Tôhoku Math. Journn.
28 (1976), 267-276.

BOUNDARIES OF COMPONENTS OF KLEINIAN GROUPS

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(Received March 17, 1975)

1. Introduction and preliminaries. Let G be a Kleinian group. A component Δ of G is a connected component of the region of discontinuity $\Omega(G)$. As is well known, the boundary $\partial\Delta$ of Δ is included in the limit set $\Lambda(G)$ of G. A maximal subgroup G_Δ of G, which makes a component Δ of G invariant, is called a component subgroup of G. Let Δ_1 and Δ_2 $(\neq \Delta_1)$ be two components of G. Then, as is easily seen, inclusion relations

$$\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) \subset \Lambda(G_{\Delta_1}) \cap \Lambda(G_{\Delta_2}) \subset \partial\Delta_1 \cap \partial\Delta_2$$

hold. In this article we shall discuss sufficient conditions in order that $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) = \partial\Delta_1 \cap \partial\Delta_2$. Maskit [6] treated the same problem and gave a condition which always holds for a finitely generated Kleinian group G. Results obtained here are extensions of Maskit's result. Further we shall give some applications of our results to function groups.

Throughout this article, G denotes a Kleinian group. If $\Lambda(G) = \emptyset$, then G is a finite group. For a proper subdomain D of the Riemann sphere $\hat{C}$, we denote by $SL'(D)$ the largest subgroup of all Möbius transformations which leaves D fixed. For every component Δ of G, the component subgroup G_Δ coincides with $G \cap SL'(\Delta)$. If $G_\Delta = G$, then Δ is called an invariant component of G, and, if G has such a component, G is called a function group. Two components Δ and Δ' $(\neq \Delta)$ of G are called conjugate to each other, if there is a $g \in G$ with $g(\Delta) = \Delta'$, and in this case they are called non-invariant components of G. For every component Δ of G, the quotient Δ/G_Δ gives a Riemann surface and, if the boundary $\partial\Delta$ of Δ has at least three points, then the Poincaré area of Δ/G_Δ is given by

$$\text{Area}\,(\Delta/G_\Delta) = \iint_{\Delta/G_\Delta} \rho(z)^2 dxdy\ ,$$

where $\rho(z)$ is the Poincaré metric on Δ. It follows from Ahlfors' finiteness theorem that if G is (non-elementary and) finitely generated, then Area $(\Delta/G_\Delta) < \infty$ for each component Δ of G.

2. Quasi-circles. A Kleinian group G is called quasi-Fuchsian with a fixed curve C if C is a directed Jordan curve in $\hat{C}$ and all elements of G leave C fixed. In this case $\Lambda(G)\subset C$ and G is called of the first kind if $\Lambda(G)=C$. If G is a quasi-Fuchsian group of the first kind, then G has two invariant components. A quasi-circle is the image of a circle or a straight line under a quasiconformal mapping. When a quasi-circle L passes through ∞, it was shown by Ahlfors [2] that if z_1, z_2, z_3 are any finite three points on L such that z_3 separates z_1, z_2, then

$$|(z_3-z_1)/(z_2-z_1)|\leqq C(K),$$

where $C(K)$ is a numerical function of the maximal dilatation K of the quasiconformal mapping. We need the following form of the above inequality.

LEMMA 1. *For any finite four points z_1, z_2, z_3, z_4 which lie on a quasi-circle L in this order, it holds that*

$$|(z_3-z_1)/(z_2-z_1)|\leqq C(K)\,|(z_4-z_3)/(z_4-z_2)|\,.$$

PROOF. We transform z_4 to ∞ by a linear transformation T: $z\mapsto (z-z_4)^{-1}$ so that $T(L)$ is a quasi-circle passing through ∞. Then we have

$$\begin{aligned}&|(z_2-z_4)(z_1-z_3)/(z_3-z_4)(z_1-z_2)|\\&\quad=|(T(z_3)-T(z_1))/(T(z_2)-T(z_1))|\leqq C(K)\,,\end{aligned}$$

which implies the desired inequality.

Now, using Lemma 1 we can prove the following lemma.

LEMMA 2. *Let L be a quasi-circle lying in $\bar{U}(=U\cup R\cup\{\infty\})$ with $0\in L$ and let I be the component, lying in U, of the complement of L. If Γ is a finitely generated Fuchsian group of the first kind acting on U, then for any $r>0$, the set $I_r=I\cap\{z\,|\,|z|<r\}$ is not contained in any fundamental set of Γ.*

PROOF. Assume that for some $r>0$ there is a fundamental set $D\subset U$ of Γ which includes I_r. Then L does not include any line segment of R so that for a sufficiently small $y>0$ it holds that $\{z\,|\,\mathrm{Im}\,z=y\}\cap L\cap\{z\,|\,|z|<r\}\neq\varnothing$. In fact, there are points z_1, z_2 of $\{z\,|\,\mathrm{Im}\,z=y\}\cap L$ such that they are separated by 0 as the points on L and such that the open segment z_1z_2 lies in I_r. Therefore we have

$$\mathrm{Area}\,(U/\Gamma)=\iint_D y^{-2}dxdy\geqq\iint_{I_r}y^{-2}dxdy\geqq\int_0^{y_0}y^{-2}dy\int_{z_1}^{z_2}dx\,,$$

where $y_0(>0)$ is sufficiently small and z_1, z_2 lie on L with $\mathrm{Im}\,z_1=\mathrm{Im}\,z_2=y<y_0$. We take and fix a point $z_4\neq 0$ on L such that points z_1, $z_3=0$,

z_2 and z_4 lie on L in this order. By Lemma 1 with $z_3 = 0$ we have

$$\int_{z_1}^{z_2} dx = |z_2 - z_1| \geqq (2C(K))^{-1}y .$$

This implies Area $(U/\Gamma) = +\infty$, which is a contradiction. Thus we have Lemma 2.

Let Γ be a Fuchsian group of the first kind which leaves the upper half plane U fixed. A quasiconformal mapping $w: \hat{C} \mapsto \hat{C}$ is called compatible with Γ if $w\Gamma w^{-1} = G$ is a Kleinian group. In this case G is a quasi-Fuchsian group of the first kind with a quasi-circle $w(\partial U)$ as a fixed curve and is called a quasiconformal deformation of Γ. From a theorem in [5] we easily see that every finitely generated quasi-Fuchsian group of the first kind is a quasiconformal deformation of a finitely generated Fuchsian group of the first kind. For later use we restate Lemma 2 in the following form.

LEMMA 2′. *Let G be a finitely generated quasi-Fuchsian group of the first kind with Δ as a component, L a closed Jordan curve lying in $\bar{\Delta}$ with $L \cap \partial\Delta \neq \varnothing$, I a component, lying in Δ, of the complement of L, and I_r the set $I \cap \{z \mid |z - z_0| < r\}$ for a point z_0 of $L \cap \partial\Delta$. If $g(I_r) \cap I_r = \varnothing$ for some $r > 0$ and for each $g \in G$ not being the identity, then L is not a quasi-circle.*

PROOF. Since G is a quasiconformal deformation of a finitely generated Fuchsian group Γ of the first kind acting on U, there is a quasiconformal mapping w of U onto Δ with $w(0) = z_0$ and $G = w\Gamma w^{-1}$. Then $w^{-1}(L)$ lies in $\bar{U}$ with $0 \in w^{-1}(L)$. Since the action of Γ on the set $w^{-1}(I)$ is the same as that of G on I, there is an $r > 0$ such that $w^{-1}(I) \cap \{z \mid |z| < r\}$ is contained in a fundamental set of Γ. Hence the closed Jordan curve $w^{-1}(L)$ is not a quasi-circle by Lemma 2. Therefore L is not a quasi-circle. Thus we have proved the lemma.

3. Auxiliary domains Δ_i^* and D_i. Let Δ_1 and Δ_2 $(\neq \Delta_1)$ be two distinct components of a Kleinian group G. For $i = 1, 2$, let Δ_i^* be a component of the complement of $\bar{\Delta}_i$ such that $\Delta_i^* \supset \Delta_{3-i}$ and let $G_{\Delta_i^*}$ be the component subgroup of G_{Δ_i} which leaves Δ_i^* fixed. Let D_i be a component of the complement of $\bar{\Delta}_i^*$ such that $D_i \supset \Delta_i$ and let G_{D_i} be the component subgroup of $G_{\Delta_i^*}$ which leaves D_i fixed. For these D_1 and D_2 we have

PROPOSITION 1. *$D_1 \cap D_2 = \varnothing$ and $\partial D_1 \cap \partial D_2 = \partial\Delta_1 \cap \partial\Delta_2$.*

PROOF. Clearly Δ_1^* lies in the exterior of D_1 and Δ_2 lies in Δ_1^*, and hence the boundary of Δ_2 lies in $\bar{\Delta}_1^*$. Hence there is a component of the complement of $\bar{\Delta}_2$ which includes D_1. This component is identical with Δ_2^*. Since D_2 lies in the exterior of Δ_2^*, D_2 lies in the exterior of D_1, which implies the first assertion. For the second assertion we note that $\partial D_1 \subset \partial \Delta_1$ and $\partial D_2 \subset \partial \Delta_2$. This implies $\partial D_1 \cap \partial D_2 \subset \partial \Delta_1 \cap \partial \Delta_2$. Next we show the converse inclusion. By definitions of D_1 and D_2, it holds that $\Delta_1 \subset D_1$ and $\Delta_2 \subset D_2$. These and the first assertion of the present proposition imply $\partial D_1 \cap \partial D_2 \supset \partial \Delta_1 \cap \partial \Delta_2$. Thus Proposition 1 is proved.

PROPOSITION 2. *If* $\Lambda(G_{D_1} \cap G_{D_2}) = \partial D_1 \cap \partial D_2$, *then* $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) = \partial \Delta_1 \cap \partial \Delta_2$.

PROOF. This follows easily from Proposition 1 and from an obvious inclusion relation $\Lambda(G_{D_1} \cap G_{D_2}) \subset \Lambda(G_{\Delta_1} \cap G_{\Delta_2})$.

Now we put the following assumptions on G:

i) $G_{\Delta_i^*}$ $(i = 1, 2)$ is a quasi-Fuchsian group of the first kind,

ii) $\partial \Delta_i^*$ $(i = 1, 2)$ is a quasi-circle and

iii) G_{D_2} is finitely generated.

The conditions i), ii) imply that $G_{\Delta_i^*}$ has two invariant components Δ_i^* and D_i with the common boundary $\partial \Delta_i^* = \partial D_i$ being a quasi-circle and that $G_{\Delta_i^*} = G_{D_i}$. This and iii) imply that G_{D_2} is a finitely generated quasi-Fuchsian group of the first kind with D_2 as a component. We note that if G is a finitely generated Kleinian group or, more generally, if Δ_i/G_{Δ_i} is a finite Riemann surface, then i), ii) and iii) are satisfied.

REMARK. There is a Kleinian group G with components Δ_1, Δ_2 satisfying i), ii) and $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) \neq \partial \Delta_1 \cap \partial \Delta_2$. Such a group is easily constructed by applying Klein's Combination Theorem to two infinitely generated Fuchsian groups of the first kind. Further, there is a Kleinian group G with components Δ_1, Δ_2 satisfying i), iii) and $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) \neq \partial \Delta_1 \cap \partial \Delta_2$. Such a group is constructed by applying Klein's Combination Theorem to the following two groups G_1 and G_2. Namely, G_1 is a finitely generated Fuchsian group of the first kind acting on the upper half plane and containing a parabolic cyclic subgroup generated by a transformation $z \mapsto z + 1$ and G_2 is an infinitely generated quasi-Fuchsian group of the first kind constructed as follows. Let C_0 (or C_0') be a circle with center $1/3 + 2i$ (or $2/3 + 2i$) and radius $1/6$ (or $1/6$), and let C_j (or C_j'), $j = 1, 2, \cdots$, be a circle with center $1/3 + (2 + 1/42 + 2j/7)i$ (or $2/3 + (2 + 1/42 + 2j/7)i$) and radius $1/7$ (or $1/7$). Denoting by g_0 a parabolic transformation which maps the exterior of C_0 onto the interior of C_0' and denoting by g_j a loxodromic transformation which maps the exterior of C_j onto the interior of C_j', $j = 1, 2, \cdots$, we represent by G_2 the group generated by g_j, $j=0, 1, 2, \cdots$. One can easily see that the group $\langle G_1, G_2\rangle$ generated by G_1 and G_2 has the desired properties.

Now we shall prove the following.

PROPOSITION 3. *Assume* i), ii) *and* iii). *If the condition*

$$(*) \qquad g(D_1)\cap D_1=\emptyset \text{ for each } g\in G_{D_2}\backslash G_{D_1}$$

holds, then $\Lambda(G_{D_1}\cap G_{D_2})=\partial D_1\cap\partial D_2$.

PROOF. It is sufficient to prove that $z_0\in\partial D_1\cap\partial D_2$ implies $z_0\in\Lambda(G_{D_1}\cap G_{D_2})$. We assume that this is not true. Then $z_0\in\Omega(G_{D_1}\cap G_{D_2})$ and z_0 is not a fixed point of an elliptic element of $G_{D_1}\cap G_{D_2}$. For, fixed points of elliptic elements of the common subgroup $G_{D_1}\cap G_{D_2}$ lie in D_1 and D_2, respectively. Therefore there is an $r>0$ such that $I_r=\{z\mid |z-z_0|<r\}\cap D_1\subset\Omega(G_{D_1}\cap G_{D_2})$ and such that $g(I_r)\cap I_r=\emptyset$ for all $g\in G_{D_1}\cap G_{D_2}$ being not the identity. This and $(*)$ imply that $g(I_r)\cap I_r=\emptyset$ for each $g\in G_{D_2}$ being not the identity. By Lemma 2′, we see that ∂D_1 is not a quasi-circle. This contradicts ii) and the proposition is proved.

4. **Theorems.** Let G be a Kleinian group and let Δ_1 and Δ_2 $(\neq\Delta_1)$ be two components of G. From Propositions 2 and 3, we see that the conditions i), ii), iii) in the preceding section and the assumption $(*)$ in Proposition 3 imply $\Lambda(G_{\Delta_1}\cap G_{\Delta_2})=\partial\Delta_1\cap\partial\Delta_2$. The conditions i), ii) and iii) are natural ones. So our task is to give a simple sufficient condition in order that $(*)$ holds. From this point of view, we can prove the following three theorems.

THEOREM 1. *If G_{Δ_1} is a quasi-Fuchsian group of the first kind with the quasi-circle $\partial\Delta_1$ and if $G_{\Delta_2^*}$ is a finitely generated quasi-Fuchsian group of the first kind, then $\Lambda(G_{\Delta_1}\cap G_{\Delta_2})=\partial\Delta_1\cap\partial\Delta_2$.*

PROOF. Under the assumptions, the conditions i), ii) and iii) hold clearly and we see $\Delta_1=D_1$, which is a component of G. Hence $g(D_1)=D_1$ or $g(D_1)\cap D_1=\emptyset$. Since $g(D_1)=D_1$ is equivalent to $g\in G_{D_1}$, the condition $(*)$ holds. From Propositions 2 and 3, we have our Theorem.

THEOREM 2. *Let $G_{\Delta_1^*}$ be a quasi-Fuchsian group of the first kind with the quasi-circle $\partial\Delta_1^*$ and let $G_{\Delta_2^*}$ be a finitely generated quasi-Fuchsian group of the first kind. If $\partial\Delta_1\cap\partial\Delta_2$ contains at least two points, then $\Lambda(G_{\Delta_1}\cap G_{\Delta_2})=\partial\Delta_1\cap\partial\Delta_2$.*

PROOF. Obviously the conditions i), ii) and iii) are satisfied and it is easy to see that $\partial D_1\cap\partial D_2$ contains at least two points. Assume that $g(D_1)\cap D_1\neq\emptyset$ for some $g\in G_{D_2}\backslash G_{D_1}$. Then $g(\Delta_1)\cap\Delta_1=\emptyset$. In fact, if $g(\Delta_1)\cap\Delta_1\neq\emptyset$, then $g\in G_{\Delta_1}$ and $g(\Delta_1^*)\cap\Delta_1^*\supset g(D_2)\cap D_2\neq\emptyset$, which implies

$g \in G_{\Delta_1^*} = G_{D_1}$, a contradiction. Hence we see that $g(\Delta_1)$ lies in a component Δ^* of the complement of $\bar{\Delta}_1$ and that $\partial g(D_1)$ lies in $\bar{\Delta}^*$. The last fact follows from $\partial D_1 \subset \partial \Delta_1$. There occur two cases.

The case where $\Delta^* \neq \Delta_1^*$. Note that in this case $g(D_1) \subset \Delta^* \subset D_1 \backslash \Delta_1$. We take and join two points in $\partial D_1 \cap \partial D_2$ by Jordan arcs in D_i, $i = 1, 2$, respectively, and have a closed Jordan curve K passing through these two points. Now $g(K)$ is a closed Jordan curve with a property $\Delta_1 \cap g(K) = \emptyset$. On the other hand, $D_1 \cap g(K) \neq \emptyset$ and $D_2 \cap g(K) \neq \emptyset$ so that $g(K) \cap \partial D_1 \neq \emptyset$. Hence both components of the complement of $g(K)$ include points of ∂D_1 which are also points of $\partial \Delta_1$. This contradicts connectedness of Δ_1.

The case where $\Delta^* = \Delta_1^*$. Note that in this case $g(D_1) \supsetneqq D_1$. Hence $g^{-1}(D_1) \subsetneqq D_1$. Therefore $g^{-1}(D_1)$ and hence $g^{-1}(\Delta_1)$ lies in another component ($\neq \Delta_1^*$) of the complement of $\bar{\Delta}_1$. It also holds that $g^{-1} \in G_{D_2} \backslash G_{D_1}$ and $g^{-1}(\Delta_1) \cap \Delta_1 = \emptyset$. Thus we can reduce this case to the case stated above.

Therefore, we see that (*) in Proposition 3 holds. By the same way as in the proof of Theorem 1, we have our Theorem 2.

Further, we can prove the following.

THEOREM 3. *If $G_{\Delta_i^*}$, $i = 1, 2$, is a finitely generated quasi-Fuchsian group of the first kind, then $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) = \partial \Delta_1 \cap \partial \Delta_2$.*

PROOF. The conditions i), ii) and iii) are obviously satisfied. If $\partial D_1 \cap \partial D_2$ contains two points, then the proof of Theorem 2 shows validity of (*) in Proposition 3 and we have the desired. So we may assume $\partial D_1 \cap \partial D_2 = \{z_0\}$. Contrary to (*) in Proposition 3, assume that $g(D_1) \cap D_1 \neq \emptyset$ for some $g \in G_{D_2} \backslash G_{D_1}$. Then, by the same reasoning as in the proof of Theorem 2, it holds that $g(\Delta_1) \cap \Delta_1 = \emptyset$ and that $\partial g(D_1)$ lies in the closure of a component Δ^* of the complement of $\bar{\Delta}_1$.

We assert that $g(z_0) = z_0$. In the case $\Delta^* \neq \Delta_1^*$, we have $g(D_1) \subset \Delta^* \subsetneqq D_1$ and $g(z_0) = g(\partial D_1 \cap \partial D_2) = \partial g(D_1) \cap \partial D_2 \subset \bar{D}_1 \cap \partial D_2 = z_0$. In the case $\Delta^* = \Delta_1^*$, we have $g(D_1) \supsetneqq D_1$ and $g(z_0) = g(\bar{D}_1 \cap \partial D_2) = g(\bar{D}_1) \cap \partial D_2 \supset \partial D_1 \cap \partial D_2 = z_0$. In both cases $g(z_0) = z_0$.

Therefore, z_0 is a fixed point of a non-elliptic element of G_{D_2}. By the quite same reasoning as above, if $g'(D_2) \cap D_2 \neq \emptyset$ for some $g' \in G_{D_1} \backslash G_{D_2}$, then $g'(z_0) = z_0$ and g' is a non-elliptic element of G_{D_1}. Since g and g' have their fixed points on ∂D_2 and ∂D_1, respectively, they are parabolic. For, otherwise G is not Kleinian. If they are not in the same parabolic cyclic group, then an invariant curve in D_1 under g' intersects an invariant curve in D_2 under g, which is impossible. Since

$g \neq g'$, there are integers m, n ($\neq m$) such that $g^m = (g')^n$ and $g^m(D_1) = (g')^n(D_1) = D_1$. As was shown already, either $g(D_1) \subsetneqq D_1$ or $g(D_1) \supsetneqq D_1$ holds. So we have $g^m(D_1) \neq D_1$, a contradiction. Thus, for at least one of $i = 1$ and 2, we have $g(D_i) \cap D_i = \varnothing$ for each $g \in G_{D_{3-i}} \backslash G_{D_i}$. By the same way as in the proof of Theorem 1, we have the required.

By using Theorem 3, Ahlfors' finiteness theorem and Maskit's theorem [Theorem 2; 5], we have immediately the following.

COROLLARY ([6]). *If Δ_i/G_{Δ_i} is a finite Riemann surface, $i = 1, 2$, then $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) = \partial\Delta_1 \cap \partial\Delta_2$.*

REMARK. It is not known whether or not the conclusion of Theorem 2 holds without the condition on the number of points of the set $\partial\Delta_1 \cap \partial\Delta_2$.

5. The case of function groups. In this section we restrict ourselves to function groups. Let G be a function group with an invariant component Δ_0. Obviously $G_{\Delta_0} = G$ and $\partial\Delta_0 = \Lambda(G)$. Hence, for any other component Δ of G, it holds that $G_{\Delta_0} \cap G_\Delta = G_\Delta$ and $\partial\Delta_0 \cap \partial\Delta = \partial\Delta$. So $\Lambda(G_{\Delta_0} \cap G_\Delta) = \partial\Delta_0 \cap \partial\Delta$ if and only if Δ is a component of G_Δ. Thus, in what follows, we consider only non-invariant components of G. Let Δ be a non-invariant component of G. By Accola's theorem [1], Δ is simply connected. Hence there is a conformal bijection h_Δ of the upper half plane U onto Δ and $\Gamma_\Delta = h_\Delta^{-1} G_\Delta h_\Delta$ is a Fuchsian group which is called a Fuchsian equivalent of G_Δ. The isomorphism $\chi_\Delta: \Gamma_\Delta \to G$ which carries $\gamma \in \Gamma_\Delta$ into $g = h_\Delta \circ \gamma \circ h_\Delta^{-1} \in G_\Delta$, is called the cannonical isomorphism. If $\gamma \in \Gamma_\Delta$ is elliptic of order ν, so is $g = \chi_\Delta(\gamma)$ and vice versa, and g has precisely one fixed point in Δ which is the image of the fixed point of γ in U under h_Δ. The mapping h_Δ can be extended to a mapping $\tilde{h}_\Delta$ of the union of U and the set of non-elliptic fixed points of Γ_Δ onto the union of Δ and the set of non-elliptic fixed points of G_Δ. Bers [3] showed this fact by using the notion of the terminal arcs, which is defined as follows: Let $\gamma \in SL'$ be parabolic or loxodromic and let C be a simple open Jordan arc with definite endpoints. Then the curve C is called a terminal arc of γ if $\gamma(C) \subset C$ and if exactly one of the endpoints of C is fixed under γ.

Now let Δ_1 and Δ_2 ($\neq \Delta_1$) be non-invariant components of a function group G. For the common subgroup of G_{Δ_1} and G_{Δ_2}, we can prove the following.

PROPOSITION 4. *The common subgroup $G_{\Delta_1} \cap G_{\Delta_2}$ of G_{Δ_1} and G_{Δ_2} is either finite or parabolic cyclic.*

PROOF. We assume that $G_{\Delta_1} \cap G_{\Delta_2}$ is not finite. Then it contains non-elliptic elements. It is obvious that if there is a loxodromic element $g \in G_{\Delta_1} \cap G_{\Delta_2}$, then both fixed points z_1 and z_2 of g lie on $\partial\Delta_1 \cap \partial\Delta_2$. Let C_1 and C_2 (or C_1' and C_2') be terminal arcs of g in Δ_1 (or in Δ_2) such that C_1 (or C_1') has an endpoint z_1 and C_2 (or C_2') has an endpoint z_2. We may assume that endpoints of C_1 and C_2 (or C_1' and C_2') different from z_1 and z_2 are identical. Thus C_1, C_2, C_1', C_2' and two points z_1 and z_2 form a closed Jordan curve K and $K \cap \Delta_0 = \emptyset$, where Δ_0 is an invariant component of G. Both the interior and the exterior of K include points of $\Lambda(G)$. Since $\partial\Delta_0 = \Lambda(G)$, the interior and the exterior of K also include points of Δ_0. This contradicts connectedness of Δ_0. Hence there is no loxodromic element in $G_{\Delta_1} \cap G_{\Delta_2}$. Next we assume that in $G_{\Delta_1} \cap G_{\Delta_2}$ there are two parabolic elements g and g' with the different fixed points. We are also able to draw a closed Jordan curve lying in $\Delta_1 \cap \Delta_2 \cap \Lambda(G)$ such that both complements of this curve with respect to $\hat{C}$ include points of Δ_0. The same argument as used just above leads us to a contradiction.

Thus we see that there exists one and only one fixed point of the parabolic elements in $G_{\Delta_1} \cap G_{\Delta_2}$. This fact implies that there is no elliptic element in $G_{\Delta_1} \cap G_{\Delta_2}$. For, Δ_1 and Δ_2 are simply connected and a conjugation of a parabolic element by an elliptic element gives another parabolic element with the different fixed point.

To complete the proof it suffices to show that if there are two parabolic elements in $G_{\Delta_1} \cap G_{\Delta_2}$ with the same fixed point, then they are powers of some parabolic element. This follows at once from the fact that Δ_1 is conformally equivalent to the upper half plane and Fuchsian equivalent Γ_{Δ_1} of G_{Δ_1} makes the upper half plane invariant. Thus we have our proposition.

On the common boundary $\partial\Delta_1 \cap \partial\Delta_2$ of non-invariant components Δ_1 and Δ_2 of G, we have the following.

PROPOSITION 5. *If each boundary of Δ_1 and Δ_2 consists of a closed Jordan curve, then $\partial\Delta_1 \cap \partial\Delta_2$ consists of at most one point.*

PROOF. First we note that G has an invariant component Δ_0. We assume that $\partial\Delta_1 \cap \partial\Delta_2$ contains two points. Connect them by a simple arc C_1 in Δ_1 and by a simple arc C_2 in Δ_2, respectively. Then we have a closed Jordan curve K consisting of C_1 and C_2 such that both the interior and the exterior of K contains points of Δ_0. This contradicts connectedness of Δ_0.

REMARK. Accola [1] gave an example of a function group G which has two invariant components Δ_0, Δ_0' and an infinite number of atoms Δ_i, $i = 1, 2, \cdots$. We can see that $\partial\Delta_0' \cap \partial\Delta_1$ is not finite set. It seems to be open whether or not there is a function group G with exactly one invariant component such that the number of points of $\partial\Delta_1 \cap \partial\Delta_2$ is greater than 1, where Δ_1 and Δ_2 are non-invariant components of G.

Now we can prove the following.

THEOREM 4. *Let G be a function group and let Δ_1 and Δ_2 $(\neq \Delta_1)$ be non-invariant components of G. Suppose that G_{Δ_1} is a quasi-Fuchsian group of the first kind with the quasi-circle $\partial\Delta_1$ and that G_{Δ_2} is a finitely generated quasi-Fuchsian group of the first kind. Then $G_{\Delta_1} \cap G_{\Delta_2}$ is a parabolic cyclic group if and only if $\partial\Delta_1 \cap \partial\Delta_2$ consists of only one point.*

PROOF. Note that Δ_2^* is the complement of $\bar{\Delta}_2$ and $G_{\Delta_2^*} = G_{\Delta_2}$. Hence by Theorem 1, it holds that $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) = \partial\Delta_1 \cap \partial\Delta_2$. By this equality and Proposition 4, if part of the theorem is obvious. Conversely, if $G_{\Delta_1} \cap G_{\Delta_2}$ is parabolic cyclic, then $\Lambda(G_{\Delta_1} \cap G_{\Delta_2}) \neq \emptyset$, so $\partial\Delta_1 \cap \partial\Delta_2 \neq \emptyset$. By Proposition 5, $\partial\Delta_1 \cap \partial\Delta_2$ consists of only one point. Thus the proof is completed.

The fixed point z_0 of a parabolic element g of G_Δ is called a cusp on Δ if there is a circle C passing through z_0 such that the interior I of C is included in Δ and the action of G_Δ on I is equivalent to the action of g. The domain I is called a half plane of g belonging to z_0. Clearly there are no more than two disjoint half planes of g belonging to the cusp z_0. It is well known that for Fuchsian groups every fixed point of parabolic elements is a cusp. (see, for example, [4], p. 61). It is also true for quasi-Fuchsian groups of the first kind. Using the above fact and Theorem 4, we can prove following two theorems.

THEOREM 5. *Let G be a function group and let $\Delta_1, \Delta_2, \Delta_3$ be non-invariant components of G. If G_{Δ_i} $(i = 1, 2, 3)$ are quasi-Fuchsian groups of the first kind with quasi-circle $\partial\Delta_i$ and if one of them is finitely generated, then $\partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3 = \emptyset$.*

PROOF. Let G_{Δ_1} be finitely generated and assume $z_0 \in \partial\Delta_1 \cap \partial\Delta_2 \cap \partial\Delta_3$. Then, by Proposition 5 and by Theorem 4, $\partial\Delta_1 \cap \partial\Delta_2$ consists of only a fixed point z_0 of the common parabolic cyclic subgroup of G_{Δ_1} and G_{Δ_2}. Therefore there are two disjoint half planes belonging to z_0, one lies in Δ_1 and the other lies in Δ_2. The same is also true for Δ_1 and Δ_3. Hence there would be three disjoint half planes belonging to z_0. Thus we have a contradiction and the theorem is proved.

THEOREM 6. *Let G be a B-group, that is, a finitely generated (non-elementary) Kleinian group with a simply connected invariant component. Let Δ_1 and Δ_2 be non-invariant components of G. If $\partial\Delta_1 \cap \partial\Delta_2 \neq \emptyset$, then Δ_1 and Δ_2 are attached at the fixed point of an accidental parabolic transformation of G.*

PROOF. An accidental parabolic transformation g is a parabolic transformation of G such that $h_{\Delta_0}^{-1} g h_{\Delta_0}$ is hyperbolic, where Δ_0 is an invariant component of G and h_{Δ_0} is a conformal mapping of the upper half plane onto Δ_0. Since G is finitely generated, the conditions on G_{Δ_1} and G_{Δ_2} in Theorem 4 is clearly satisfied. Hence by Theorem 4, $\partial\Delta_1 \cap \partial\Delta_2 = z_0$ is a fixed point of a parabolic element g of G. If g is not accidental parabolic, then $h_{\Delta_0}^{-1} g h_{\Delta_0}$ is parabolic. Considering the image of a half plane belonging to a cusp $\tilde{h}_{\Delta_0}^{-1}(z_0)$ on the upper half plane, we can easily find a half plane, in Δ_0, belonging to z_0. Thus there would be three disjoint half planes belonging to z_0. This contradiction proves our theorem.

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