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ON A PROPERTY OF BRIESKORN MANIFOLDS

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1. Introduction. A Brieskorn manifold is by definition a $(2n-1)$-dimensional submanifold $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ in a complex space C^{n+1} with complex coordinates $z_0, z_1, \cdots, z_n$ which is defined by equations

$$z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0 \tag{1.1}$$

and

$$z_0\bar{z}_0 + z_1\bar{z}_1 + \cdots + z_n\bar{z}_n = 1\ , \tag{1.2}$$

where $a_0, a_1, \cdots, a_n$ are positive integers.

Recently, K. Abe [1] introduced an almost contact structure for every Brieskorn manifold, i.e. a triple (ϕ, ξ, η) of a (1, 1)-tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 X = -X + \eta(X)\xi\ , \quad \eta(\xi) = 1\ . \tag{1.3}$$

He studied the structure with special emphasis of the non-regularity of the 1-dimensional foliation generated by the vector field ξ in general.

A differentiable manifold M^{2n-1} is said to be a contact manifold if there exists a 1-form ζ on M^{2n-1} such that

$$\zeta \wedge (d\zeta)^{n-1} \neq 0 \tag{1.4}$$

and ζ is called a contact form. A contact manifold admits an almost contact structure closely related with the contact form.

The main result of this paper is the following

MAIN THEOREM. *Every Brieskorn manifold is a contact manifold.*

It is well known that the set of all Brieskorn manifolds of dimension $2n-1$ $(n \geqq 2)$ contains all homotopy $(2n-1)$-spheres which are boundaries of compact orientable parallelizable manifolds. [2] [3]

In § 2, we shall find a candidate of a contact form on $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$. In § 3, we shall prove the main theorem by showing that the candidate is really a contact form.

Besides the almost contact structure (ϕ, ξ, η) defined by K. Abe on $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$, we can naturally define an almost contact structure (ϕ', ξ', η') on the same Brieskorn manifold as the latter is a hypersurface of a Kählerian manifold. In § 4, we give necessary and sufficient condition for the coincidence of two 1-dimensional foliations generated by the vector fields ξ and ξ'.

2. To find a candidate of a contact form. We denote the hypersurface in C^{n+1} defined by (1.1) by V. If all $a_\alpha \geqq 2$ ($\alpha = 0, 1, \cdots, n$), then V has an isolated singularity at the origin O. We call $V - \{0\}$ a Brieskorn variety and denote it by $B^{2n}(a_0, a_1, \cdots, a_n)$ or simply by B^{2n}. The Brieskorn manifold $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ is the intersection of B^{2n} with the unit hypersphere S^{2n+1}. We denote it simply by Σ^{2n-1} too.

Let us consider the C-action on C^{n+1} defined by

$$z'_\alpha = e^{mw/a_\alpha} z_\alpha \ , \tag{2.1}$$

where m is the least common multiple of the integers $a_0, a_1, \cdots, a_n$ and w is a complex variable. We can easily see that the C-action fixes the origin O and transforms B^{2n} onto itself. Therefore, restricting w to its real part s and differentiating $z'_\alpha(s)$ at $s = 0$ we see that

$$u_1 = \left(\frac{m}{a_\alpha} z_\alpha\right) \qquad z \in B^{2n} \tag{2.2}$$

is a tangent vector of B^{2n} at z. In the same way, restricting w to its purely imaginary part it (t: real), we see that

$$u_2 = iu_1 = \left(\frac{m}{a_\alpha} i z_\alpha\right) \qquad z \in B^{2n} \tag{2.3}$$

is a tangent vector of B^{2n} at z orthogonal to u_1. When we restrict w to it, (2.1) gives a S^1-action on C^{n+1} and the S^1-action leaves B^{2n}, S^{2n+1} and so their intersection Σ^{2n-1}. Therefore, if $z \in \Sigma^{2n-1}$, the orbit of the point z under this action lies on Σ^{2n-1} and so u_2 is a tangent vector of Σ^{2n-1}.

Now, denoting the differential at a point z on B^{2n} by dz, we get by (1.1)

$$\sum \frac{\partial f}{\partial z_\alpha} dz_\alpha = 0 \ , \tag{2.4}$$

where $f(z_0, z_1, \cdots, z_n)$ means the polynomial on the left hand side of (1.1). (2.4) is equivalent with $\langle \overline{\partial f/\partial z}, dz \rangle = 0$, where the bracket means the inner product of two vectors $\overline{\partial f/\partial z}$ (the complex conjugate of $\partial f/\partial z$) and dz in C^{n+1}. So, we have

$$\Re\left\langle \overline{\frac{\partial f}{\partial z}}, dz \right\rangle = 0 \ , \qquad \Re\left\langle i\overline{\frac{\partial f}{\partial z}}, dz \right\rangle = 0 \ .$$

These equations tell us that

$$(2.5)\qquad v_1 \equiv \left(\frac{\overline{\partial f}}{\partial z_\alpha}\right) = (a_\alpha \bar{z}_\alpha^{a_\alpha - 1}) ,$$

$$v_2 \equiv \left(i\frac{\overline{\partial f}}{\partial z_\alpha}\right) = (ia_\alpha \bar{z}^{a_\alpha - 1}) = iv_1$$

are normal vectors of B^{2n} at the point z. We can easily show that u_1, u_2, v_1 and v_2 are mutually orthogonal.

Let us restrict the point z to the one on Σ^{2n-1}. Then the unit normal vector n of S^{2n+1} has z_α as its components. v_1, v_2 and n are normals to Σ^{2n-1} in C^{n+1}.

They are linearly independent. For if there is a relation of the form $n = \rho v_1 + \sigma v_2$, then we have

$$z_\alpha = (\rho + \sigma i) a_\alpha \bar{z}_\alpha^{a_\alpha - 1} ,$$

which shows us that

$$\sum \frac{z_\alpha \bar{z}_\alpha}{a_\alpha} = (\rho + \sigma i)(\sum \bar{z}_\alpha^{a_\alpha}) = 0$$

and so $z_\alpha = 0$, contradictory to the fact that $z \in \Sigma^{2n-1}$. We define λ, μ by

$$(2.6)\qquad \lambda = -\frac{\Re\mathrm{e}(\sum a_\alpha z_\alpha^{a_\alpha})}{\langle v_1, v_1\rangle} , \quad \mu = \frac{\Im\mathrm{m}(\sum a_\alpha z_\alpha^{a_\alpha})}{\langle v_2, v_2\rangle} .$$

Then, we can easily verify that v_1, v_2 and

$$(2.7)\qquad v = n + \lambda v_1 + \mu v_2$$

are normal vectors of Σ^{2n-1} in C^{n+1} orthogonal with each other. Hence, v is a normal vector of Σ^{2n-1} which lies in the tangent space of B^{2n} at each point $z \in \Sigma^{2n-1}$.

B^{2n} inherits the complex structure from that of C^{n+1}. If we denote the Kählerian inner product by $\langle\langle\ ,\ \rangle\rangle$, we have

$$\langle\langle iv, dz\rangle\rangle = \Re\mathrm{e}\langle iv, dz\rangle .$$

On account of (2.4) and (2.5), this reduces to

$$\langle\langle iv, dz\rangle\rangle = \frac{i}{2}\sum_{\alpha=0}^{n}(z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha) .$$

The real 1-form ζ on $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ defined by

$$(2.8)\qquad \zeta = \frac{i}{2}\sum(z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha)$$

i.e. the restriction of the real 1-form on C^{n+1} defined by the right hand side of (2.8) to Σ^{2n-1} is a candidate of a contact form for the Brieskorn manifold in consideration. The geometrical meaning of ζ is given as

$$\zeta = \langle\langle iv, dz\rangle\rangle = \langle\langle in, dz\rangle\rangle \ . \tag{2.9}$$

3. A proof of the main theorem. We shall show that the 1-form ζ on $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ defined by (2.8) is a contact form.

From (2.8) we have

$$d\zeta = i \sum_{\alpha=0}^{n} dz_\alpha \wedge d\bar{z}_\alpha \ . \tag{3.1}$$

So, we get

$$\begin{aligned} \zeta \wedge (d\zeta)^{n-1} &= \frac{i^n}{2}\left\{\sum_{\alpha=0}^{n}(z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha)\right\} \wedge \left(\sum_{\beta=0}^{n} dz_\beta \wedge d\bar{z}_\beta\right)^{n-1} \\ &= \frac{(n-1)!\, i^n}{2}\Bigg[\left\{\sum_{\alpha=0}^{n}(z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha)\right\} \\ &\quad \wedge \Bigg\{\sum_{\beta<\gamma}(dz_0 \wedge d\bar{z}_0) \wedge \cdots \wedge \widehat{(dz_\beta \wedge d\bar{z}_\beta)} \\ &\quad \wedge \cdots \wedge \widehat{(dz_\gamma \wedge d\bar{z}_\gamma)} \wedge \cdots \wedge (dz_n \wedge d\bar{z}_n)\Bigg\}\Bigg] , \end{aligned} \tag{3.2}$$

where roofs mean factors which should be omitted.

To show (1.4), we may first restrict ourselves on the domain D_n on Σ^{2n-1} where $z_n \neq 0$.

On D_n we have by (1.1)

$$dz_n = -\sum_{p=0}^{n-1} l_p dz^p \ , \tag{3.3}$$

where we have put

$$l_p = \frac{t_p}{t_n} \ , \qquad t_\alpha = a_\alpha z_\alpha^{a_\alpha - 1} \ . \tag{3.4}$$

We denote the equation complex conjugate to (3.3) by $\overline{(3.3)}$. On the other hand, we have by (1.2)

$$\sum_{\alpha=0}^{n}(z_\alpha d\bar{z}_\alpha + \bar{z}_\alpha dz_\alpha) = 0$$

on B^{2n-1}. Putting (3.3) and $\overline{(3.3)}$ into the last equation, we have

$$\sum_{p=0}^{n-1}(m_p dz_p + \bar{m}_p d\bar{z}_p) = 0 \ , \tag{3.5}$$

where we have put

(3.6) $$m_p = \bar{z}_p - \bar{z}_n l_p\,, \qquad \bar{m}_p = z_p - z_n \bar{l}_p\,.$$

The functions $m_0, m_1, \cdots, m_{n-1}$ defined on D_n can not vanish simultaneously at any point of D_n. For, if $m_0, m_1, \cdots, m_{n-1}$ vanish simultaneously at a point z on D_n, we have

(3.7) $$\frac{t_0}{\bar{z}_0} = \frac{t_1}{\bar{z}_1} = \cdots = \frac{t_n}{\bar{z}_n}\,,$$

which tells us that

$$\frac{z_0^{a_0}}{\frac{z_0\bar{z}_0}{a_0}} = \frac{z_1^{a_1}}{\frac{z_1\bar{z}_1}{a_1}} = \cdots = \frac{z_n^{a_n}}{\frac{z_n\bar{z}_n}{a_n}} = \frac{\sum z_\alpha^{a_\alpha}}{\sum\frac{z_\alpha\bar{z}_\alpha}{a_\alpha}} = 0$$

by (1.1). This implies that z is the origin of C^{n+1}, contrary to our assumption that $z \in D_n$. Hence we may consider the subdomain $D_{n,n-1}$ in D_n such that

(3.8) $$\bar{m}_{n-1} \neq 0\,.$$

Then, we see that

(3.9) $$d\bar{z}_{n-1} = -\frac{1}{\bar{m}_{n-1}}\left(\sum_{p=0}^{n-1} m_p dz_p + \sum_{k=0}^{n-2} \bar{m}_k d\bar{z}_k\right)$$

holds good on $D_{n,n-1}$.

Now, if we pay attention to the domain $D_{n,n-1}$ on Σ^{2n-1}, (3.2) can be written as

(3.10) $$\zeta \wedge (d\zeta)^{n-1} = \frac{(n-1)!\,i^n}{2}(A + B + C)$$

where A, B and C are $(2n-1)$-forms defined as follows:

A: the sum of monomials each of which contains $z_k d\bar{z}_k - \bar{z}_k dz_k$ ($k = 0, 1, \cdots, n-2$) as its factor,

B: the sum of monomials each of which contains $z_{n-1}d\bar{z}_{n-1} - \bar{z}_{n-1}dz_{n-1}$ as its factor, and

C: the sum of monomials each of which contains $z_n d\bar{z}_n - \bar{z}_n dz_n$ as its factor.

We shall calculate A, B and C on $D_{n,n-1}$. For the convenience of printing, we put

(3.11) $$\omega_\alpha = dz_\alpha \wedge d\bar{z}_\alpha\,.$$

(i) Calculation of A. If we fix the value of k, any non-zero monomial in (3.2) which contains $z_k d\bar{z}_k - \bar{z}_k dz_k$ does not contain $dz_k \wedge d\bar{z}^k$ as its factor. So A can be written as

$$A = A_1 + A_2 + A_3 , \tag{3.12}$$

where A_1, A_2 and A_3 are $(2n-1)$-forms with the following additional properties:

A_1: the sum of monomials each of which contains $dz_{n-1} \wedge d\bar{z}_{n-1}$ as its factor, but does not contain $dz_n \wedge d\bar{z}_n$ as its factor,

A_2: the sum of monomials each of which contains $dz_n \wedge d\bar{z}_n$ as its factor, but does not contain $dz_{n-1} \wedge d\bar{z}_{n-1}$ as its factor,

A_3: the sum of monomials each of which contains both of $dz_{n-1} \wedge d\bar{z}_{n-1}$ and $dz_n \wedge d\bar{z}_n$ as its factors.

First, we see easily that

$$A_1 = \sum_{k=0}^{n-2} \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge (z_k d\bar{z}_k - \bar{z}_k dz_k) \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1} .$$

Substituting (3.9) into the last equation, we get

$$A_1 = \frac{-1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2} (z_k m_k + \bar{z}_k \bar{m}_k) \Omega , \tag{3.13}$$

where we have put

$$\Omega = \omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \wedge dz_{n-1} . \tag{3.14}$$

Next, we see that

$$A_2 = \sum_{k=0}^{n-2} \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge (z_k d\bar{z}_k - \bar{z}_k dz_k) \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-2} \wedge \omega_n .$$

Substituting (3.3) and $(\overline{3.3})$ into the last equation we get

$$\begin{aligned} A_2 = \sum_{k=0}^{n-2} \{ & -z_k l_k \bar{l}_{n-1} \omega_0 \wedge \cdots \wedge \omega_{n-2} \wedge d\bar{z}_{n-1} \\ & + z_k l_{n-1} \bar{l}_{n-1} \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge d\bar{z}_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1} \\ & + \bar{z}_k l_{n-1} \bar{l}_{n-1} \omega_0 \wedge \cdots \wedge \omega_{n-2} \wedge dz_{n-1} \\ & - \bar{z}_k l_{n-1} \bar{l}_{n-1} \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge dz_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1} \} . \end{aligned}$$

By virtue of (3.9) this is transformed to

$$\begin{aligned} A_2 = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2} \{ & z_k \bar{l}_{n-1} (l_k m_{n-1} - l_{n-1} m_k) \\ & + \bar{z}_k l_{n-1} (\bar{l}_k \bar{m}_{n-1} - \bar{l}_{n-1} \bar{m}_k) \} \Omega . \end{aligned} \tag{3.15}$$

Thirdly, A_3 can be written as

$$A_3 = A_3' + A_3'' , \tag{3.16}$$

where we have put

$$A_3' = \sum_{k=0}^{n-2}\sum_{j=k+1}^{n-2} \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge (z_k d\bar{z}_k - \bar{z}_k dz_k)$$
$$\wedge\, \omega_{k+1} \wedge \cdots \wedge \hat{\omega}_j \wedge \cdots \wedge \omega_{n-2} \wedge \omega_{n-1} \wedge \omega_n\,,$$
$$A_3'' = \sum_{k=0}^{n-2}\sum_{h=0}^{k-1} \omega_0 \wedge \cdots \wedge \hat{\omega}_h \wedge \cdots \wedge \omega_{k-1}$$
$$\wedge\, (z_k d\bar{z}_k - \bar{z}_k dz_k) \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-2} \wedge \omega_{n-1} \wedge \omega_n\,.$$

Substituting (3.3) and $(\overline{3.3})$ into A_3' we get

$$\begin{aligned} A_3' = \sum_{k=0}^{n-2}\sum_{j=k+1}^{n-2} \{ & -z_k l_k \bar{l}_j \omega_0 \wedge \cdots \wedge \omega_{j-1} \wedge d\bar{z}_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_{n-1} \\ & + z_k l_j \bar{l}_j \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge d\bar{z}_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1} \\ & + \bar{z}_k l_j \bar{l}_k \omega_0 \wedge \cdots \wedge \omega_{j-1} \wedge dz_j \wedge \omega_{j+1} \wedge \cdots \wedge \omega_{n-1} \\ & - \bar{z}_k l_j \bar{l}_j \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge dz_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1}\}\,. \end{aligned}$$

By virtue of (3.9), the last equation is transformed to

$$A_3' = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2}\sum_{j=k+1}^{n-2} \{z_i \bar{l}_j (l_i m_j - l_j m_i) + \bar{z}_i l_j (\bar{l}_i \bar{m}_j - \bar{m}_i \bar{l}_j)\}\Omega\,. \tag{3.17}$$

In the same way A_3'' is transformed to

$$A_3'' = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2}\sum_{h=0}^{k-1} \{z_k \bar{l}_h (l_k m_h - l_h m_k) + \bar{z}_k l_h (\bar{l}_k \bar{m}_h - \bar{l}_h \bar{m}_k)\}\Omega\,.$$

However, this can be written also as

$$A_3'' = \frac{1}{\bar{m}_{n-1}} \sum_{h=0}^{n-2}\sum_{k=h+1}^{n-2} \{z_k \bar{l}_h (l_k m_h - l_h m_k) + \bar{z}_k l_h (\bar{l}_k \bar{m}_h - \bar{l}_h \bar{m}_k)\}\Omega\,.$$

Changing indices h and k to k and j respectively we have

$$A_3'' = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2}\sum_{j=k+1}^{n-2} \{z_j \bar{l}_k (l_j m_k - l_k m_j) + \bar{z}_j l_k (\bar{l}_j \bar{m}_k - \bar{l}_k \bar{m}_j)\}\Omega\,. \tag{3.18}$$

So, by (3.15) $\sim$ (3.17), we get

$$\begin{aligned} A_3 = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2}\sum_{j=k+1}^{n-2} \{ & (l_k m_j - l_j m_k)(z_k \bar{l}_j - z_j \bar{l}_k) \\ & + (\bar{l}_k \bar{m}_j - \bar{m}_k \bar{l}_j)(\bar{z}_k l_j - \bar{z}_j l_k)\}\Omega\,. \end{aligned} \tag{3.19}$$

(ii) Calculation of B. Clearly B can be written as

$$B = B_1 + B_2\,, \tag{3.20}$$

where B_1 and B_2 are $(2n-1)$-forms with the following additional properties:

B_1: the monomial which contains $z_{n-1} d\bar{z}_{n-1} - \bar{z}_{n-1} dz_{n-1}$ as its factor, but does not contain $dz_n \wedge d\bar{z}_n$ as its factor,

B_2: the sum of monomials each of which contains both of $z_{n-1}d\bar{z}_{n-1} - \bar{z}_{n-1}dz_{n-1}$ and $dz_n \wedge d\bar{z}_n$ as its factors.

First, we see that

$$B_1 = \omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_{n-2} \wedge (z_{n-1}d\bar{z}_{n-1} - \bar{z}_{n-1}dz_{n-1}) .$$

Substituting (3.9) in it, we get

$$B_1 = \frac{-1}{\bar{m}_{n-1}}(z_{n-1}m_{n-1} + \bar{z}_{n-1}\bar{m}_{n-1})\Omega . \tag{3.21}$$

Next, we see that

$$B_2 = \sum_{k=0}^{n-2} \omega_0 \wedge \cdots \wedge \hat{\omega}_k \wedge \cdots \wedge \omega_{n-2} \wedge (z_{n-1}d\bar{z}_{n-1} - \bar{z}_{n-1}dz_{n-1}) \wedge \omega_n .$$

Substituting (3.3) and $(\overline{3.3})$ into the last equation we have

$$\begin{aligned} B_2 = \sum_{k=0}^{n-2} \{ & z_{n-1}l_k\bar{l}_k\omega_0 \wedge \cdots \wedge \omega_{n-2} \wedge d\bar{z}_{n-1} \\ & - z_{n-1}\bar{l}_k l_{n-1}\omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge d\bar{z}_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1} \\ & - \bar{z}_{n-1}l_k\bar{l}_k\omega_0 \wedge \cdots \wedge \omega_{n-2} \wedge dz_{n-1} \\ & + \bar{z}_{n-1}l_k\bar{l}_{n-1}\omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge dz_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1}\} . \end{aligned}$$

By virtue of (3.9), this is transformed to

$$\begin{aligned} B_2 = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2} \{ & z_{n-1}\bar{l}_k(l_{n-1}m_k - l_k m_{n-1}) \\ & + \bar{z}_{n-1}l_k(\bar{l}_{n-1}\bar{m}_k - \bar{l}_k\bar{m}_{n-1})\}\Omega . \end{aligned} \tag{3.22}$$

(iii) Calculation of C. Clearly, C can be written as

$$C = C_1 + C_2 , \tag{3.23}$$

where C_1 and C_2 are $(2n-1)$-forms with the following additional properties:

C_1: the monomial which contains $z_n d\bar{z}_n - \bar{z}_n dz_n$ as its factor, but does not contain $dz_{n-1} \wedge d\bar{z}_{n-1}$ as its factor:

C_2: the sum of monomials each of which contains both of $z_n d\bar{z}_n - \bar{z}_n dz_n$ and $dz_{n-1} \wedge d\bar{z}_{n-1}$ as its factors.

First, we see that

$$C_1 = \omega_0 \wedge \cdots \wedge \omega_{n-2} \wedge (z_n d\bar{z}_n - \bar{z}_n dz_n) .$$

Substituting (3.3) and $(\overline{3.3})$ into the last equation, we have

$$\begin{aligned} C_1 = & -z_n\bar{l}_{n-1}\omega_0 \wedge \cdots \wedge \omega_{n-2} \wedge d\bar{z}_{n-1} \\ & + \bar{z}_n l_{n-1}\omega_0 \wedge \cdots \wedge \omega_{n-2} \wedge dz_{n-1} . \end{aligned}$$

By virtue of (3.9), this reduces to

$$C_1 = \frac{1}{\bar{m}_{n-1}}(z_n\bar{l}_{n-1}m_{n-1} + \bar{z}_n l_{n-1}\bar{m}_{n-1})\Omega . \tag{3.24}$$

Next, we see that

$$\begin{aligned} C_2 = \sum_{k=1}^{n-2} \omega_0 \wedge \cdots \wedge \hat{\omega}_k \wedge \cdots \wedge \omega_{n-2} \\ \wedge \omega_{n-1} \wedge (z_n d\bar{z}_n - \bar{z}_n dz_n) . \end{aligned}$$

Substituting (3.3) and $(\overline{3.3})$ into the last equation, we have

$$\begin{aligned} C_2 = \sum_{k=0}^{n-2} (z_n \bar{l}_k \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge d\bar{z}_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1} \\ - \bar{z}_n l_k \omega_0 \wedge \cdots \wedge \omega_{k-1} \wedge dz_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1}) . \end{aligned}$$

By virtue of (3.9), this is transformed to

$$C_2 = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2} (z_n \bar{l}_k m_k + \bar{z}_n l_k \bar{m}_k)\Omega . \tag{3.25}$$

Now, we define a function F on $D_{n,n-1}$ by

$$\zeta \wedge (d\zeta)^{n-1} = \frac{(n-1)!(i)^n}{2} F\Omega . \tag{3.26}$$

Then, by (3.10), (3.12), (3.20) and (3.23) we have

$$\begin{aligned} F\Omega &= A + B + C \\ &= (A_1 + B_1) + (C_1 + C_2) + \{A_3 + (A_2 + B_2)\} . \end{aligned} \tag{3.27}$$

To show (1.4) on $D_{n,n-1}$, it is sufficient to show that $F \neq 0$. By (3.13), (3.21), (3.24) and (3.25), we have

$$A_1 + B_1 = -\frac{1}{\bar{m}_{n-1}} \sum_{p=0}^{n-1} (z_p m_p + \bar{z}_p \bar{m}_p)\Omega , \tag{3.28}$$

$$C_1 + C_2 = \frac{1}{\bar{m}_{n-1}} \left(z_n \sum_{p=0}^{n-1} \bar{l}_p m_p + \bar{z}_n \sum_{p=0}^{n-1} l_p \bar{m}_p \right) \Omega . \tag{3.29}$$

Similarly, we have by (3.15) and (3.22)

$$\begin{aligned} A_2 + B_2 = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2} \{(l_k m_{n-1} - l_{n-1} m_k)(z_k \bar{l}_{n-1} - z_{n-1} \bar{l}_k) \\ + (\bar{l}_k \bar{m}_{n-1} - \bar{l}_{n-1} \bar{m}_k)(\bar{z}_k l_{n-1} - \bar{z}_{n-1} l_k)\}\Omega . \end{aligned}$$

So, we get by (3.19)

$$(3.30)\qquad A_3 + (A_2 + B_2) = \frac{1}{\bar{m}_{n-1}} \sum_{k=0}^{n-2} \sum_{j=k+1}^{n-1} \{(l_k m_j - l_j m_k)(z_k \bar{l}_j - z_j \bar{l}_k) + (\bar{l}_k \bar{m}_j - \bar{m}_k \bar{l}_j)(\bar{z}_k l_j - \bar{z}_j l_k)\}\Omega \,.$$

Putting (3.28) $\sim$ (3.30) into (3.27) and substituting m_p, $\bar{m}_p$ by (3.6), we get

$$\frac{1}{2}F = -\sum_{p=0}^{n-1} z_p \bar{z}_p + \sum_{p=0}^{n-1} z_p l_p \bar{z}_n + \sum_{p=0}^{n-1} \bar{z}_p \bar{l}_p z_n - \sum_{p=0}^{n-1} l_p \bar{l}_p z_n \bar{z}_n - \sum_{k=0}^{n-2} \sum_{j=k+1}^{n-1} (z_k \bar{l}_j - z_j \bar{l}_k)(\bar{z}_k l_j - \bar{z}_j l_k) \,.$$

By virtue of (3.4), this is transformed to

$$\begin{aligned} \frac{1}{2} t_n \bar{t}_n F &= -\sum_{p=0}^{n-1} | t_n \bar{z}_p |^2 - \sum_{p=0}^{n-1} | \bar{t}_p z_n |^2 + 2\sum_{p=0}^{n-1} \Re\mathrm{e}((t_n \bar{z}_p) \cdot (\bar{t}_p z_n)) \\ &\quad - \sum_{k=0}^{n-2} \sum_{j=k+1}^{n-1} | z_k \bar{t}_j - z_j \bar{t}_k |^2 \\ &= -\sum_{p=0}^{n-1} \{\Re\mathrm{e}(t_n \bar{z}_p) - \Re\mathrm{e}(\bar{t}_p z_n)\} - \sum_{p=0}^{n-1} \{\Im\mathrm{m}(t_n \bar{z}_p) \\ &\quad + \Im\mathrm{m}(\bar{t}_p z_n)\}^2 - \sum_{k=0}^{n-2} \sum_{j=k+1}^{n-1} | z_k \bar{t}_j - z_j \bar{t}_k |^2 \,. \end{aligned}$$

Thus, we see that $F \leqq 0$ on $D_{n,n-1}$.

We want to show that F does not vanish at any point on $D_{n,n-1}$ by reduction ad absurdum. For the purpose we assume that $F = 0$. Then, we have

$$\Re\mathrm{e}(t_n \bar{z}_p) = \Re\mathrm{e}(\bar{t}_p z_n) \,, \quad \Im\mathrm{m}(t_n \bar{z}_p) = -\Im\mathrm{m}(\bar{t}_p z_n)$$

for $p = 0, 1, \cdots, n-1$ and

$$z_k \bar{t}_j = z_j \bar{t}_k$$

for $k = 0, 1, \cdots, n-2$ and $j = k+1, \cdots, n-1$. As we can easily see, these relations are equivalent with the conjugate of (3.7). So, in the same way as the proof that $m_0, m_1, \cdots, m_{n-1}$ do not vanish simultaneously, we arrive at a contradiction. Therefore, $F < 0$ and so (1.4) holds on $D_{n,n-1}$.

Quite the same argument can be performed for other domains $D_{n,k} (k = 0, 1, \cdots, n-2)$ similarly defined as $D_{n,n-1}$. So, (1.4) holds on D_n.

In the same way, we can show that (1.4) holds for domains $D_0, D_1, \cdots, D_{n-1}$ on Σ^{2n-1} similary defined as D_n. Consequently, we can conclude that (1.4) holds over the whole Σ^{2n-1}. This completes the proof.

N.B. It will be an interesting problem to study whether odd dimensional homotopy spheres which are not boundaries of compact orientable parallelisable manifolds are contact manifolds or not.

4. A characterization of Brieskorn manifolds with $a_0 = a_1 = \cdots = a_n$. The almost contact structure (ϕ, ξ, η) on $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ introduced by K. Abe has the property that $\xi = u_2$. Making use of the fact that the vector field u_2 generates a 1-dimensional foliation each of whose orbits is a closed curve, he proved that his almost contact structure (the foliation) is in general non-regular.

On the other hand, we can introduce naturally an almost contact structure (ϕ', ξ', η') on the same Brieskorn manifold as follows:

$$\phi' X = JX - \langle JX, n_1 \rangle n_1 \, ,$$
$$\xi' = Jn_1 \, , \qquad \eta'(X) = \langle \xi', X \rangle \, ,$$

where J is the complex structure of the Brieskorn variety B^{2n}, X is an arbitrary tangent vector of Σ^{2n-1} and $n_1 = v/\langle v, v \rangle$. Thus, we have interest to study the condition under which two foliations generated by the vector fields ξ and ξ' coincide.

THEOREM. *The two vector fields ξ and ξ' generate the same 1-dimensional foliation in $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ if and only if $a_0 = a_1 = \cdots = a_n$.*

PROOF. The two foliations coincide if and only if the vector fields iv and u_2 on Σ^{2n-1} are linearly dependent at each point of Σ^{2n-1} and so they coincide if and only if the vector field u_1 is normal to Σ^{2n-1}. Thus, the condition for the coincidence is that

$$\Re e \langle u_1, X \rangle = 0$$

is satisfied for any X which satisfies

$$\left\langle \overline{\frac{\partial f}{\partial z}}, X \right\rangle = 0 \, , \qquad \Re e \langle z, X \rangle = 0 \, .$$

Considering a special point $z' = (z_0, z_1, 0, \cdots, 0)$, and X such that $X_0 \neq 0$, we can easily deduce from these equations that $a_0 = a_1$. In the same way, we get $a_i = a_j$ $(i \neq j)$ $i = 0, 1, \cdots, n$. q.e.d.

N.B. 1. As a corollary of the last theorem, we can see that the two almost contact structures (ϕ, ξ, η) and (ϕ', ξ', η') defined on the same Brieskorn manifold $\Sigma^{2n-1}(a_0, a_1, \cdots, a_n)$ coincide if and only if $a_0 = a_1 = \cdots = a_n$.

N.B. 2. Brieskorn manifold Σ^{2n-1} with $a_0 = a_1 = \cdots = a_n$ is a principal circle bundle over the $(2n-2)$-dimensional manifold (1.1) in CP^n and

(ϕ', ξ', η') with the induced Riemannian metric g' from C^{n+1} is a normal contact metric structure.

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