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A NOTE ON THE CONTINUITY OF AUTOMORPHIC REPRESENTATIONS OF GROUPS

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Let A be a C^*-algebra with identity operator 1 acting on a *not necessarily separable* Hilbert space $\mathfrak{H}$ and M be the von Neumann algebra generated by A. Let G be a topological group and $g \to \alpha_g$ be a representation of G into the group $\mathrm{Aut}^*(A)$ of $*$-automorphisms of A. Suppose that for each $g \in G$, α_g is extendable on M as a $*$-automorphism β_g of M (it is easy to check that $g \to \beta_g$ is also a representation of G into $\mathrm{Aut}^*(M)$) and suppose that the mapping $g \to (\alpha_g(a)\xi, \eta)$ is continuous on G for each $a \in A$, ξ, $\eta \in \mathfrak{H}$. Then it is natural to ask whether the mapping $g \to (\beta_g(b)\xi, \eta)$ is continuous on G for each $b \in M$, ξ, $\eta \in \mathfrak{H}$ or not. For the physical background of this problem, see [3] and [5].

In this paper, we will introduce two assumptions on the action of G ((α) and (β) below) and then under the condition that G is topologized by a complete metric, will give the affirmative answer to the above problem. Next we will apply this to a result for non separable case which plays a rôle in the theorem concerning the existence of invariant traces [8].

Our first assumption is

(α). For each $g \in G$, there exists a unitary operator u_g on such that $\alpha_g = \mathrm{Ad}\, u_g \mid A$ (where $(\mathrm{Ad}\, u_g \mid A)(x) = u_g x u_g^*$ for all $x \in A$).

We do not assume that the map $G \ni g \to u_g$ is either a representation or continuous, however we suppose

(β). There is a sequence $\{a(n)\}$ of weakly continuous mapping of G into A such that $u_g \in \{a(n)(g),\ n = 1, 2, \cdots\}''$ (the double commutant of $\{a(n)(g),\ n = 1, 2, \cdots\}$ [4]) for each $g \in G$.

The key point of the proof is, roughly speaking, how to reduce the problem to the case that $\mathfrak{H}$ is separable. To do this, the Pedersen's theorem ([9]) concerning "monotone closures in operator algebras" plays an important rôle. In the case that *$\mathfrak{H}$ is separable* and the topology of G is given by a complete metric, R. R. Kallman ([6]) solved the problem in the affirmative.

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Under a certain joint continuity on the map $G \times A \ni (g, a) \rightarrow \alpha_g(a) \in A$, J. F. Aarnes ([1]) gave the affirmative answer to the question.

2. Proofs of Theorem. First of all, we shall show the following generalization of the Kallman's result.

PROPOSITION. *Keeping the notations as above, instead of the postulates* (α) *and* (β), *we suppose that* M *is* σ*-finite. If* G *is topologized by a complete metric, then* $g \rightarrow (\beta_g(x)\xi, \eta)$ *is continuous on* G *for each* $x \in M$, ξ *and* η *in* $\mathfrak{H}$.

PROOF. We have only to show that for any sequence $\{g_m\}$ in G which converges to g_0 in G and any self-adjoint element $z(\|z\| \leqq 1)$ in M, $\beta_{g_m}(z) \rightarrow \beta_{g_0}(z)$ weakly in M. By the Kaplansky's density theorem ([4]), we can choose a sequence $\{z_n\}$ in the self-adjoint part of $A(\|z_n\| \leqq 1,\ n = 1, 2, 3, \cdots)$ such that $z_n \rightarrow z\ (n \rightarrow \infty)$ strongly. Let G_∞ be the group generated algebraically by $\{g_m;\ m = 0, 1, 2, \cdots\}$. Let H be the closure of G_∞ in G. Then H is a complete separable metric group with the countable dense subgroup G_∞. Let B be the C^*-subalgebra of A generated by $\{\alpha_h(z_n);\ h \in H,\ n = 1, 2, \cdots;\ 1 \text{ (the identity operator on } \mathfrak{H})\}$. Then one can easily check that $\alpha_h(B) = B$ for each $h \in H$ and $h \rightarrow (\alpha_h(a)\xi, \eta)$ is continuous on H for all $a \in B$ and ξ, $\eta \in \mathfrak{H}$. Since M is σ-finite, the weak closure N of B is also σ-finite and there exists a faithful normal state ϕ on N. Let $(\pi_\phi, \mathfrak{H}_\phi, \xi_\phi)$ be the cyclic representation of N induced by ϕ. Then π_ϕ is a $*$-isomorphism of N onto the von Neumann algebra $\pi_\phi(N)$ on $\mathfrak{H}_\phi$. Noting that $\mathfrak{N} \equiv \{p(\alpha_{h_1}(z_{n_1}), \cdots, \alpha_{h_k}(z_{n_k}));\ h_i \in G_\infty,\ i = 1, 2, \cdots, k;\ z_{n_1}, \cdots, z_{n_k} \in \{z_n\}_{n=1}^\infty;\ p(\lambda_1, \cdots, \lambda_k)$ a polynomial with rational coefficients in $\lambda_1, \cdots, \lambda_k\}$ is countable and $\{\pi_\phi(a)\xi_\phi,\ a \in \mathfrak{N}\}$ is uniformly dense in $\mathfrak{H}_\phi$, by the normality of π_ϕ and the continuity of the action α_g on A of H, we can easily show that $\mathfrak{H}_\phi$ is separable. Let $\gamma_g = \pi_\phi \circ \beta_g \circ \pi_\phi^{-1}$ on $\pi_\phi(N)$ for each $g \in H$. Then $g \rightarrow \gamma_g$ is a representation of H into $\mathrm{Aut}^*(\pi_\phi(N))$ such that $\gamma_g(\pi_\phi(a)) = \pi_\phi(\alpha_g(a))$ for each $g \in H$ and $a \in B$. Thus $g \rightarrow \gamma_g \mid \pi_\phi(B)$ is weakly continuous on H and hence $\{\pi_\phi(B),\ \widetilde{\pi_\phi(B)}\ (=\pi_\phi(N)),\ \mathfrak{H}_\phi,\ \gamma,\ H\}$ satisfies the assumption of the Kallman's Theorem ([6]). Hence by [6], for each $a \in \pi_\phi(N)$, $g \rightarrow \gamma_g(a)$ is weakly continuous on H. This implies by the normality of π_ϕ, $g \rightarrow (\beta_g(b)\xi, \eta)$ is continuous on H for each $b \in N$, ξ, $\eta \in \mathfrak{H}$. Since $\{g_m, g_0\}_{m=1}^\infty \subset H$ and $z \in N$, we can show the desired statement cited in the first paragraph of the proof. The Proposition follows.

THEOREM. *Keeping the notations in Proposition, we do not assume that* M *is* σ*-finite but suppose the postulates* (α) *and* (β). *Then, under* *the condition that the topology of G is given by a complete metric, $g \to (\beta_g(a)\xi, \eta)$ is continuous for any $a \in M$ and $\xi, \eta \in \mathfrak{H}$.*

PROOF. First of all, we shall show that for any non-negative element z in M such that there is a sequence $\{z_n\}(\|z_n\| \leqq 1, n = 1, 2, \cdots)$ in the non-negative part of A with $z_n \to z$ strongly, and for any normal state ϕ of M, $g \to \phi(\beta_g(z))$ is continuous on G. To do this, it is sufficient to show that $\phi(\beta_{g_m}(z)) \to \phi(\beta_{g_0}(z))$ $(m \to \infty)$ for any sequence $\{g_n\}$ in G with $g_n \to g_0$ in G. By the same way as that in Proposition, let G_∞ be the countable subgroup of G generated algebraically by $\{g_m, m = 0, 1, 2, \cdots\}$ and H be the closure of G_∞ in G. Let C be the C^*-subalgebra of A generated by $\{\beta_h(a(n)(g)), h \in H; n = 1, 2, \cdots; g \in G_\infty\}$, $\{\beta_h(z_n), n = 1, 2, \cdots; h \in H\}$ and $1_{\mathfrak{H}}$. Then we can easily check that $\beta_g(C) = C$ for all $g \in H$. Since $C \subset A$, $g \to (\beta_g(a)\xi, \eta)$ is continuous on H for any $a \in C$, $\xi, \eta \in \mathfrak{H}$. Let $\tilde{C}$ be the weak closure of C and let $\psi = \phi \mid \tilde{C}$ (where ϕ is an arbitrarily given normal state of M) and e_ψ be the support of ψ in $\tilde{C}$. Then e_ψ is σ-finite in $\tilde{C}$. Next we shall show that the central support (in $\tilde{C}$) z_ψ of e_ψ is also σ-finite in $\tilde{C}$. At first, we claim that $\tilde{C}$ is countably generated. In fact, since $\mathfrak{A} = \{\beta_h(a(n)(g)); 1_{\mathfrak{H}}; h \in G_\infty; n = 1, 2, \cdots; g \in G_\infty\}$ and $\mathfrak{B} = \{\beta_n(z_n), n = 1, 2, 3, \cdots; h \in G_\infty\}$ are countable sets, the set $\mathfrak{L} = \{a_1b_1a_2b_2 \cdots a_kb_k, a_i \in \mathfrak{A}, b_i \in \mathfrak{B}, i = 1, 2, \cdots, k\ k \in N\}$ (where N is the set of natural numbers) is countable and by the continuity of the action α_g of G in $\mathrm{Aut}^*(B)$, the countable set of rational linear combinations of elements in $\mathfrak{L}$ is strongly dense in $\tilde{C}$. The statement follows. Hence there exists a strongly dense countable subset $\{a_n\}_{n=1}^\infty$ in $\tilde{C}$. Let $f_n(\in \tilde{C})$ be the range projection of a_ne_ψ. Then $f_n \prec e_\psi$ for each n and thus, f_n is σ-finite in $\tilde{C}$ for each n. Let $f = \bigvee_{k=1}^\infty f_n$ in $\tilde{C}$. Then one can easily check that f is also σ-finite in $\tilde{C}$. We claim that $f = z_\psi$. Since $z_\psi\mathfrak{H} = [\tilde{C}e_\psi\mathfrak{H}]$ ([4]) (where $[\mathfrak{K}]$ is the closed linear span of $\mathfrak{K}$ in $\mathfrak{H}$), $z_\psi \geqq f$ follows. Conversely, each $ae_\psi\eta$ $(a \in \tilde{C}\eta \subset \mathfrak{H})$ can be approximated in the norm by an element of the form $a_{n_i}e_\psi\eta(a_{n_i} \in \{a_n\}_{n=1}^\infty)$, $z_\psi \leqq f$ follows immediately and $z_\psi = f$ is σ-finite in $\tilde{C}$. Since for each $g \in G_\infty$, $u_g \in \{a(n)(g); n = 1, 2, \cdots\}''$, $u_g \in \tilde{C}$ for all $g \in G_\infty$. For any $g \in H$, there is a sequence $\{g_m\}$ in G_∞ such that $d(g_m, g) \to 0$ $(m \to \infty)$. By Postulate (β), $a(n)(g_m) \to a(n)(g)$ $(m \to \infty)$ weakly in $\tilde{C}$ for each n. Thus $\{a(n)(g), n = 1, 2, \cdots\}'' \subset \{a(n)(g_m); n = 1, 2, \cdots; m = 1, 2, \cdots\}'' \subset \tilde{C}$ and $u_g \in \tilde{C}$ for any $g \in H$ by Postulate (β). This implies by Postulate (α), $\beta_g(z_\psi) = z_\psi$ for all $g \in H$.

For any $a \in \tilde{C}z_\psi$ ([4]), let $\gamma_g(az_\psi) = \beta_g(a)z_\psi$, then noting that $\beta_g(az_\psi) = \beta_g(a)z_\psi$, it is well-defined and $g \to \gamma_g$ is a representation of H into $\mathrm{Aut}^*(\tilde{C}z_\psi)$. Moreover, $\gamma_g(Cz_\psi) = Cz_\psi$ for each $g \in H$ and $g \to \gamma_g \mid Cz_\psi$

is strongly continuous. Thus the system $\{Cz_\psi, \widetilde{C}z_\psi\ (=\widetilde{Cz_\psi}), \gamma, H\}$ satisfies the condition of Proposition, because z_ψ is σ-finite in $\widetilde{C}$. Hence, noting that $z \in \widetilde{C}$ and $\{g_m, g_0\} \subset H$, $\gamma_{g_m}(zz_\psi) \to \gamma_{g_0}(zz_\psi)$ (strongly) as $m \to \infty$, that is, $\beta_{g_m}(z)z_\psi \to \beta_{g_0}(z)z_\psi$ $(m \to \infty)$ strongly. Since $z_\psi \geqq e_\psi$ and $\psi = \psi(z_\psi \cdot)$, $\varphi(\beta_{g_m}(z)) = \psi(\beta_{g_m}(z)) = \psi(\beta_{g_m}(z)z_\psi) \to \psi(\beta_{g_0}(z)z_\psi) = \psi(\beta_{g_0}(z)) = \varphi(\beta_{g_0}(z))$ $(m \to \infty)$. Denote the predual action of G by $g \to T_g$ $(g \in G)$ (that is, $(T_g\psi)(x) = \psi(\alpha_g(x))$ for all $x \in M$, $g \in G$), this implies $T_{g_m}\phi \to T_{g_0}\phi$ in A_σ (where A_σ is the set of operators in $\mathfrak{B}(\mathfrak{H})$ which can be strongly approximated by a monotone increasing sequence from A [9]) $(m \to \infty)$. Now, the argument of Akemann, Dodds and Gamlen ([2] Theorem 3.2. See also [9].) tells us that $T_{g_m}\phi \to T_{g_0}\phi$ $(\mathrm{m} \to \infty)$ weakly in M_* (the predual of M) for each $\phi \in M_*$. This implies that $\alpha_{g_m}(a) \to \alpha_{g_0}(a)$ weakly $(m \to \infty)$ for any $a \in M$. This completes the proof of Theorem.

3. Applications. This section concerns with an application of the above discussions to a result which plays a rôle in a theorem concerning the existence of invariant traces in semi-finite von Neumann algebras [8]. Before going into discussions, we prepare the following lemma which is a modification of [8, Lemma 2].

LEMMA. *Let M be a von Neumann algebra acting on a not necessarily separable Hilbert space $\mathfrak{H}$, N be a von Neumann subalgebra such that there is a normal projection of norm one Φ from M onto N. Let u be a unitary operator in M such that $u^{-1/n}\Phi(u^{1/n}) \in Z$ $(n = 1, 2, \cdots)$ (where Z is the center of M). Then there exists a unitary operator ν in $\{\Phi(u^m),\ m = 0, \pm 1, \pm 2, \cdots\}''$ $(\equiv \mathfrak{N} \subset N)$ satisfying that* $\mathrm{Ad}\, u = \mathrm{Ad}\, \nu$ *on M (that is, u and ν implement the same $*$-automorphism of M).*

PROOF. Let $b_n = \Phi(u^{1/n})$ and $c_n = u^{-1/n}b_n$ for each natural number n. Then $b_n \ni \mathfrak{N}$, $c_n \in Z$ $n = 1, 2, \cdots$. Let $b_n = \nu_n|b_n|$ (resp. $c_n = u_n|c_n|$) be the polar decomposition of b_n (resp. c_n). The unicity of the polar decomposition implies that $|b_n| = |c_n| \in Z \cap \mathfrak{N}$ and $u^{1/n}u_n = \nu_n \in \mathfrak{N}$. Let q_n be the range projection of $|c_n|$ (and thus, of $|b_n|$). Then $q_n \in \mathfrak{N} \cap Z$. Note that $u_n \in Z$, we have $\mathrm{Ad}\,(u^{1/n}) = \mathrm{Ad}\,(\nu_n)$ on Mq_n. Since the normality of Φ tells us that $c_n \to 1$ strongly $(n \to \infty)$, there exists a positive integer n_0 such that $c_{n_0} \neq 0$ and $q_{n_0} \neq 0$. It follows that $\mathrm{Ad}\,(uq_{n_0}) = \mathrm{Ad}\,(\nu_{n_0}^{n_0})$ on Mq_{n_0}. Let $\{\nu_\alpha, q_\alpha\}$ be a maximal family such that ν_α is a unitary element in $\mathfrak{N} \cap Mq_\alpha$, q_α is a projection in $Z \cap \mathfrak{N}$ for each α, $q_\alpha q_{\alpha'} = 0$ $(\alpha \neq \alpha')$ and $\mathrm{Ad}\,(\nu_\alpha) = \mathrm{Ad}\,(uq_\alpha)$ on Mq_α for each α. Then, since $\Phi(axb) = a\Phi(x)b$ for any $x \in M$, $a, b \in N$, the above arguments implies that $\sum_\alpha q_\alpha = 1$. Let $\nu = \sum_\alpha \nu_\alpha q_\alpha$ $(\in \mathfrak{N})$. Then $\mathrm{Ad}\,(\nu) = \mathrm{Ad}\,(u)$ on M. The lemma follows.

Keep the notations as above, let G be a topological group whose topology is given by a complete metric, and let $g \to u_g$ be a σ-weakly continuous unitary representation of G into M such that $u_g^{-1/n}\Phi(u_g^{1/n}) \in Z$ for each $g \in G$ and positive integer n. Let $\mathfrak{M} = (M, N')''$. Then by the above lemma, for each $g \in G$, there is a unitary element ν_g in $\{\Phi(u_g^m); m = 0, \pm 1, \pm 2, \cdots\}''$ $(\subset N)$ such that $\mathrm{Ad}\,\nu_g = \mathrm{Ad}\,u_g$ on M. Let $\sigma_g = \mathrm{Ad}\,\nu_g \,|\, \mathfrak{M}$, and let A be the C^*-algebra generated by M and $N'(\tilde{A} = \mathfrak{M})$. Then by the arguments of [8], $g \to \sigma_g$ $(g \in G)$ is a representation of G into $\mathrm{Aut}^*(\mathfrak{M})$ such that $\sigma_g(A) = A$ for each g and $g \to \sigma_g(a)$ is weakly continuous on G for each $a \in A$.

In fact, let $a_1, a_2, \cdots, a_n \in M$ and $b_1, b_2, \cdots, b_n \in N'$. Then, noting that $\nu_g \in N$ for each g, we get that $\sigma_g\sigma_h(a_1b_1a_2b_2 \cdots a_nb_n) = \nu_g\nu_h(a_1b_1a_2b_2 \cdots a_nb_n)(\nu_g\nu_h)^* = u_gu_ha_1(u_gu_h)^*b_1 \cdots (u_gu_h)a_n(u_gu_h)^*b_n = u_{gh}a_1u_{gh}{}^*b_1 \cdots u_{gh}a_nu_{gh}{}^*b_n = \nu_{gh}a_1\nu_{gh}{}^*b_1 \cdots \nu_{gh}a_n\nu_{gh}{}^*b_n = \sigma_{gh}(a_1b_1 \cdots a_nb_n)$. Thus $g \to \sigma_g$ is an automorphic representation of G on $\mathfrak{H}$ such that $\sigma_g(A) = A$ for each g. Since $g \to u_g$ is a strongly continuous unitary representation of G into M, $\sigma_g(a_1b_1a_2b_2 \cdots a_nb_n) = u_ga_1u_g{}^*b_1 \cdots u_ga_nu_g{}^*b_n$ implies that $g \to \sigma_g(a)$ is strongly continuous on G for each $a \in A$. Note that $\nu_g \in \{\Phi(u_g^m); m = 0, \pm 1, \pm 2, \cdots\}''$ and $g \to \Phi(u_g^m)$ is σ-weakly continuous on G for each m, by Theorem, we get that $g \to (\sigma_g(b)\xi, \eta)$ is continuous on G for each $b \in \mathfrak{M}$ and $\xi, \eta \in \mathfrak{H}$. Thus we have:

COROLLARY. *Let M be a von Neumann algebra with center Z acting on a not necessarily separable Hilbert space $\mathfrak{H}$ and let G be a group of $*$-automorphisms of M with the action $g \to \alpha_g \in \mathrm{Aut}^*(M)$ for each $g \in G$.*

Suppose that there is a faithful normal G-invariant state φ (thus, by a Theorem of Kovacs-Szücs [7], *there exists a faithful normal G-invariant projection Φ of norm one from M onto the fixed subalgebra M^G of M under G) and M is semi-finite. Let τ be a normal faithful semi-finite trace of M. Then by Radon-Nikodym theorem* (*see* [4]), *there is a non-negative self-adjoint densely defined operator h affiliated with M such that $\phi(a) = \tau(ha)$ for each $a \in M$. Since h^{it} and $\alpha_g(h^{it})$ implement the same automorphism of M for each real number t, $h^{-it}\Phi(h^{it}) \in Z$ for each t. Let v_t be the unitary operator in $\{\Phi(\{h^{it}\}'')\}''$ given in Lemma and let $\mathfrak{M} = (M, (M^G)')''$. Then $t \to \mathrm{Ad}\,v_t \,|\, \mathfrak{M}$ is a strongly continuous one-parameter automorphism group of $\mathfrak{M}$ (In* [8], *Nest assumes that $\mathfrak{H}$ is separable).*

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