVERSITY
COLLEGE OF GENERAL EDUCATION
KAWAUCHI, SENDAI, JAPAN