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A CONVOLUTION MEASURE ALGEBRA ON THE UNIT DISC

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1. Introduction. Let D be the unit disc $D=\{z=x+iy; x^2+y^2\leqq 1\}$ and m_α be the positive measure of total mass one on D defined by

$$dm_\alpha(z)=\frac{\alpha+1}{\pi}(1-x^2-y^2)^\alpha dxdy\ ,$$

where α is a positive real number. Let $M(D)$ be the space of all bounded regular complex valued Borel measures on D. $M(D)$ is a Banach space with the total variation norm $\|\mu\|=\int_D d|\mu|(z)$ for $\mu\in M(D)$. Denote $L^1_\alpha=L^1(D, m_\alpha)$. Then L^1_α is identified with a subspace of $M(D)$ by the map $f\mapsto fdm_\alpha$ of L^1_α to $M(D)$. The mapping is isometric, since $\|f\|_{L^1_\alpha}=\int_D|f(z)|\,dm_\alpha(x)=\|fdm_\alpha\|$.

For each point z in D, the operator T_z, called generalized translation, is defined by

$$(1)\qquad T_zf(\zeta)=\frac{\alpha}{\alpha+1}\int_D f\Big(\bar{z}\zeta+\sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}\,\xi\Big)\frac{dm_\alpha(\xi)}{1-|\xi|^2}\ ,$$

for f in the space of all continuous functions $C(D)$. By a change of variable, if z and ζ are in the interior of D, we obtain

$$T_zf(\zeta)=\int_D f(\xi)E_\alpha(z,\zeta,\xi)dm_\alpha(\xi)\ ,$$

where

$$E_\alpha(z,\zeta,\xi)=\begin{cases}\dfrac{\alpha}{\alpha+1}\dfrac{(1-|z|^2-|\zeta|^2-|\xi|^2+2\Re(\bar{z}\zeta\bar{\xi}))^{\alpha-1}}{(1-|z|^2)^\alpha(1-|\zeta|^2)^\alpha(1-|\xi|^2)^\alpha}\ ,\\ 0.\end{cases}$$

The first value is assigned only if ξ is in the disc of the center $\bar{z}\zeta$ and of radius $\sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}$. By the definition,

$$(2)\qquad E_\alpha(z,\zeta,\xi)\geqq 0\ ,\quad z,\zeta\in\text{interior of } D,\ \xi\in D\ ,$$

$$(3)\qquad \int_D E_\alpha(z,\zeta,\xi)dm_\alpha(\xi)=1\ .$$

If μ and ν are in $M(D)$, $\mu \underset{\alpha}{*} \nu$ is defined implicitly by the relation

$$\int_D f(t) d(\mu \underset{\alpha}{*} \nu)(t) = \int_D \int_D T_z f(\zeta) d\mu(z) d\nu(\zeta) \qquad (f \in C(D)).$$

In the following, we will leave off the index α when there occurs no confusion.

$\sup\{|T_z f(\zeta)|; z \in D, \zeta \in D\} \leqq \|f\|_{C(D)}$ by the definition (1) of T_z. Therefore, if $\mu, \nu \in M(D)$, then $\mu * \nu \in M(D)$ and $\|\mu * \nu\| \leqq \|\mu\| \cdot \|\nu\|$ by the Riesz representation theorem. The convolution $*$ is commutative and associative. Let δ_1 be the measure with the unit mass at the point 1, then it is the unit with respect to the convolution $*$.

$M(D)$ with the convolution $\underset{\alpha}{*}$ will be denoted by $M_\alpha(D)$. $M_\alpha(D)$ is a commutative Banach algebra with a unit. If $f \in L^1_\alpha$ and $g \in L^1_\alpha$, $f \underset{\alpha}{*} g$ will be defined by $(f dm_\alpha) \underset{\alpha}{*} (g dm_\alpha)$. Then, we obtain

$$\begin{aligned} f \underset{\alpha}{*} g(\zeta) &= \int_D T_z f(\zeta) g(z) dm_\alpha(z) \\ &= \int_D \int_D f(\xi) g(z) E_\alpha(z, \zeta, \xi) dm_\alpha(\xi) dm_\alpha(z) . \end{aligned}$$

By (2) and (3), $f \underset{\alpha}{*} g$ is in L^1_α. In fact, L^1_α is a closed ideal in $M_\alpha(D)$.

If α is a positive integer, this convolution $\underset{\alpha}{*}$ corresponds to the convolution of the zonal measure algebra on the unitary group $U(\alpha + 2)$.

The object of this paper is to determine the maximal ideal space of the Banach algebra $M_\alpha(D)$ and using it, to give a characterization of idempotent measures and to show a theorem of F. and M. Riesz type. To prove the last one, we will define a Poisson kernel and give an integral representation of it.

2. Idempotent measures and maximal ideal space of $M_\alpha(D)$. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n, order (α, β), $\alpha, \beta > -1$ defined by

$$(1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1-x)^{n+\alpha}(1+x)^{n+\beta}] ,$$

or

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)} F[-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2] ,$$

where

$$F[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!\,(c)_n} z^n ,$$

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1) , \quad (a)_0 = 1 .$$

The following will be used later (see G. Szegö [10]),

$$(4)\qquad P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n},\quad P_n^{(\alpha,\beta)}(-1)=(-1)^n P_n^{(\beta,\alpha)}(1),$$

$$(5)\qquad \int_{-1}^{1}\{P_n^{(\alpha,\beta)}(x)\}^2(1-x)^\alpha(1+x)^\beta dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

Define $R_n^{(\alpha,\beta)}(x)=P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$. Let $R_{m,n}^{(\alpha)}$ be the polynomial of degree $m+n$ in x and y defined by

$$R_{m,n}^{(\alpha)}(re^{i\theta})=r^{|m-n|}e^{i(m-n)\theta}R_{m\wedge n}^{(\alpha,|m-n|)}(2r^2-1),$$

where $re^{i\theta}=x+iy$ and $m\wedge n=\min\{m,n\}$. From the orthogonality of Jacobi polynomials, it follows that the system $\{R_{m,n}^{(\alpha)}\}_{m,n=0}^{\infty}$ constitutes an orthogonal system in $L^2(D, m_\alpha)$. Since polynomials of $R_{m,n}^{(\alpha)}$ are dense in $C(D)$, the system is complete. From the product formula for Jacobi polynomials (see T. Koornwinder [6]), it follows that

$$(6)\qquad T_z R_{m,n}^{(\alpha)}(\zeta)=\frac{\alpha}{\alpha+1}\int_D R_{m,n}^{(\alpha)}\left(\bar z\zeta+\sqrt{1-|z|^2}\sqrt{1-|\zeta|^2}\,\xi\right)\frac{dm_\alpha(\xi)}{1-|\xi|^2} = R_{m,n}^{(\alpha)}(\bar z)R_{m,n}^{(\alpha)}(\zeta).$$

For $\mu\in M_\alpha(D)$, let $\hat\mu(m,n)$ be the coefficient defined by

$$\hat\mu(m,n)=\int_D R_{m,n}^{(\alpha)}(\bar z)d\mu(z).$$

In particular, if $f\in L_\alpha^1$,

$$\hat f(m,n)=\int_D f(z)R_{m,n}^{(\alpha)}(\bar z)dm_\alpha(z).$$

By (6), if $\mu\in M_\alpha(D)$ and $\nu\in M_\alpha(D)$, it follows that

$$(a\mu+b\nu)^\wedge(m,n)=a\hat\mu(m,n)+b\hat\nu(m,n)\quad(a,b\in C),$$
$$(\mu*\nu)^\wedge(m,n)=\hat\mu(m,n)\hat\nu(m,n).$$

Therefore the map $\mu\mapsto\hat\mu(m,n)$ gives a nonzero multiplicative linear functional on $M_\alpha(D)$.

Define $h_{m,n}^{(\alpha)}=\left[\int_D|R_{m,n}^{(\alpha)}(z)|^2dm_\alpha(z)\right]^{-1}$. Then, by (5),

$$(7)\qquad h_{m,n}^{(\alpha)}=\frac{1}{(\alpha+1)\Gamma(\alpha+1)^2}\times\frac{\Gamma(m\wedge n+\alpha+1)\Gamma(m\wedge n+\alpha+|m-n|+1)}{\Gamma(m\wedge n+1)\Gamma(m\wedge n+|m-n|+1)}\times(2m\wedge n+\alpha+|m-n|+1),$$

and every $\mu \in M_\alpha(D)$ is expanded in the formal series

$$\mu \sim \sum_{m,n} h_{m,n}\hat{\mu}(m, n)R_{m,n}^{(\alpha)}(z) .$$

H. Annabi and K. Trimèche proved the following.

THEOREM 1 ([1]). *For every couple* (m, n) *of nonnegative integers, the map* $f \mapsto \hat{f}(m, n)$ *is a nonzero multiplicative linear functional on the Banach algebra* L_α^1. *Conversely, if* χ *is a nonzero multiplicative linear functional, then there exists a couple* (m, n) *of nonnegative integers such that* $\chi(f) = \hat{f}(m, n)$ $(f \in L_\alpha^1)$.

Now we can describe the maximal ideal space of the Banach algebra $M_\alpha(D)$. Let

$$M_\alpha(D^0) = \{\mu \in M_\alpha(D);\ \mu \text{ is concentrated on } D^0\} ,$$
$$M_\alpha(\partial D) = \{\mu \in M_\alpha(D);\ \mu \text{ is concentrated on } \partial D\} ,$$

where D^0 is the interior of D and ∂D is the boundary of D. Then we obtain a decomposition of $M_\alpha(D)$ into $M_\alpha(D) = M_\alpha(D^0) \oplus M_\alpha(\partial D)$. By the definition of the convolution, it follows that $M_\alpha(D^0)$ is a closed ideal in $M_\alpha(D)$ and $M_\alpha(\partial D)$ is a subalgebra of $M_\alpha(D)$. Therefore if we denote by $\Delta(M_\alpha(D))$ the maximal ideal space of $M_\alpha(D)$, it is the disjoint union

$$\Delta(M_\alpha(D)) = \Delta(M_\alpha(D^0)) \cup \Delta(M_\alpha(\partial D)) .$$

Let $M(\boldsymbol{T})$ be the space of all bounded regular Borel measures on the circle group $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$. Then $M_\alpha(\partial D) = M(\boldsymbol{T})$ as a set. Since for $\mu \in M_\alpha(\partial D)$ and $\nu \in M_\alpha(\partial D)$,

$$\int_D f(t)d\mu \underset{\alpha}{*} \nu(t) = \int_{\partial D}\int_{\partial D} f(z\zeta)d\mu(z)d\nu(z) \quad (f \in C(D)) ,$$

the convolution $\underset{\alpha}{*}$ coincides with the convolution on the circle group $\boldsymbol{T}$ for all $\alpha > 0$. So that $M_\alpha(\partial D)$ is identified with the convolution measure algebra $M(\boldsymbol{T})$ as a Banach algebra. Moreover, for $\mu \in M_\alpha(\partial D)$, $\hat{\mu}(m, n) = \hat{\mu}(m - n)$ where the righthand side is the Fourier-Stieltjes transform of μ which is regarded as an element in $M(\boldsymbol{T})$.

The maximal ideal spaces of measure algebras on locally compact abelian groups are studied in detail by Yu. A. Šreider [9], J. L. Taylor [11] and etc.

Nothing remains but to determine the maximal ideal space $\Delta(M_\alpha(D^0))$ of the Banach algebra $M_\alpha(D^0)$. Because of the special nature of the convolution in $M_\alpha(D)$, we can relate the maximal ideal space of $M_\alpha(D^0)$ to that of L_α^1. The following lemma is the key to this relation.

LEMMA 2. *Let* $\alpha > 0$. *If* μ *and* ν *are in* $M_\alpha(D^0)$, *then* $\mu \underset{\alpha}{*} \nu \in L_\alpha^1$.

PROOF. Let $\mu, \nu \in M_\alpha(D^0)$. Then,

$$\int_D f(t) d\mu \underset{\alpha}{*} \nu(t) = \int_D \int_D T_{\bar{z}} f(\zeta) d\mu(z) d\nu(\zeta)$$
$$= \int_D \int_D \int_D f(\xi) E_\alpha(\bar{z}, \zeta, \xi)\, dm_\alpha(\xi) d\mu(z) d\nu(\xi) ,$$

for $f \in C(D)$. Let F be a Borel set such that $m_\alpha(F) = 0$. By the regulality of measures, we can replace f with the characteristic function of F. For any z and ζ in D^0, $E_\alpha(\bar{z}, \zeta, \cdot)$ is absolutely continuous with respect to m_α, and so $\mu \underset{\alpha}{*} \nu(F) = 0$. Thus $\mu \underset{\alpha}{*} \nu$ is absolutely continuous with respect to m_α.

THEOREM 3. *Let $\alpha > 0$. Then $\Delta(M_\alpha(D))$ can be identified with the disjoint union $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+ \cup \Delta(M(\boldsymbol{T}))$, where $\boldsymbol{Z}^+$ denotes the set of nonnegative integers.*

PROOF. From the above arguments, it suffices to prove that $\Delta(M_\alpha(D^0))$ can be identified with $\boldsymbol{Z}^+ \times \boldsymbol{Z}^+$. Let χ be a nonzero multiplicative linear functional on $M_\alpha(D^0)$. Then there exists μ in $M_\alpha(D^0)$ such that $\chi(\mu) \neq 0$. $\chi(\mu * \mu) = \chi(\mu)^2 \neq 0$. For any $\nu \in M_\alpha(D^0)$, $\nu * (\mu * \mu) \in L^1_\alpha$ and $\mu * \mu \in L^1_\alpha$ by Lemma 2. By Theorem 1, there exists a couple (m, n) of nonnegative integers such that

$$\chi(\nu * (\mu * \mu)) = (\nu * (\mu * \mu))^\wedge(m, n)$$

and

$$\chi(\mu * \mu) = (\mu * \mu)^\wedge(m, n) .$$

Thus

$$\begin{aligned} \chi(\nu)(\mu * \mu)^\wedge(m, n) &= \chi(\nu)\chi(\mu * \mu) \\ &= \chi(\nu * (\mu * \mu)) \\ &= (\nu * (\mu * \mu))^\wedge(m, n) \\ &= \hat{\nu}(m, n) \cdot (\mu * \mu)^\wedge(m, n) . \end{aligned}$$

Thus $\chi(\nu) = \hat{\nu}(m, n)$ which proves the theorem.

For $\mu \in M_\alpha(D)$, if $\mu \underset{\alpha}{*} \mu = \mu$, it is called an idempotent measure in $M_\alpha(D)$.

H. Helson [5] has given a characterization of the idempotent measures in $M(\boldsymbol{T})$ and P. J. Cohen [3] has obtained a characterization of the idempotent measures in the convolution measure algebra on a locally compact abelian group. We will show that the idempotent measures in $M_\alpha(D)$ are essentially those in $M(\boldsymbol{T})$.

THEOREM 4. *If μ is an idempotent measure in $M_\alpha(D)$, then μ has the form*

$$\mu = \mu_0 + \mu_1 ,$$

where μ_0 is an idempotent measure in $M(\boldsymbol{T})$ and μ_1 is a finite sum $\sum_{m,n} h_{m,n} a_{m,n} R_{m,n}^{(\alpha)}(z)$ with $a_{m,n} = 0$ or ± 1.

PROOF. Let μ be an idempotent measure in $M_\alpha(D)$. Then μ is decomposed as $\mu = \mu_0 + \mu_1$ where $\mu_0 \in M_\alpha(\partial D)$ and $\mu_1 \in M_\alpha(D^0)$. The decomposition is unique. By the convolution equation $\mu * \mu = \mu$,

$$\mu_0 + \mu_1 = \mu_0 * \mu_0 + 2\mu_0 * \mu_1 + \mu_1 * \mu_1 .$$

Since $M_\alpha(\partial D)$ is a subalgebra and $M_\alpha(D^0)$ is an ideal in $M_\alpha(D)$, $\mu_0 = \mu_0 * \mu_0$. That is if μ is idempotent in $M_\alpha(D)$, so is μ_0 in $M_\alpha(\partial D)$, i.e., in $M(\boldsymbol{T})$. Since $\mu = \mu_0 + \mu_1$ and μ_0 is itself idempotent, $\hat{\mu}_1(m, n)$ takes values 0, 1, or -1. It is clear that for $f \in L_\alpha^1$, $\hat{f}(m, n) \to 0$ as $m + n \to \infty$. By Lemma 2, $\mu_1 * \mu_1 \in L_\alpha^1$ and so $(\mu_1 * \mu_1)\hat{}(m, n) \to 0$ as $m + n \to \infty$. That is $\hat{\mu}_1(m, n) \to 0$ as $m + n \to \infty$. From this it follows that all of $\hat{\mu}_1(m, n)$ vanish except a finite number of (m, n). Therefore μ must have the form described in the theorem. The proof is complete.

Related results to Theorems 3 and 4 will be found in C. F. Dunkl [4], D. L. Ragozin [7] and A. Schwartz [8]. They are concerned with the special orthogonal group $SO(n)$ and radial measures on $\boldsymbol{R}^n$, etc.

3. The Poisson kernel. In this section, a Poisson kernel on $D \times [0, 1)$ is defined which possesses the same good properties as the usual Poisson kernel on the unit disc.

DEFINITION. We call the series

$$\boldsymbol{P}_s^{(\alpha)}(z) = \sum_{m,n} s^{|m-n|+m\wedge n} h_{m,n} R_{m,n}^{(\alpha)}(z) ,$$

Poisson kernel for polynomials $R_{m,n}^{(\alpha)}$ of index $\alpha > 0$, where $0 \leqq s < 1$ and $z \in D$.

For $0 \leqq s < 1$, the series in the right hand side converges uniformly in D by (7) and the inequality $|R_{m,n}^{(\alpha)}(z)| \leqq 1$ $(z \in D)$.

THEOREM 5. *Let $0 < |z| \leqq 1$, $0 \leqq s < 1$. Then the Poisson kernel has integral representation*

$$\boldsymbol{P}_s^{(\alpha)}(z) = \frac{1 - s}{\pi(1 + s)^{\alpha+2}} \int_0^{2\pi} P_{\sqrt{s}}(\theta - \tau)\left(1 - \frac{r}{k} \cos \tau\right)^{-\alpha-2} d\tau ,$$

where $z = re^{i\theta}$, $k = (s^{1/2} + s^{-1/2})/2$ and $P_r(x)$ is the Poisson kernel for the *trigonometric polynomials, i.e.,* $P_r(x) = 1/2 + \sum_{n=1}^{\infty} r^n \cos nx$. *In particular, we have*

$$\boldsymbol{P}_s^{(\alpha)}(z) \geqq 0 \quad (z \in D) ,$$

$$\int_D \boldsymbol{P}_s^{(\alpha)}(z) dm_\alpha(z) = 1 ,$$

and

$$\boldsymbol{P}_r^{(\alpha)} * \boldsymbol{P}_s^{(\alpha)} = \boldsymbol{P}_{rs}^{(\alpha)} .$$

Most of this section is devoted to proving the first part of the theorem.

Let $z = re^{i\theta}$. Then

$$\begin{aligned}\boldsymbol{P}_s^{(\alpha)}(z) &= \sum_{m,n} s^{|m-n|+m\wedge n} h_{m,n} R_{m,n}^{(\alpha)}(z) \\ &= 2\Re\mathrm{e}\left\{\frac{1}{2}\sum_{n=0}^{\infty} h_{n,n} s^n R_n^{(\alpha,0)}(2r^2-1)\right. \\ &\quad \left. + \sum_{\beta=1}^{\infty}\left(\sum_{n=0}^{\infty} h_{n+\beta,n} s^n R_n^{(\alpha,\beta)}(2r^2-1)\right) s^\beta z^\beta\right\} .\end{aligned}$$

From (4) and (7), for $\beta \geqq 0$,

$$\begin{aligned}&h_{n+\beta,n} R_n^{(\alpha,\beta)}(2r^2-1) \\ &= \frac{1}{\Gamma(\alpha+2)} \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)} (2n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(2r^2-1) .\end{aligned}$$

Thus it follows that

$$\begin{aligned}\boldsymbol{P}_s^{(\alpha)}(z) = \frac{2}{\Gamma(\alpha+2)} \Re\mathrm{e}&\left\{\frac{1}{2}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\right. \\ &\times (2n+\alpha+1) P_n^{(\alpha,0)}(2r^2-1) s^n \\ &+ \sum_{\beta=1}^{\infty}\left(\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)}\right. \\ &\left.\left. \times (2n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(2r^2-1) s^n\right) s^\beta z^\beta\right\} .\end{aligned} \tag{8}$$

Put

$$A(\beta) = \sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)} (2n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(2r^2-1) s^n .$$

It is easy to see that

$$\begin{aligned}A(\beta) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} &\sum_{n=0}^{\infty}\frac{n!(\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+1)_n} \\ &\times (2n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(1) P_n^{(\alpha,\beta)}(2r^2-1) s^n .\end{aligned}$$

We have

$$(9)\qquad A(\beta)=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\left\{\frac{(\alpha+\beta+1)(1-s)}{(1+s)^{\alpha+\beta+2}}\right.$$
$$\left.\times F_4\left[\frac{1}{2}(\alpha+\beta+2),\frac{1}{2}(\alpha+\beta+3);\alpha+1,\beta+1;0,\frac{r^2}{k^2}\right]\right\},$$

where $k=(s^{1/2}+s^{-1/2})/2$ (see Bailey [2] p. 102). F_4 is Appell's hypergeometric function of two variables defined by

$$F_4[\alpha,\beta;\gamma,\gamma';x,y]=\sum\sum\frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!n!(\gamma)_m(\gamma')_n}x^m y^n .$$

By the definition of F_4, we have

$$(10)\qquad F_4\left[\frac{1}{2}(\alpha+\beta+2),\frac{1}{2}(\alpha+\beta+3);\alpha+1,\beta+1;0,\frac{r^2}{k^2}\right]$$
$$=\sum_{n=0}^{\infty}\frac{((\alpha+\beta+2)/2)_n((\alpha+\beta+3)/2)_n}{n!(\beta+1)_n}\left(\frac{r^2}{k^2}\right)^n$$

and further

$$(11)\qquad \frac{((\alpha+\beta+2)/2)_n((\alpha+\beta+3)/2)_n}{n!(\beta+1)_n}=\frac{\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+2)}{2^{2n}n!\Gamma(\alpha+\beta+2)\Gamma(n+\beta+1)} .$$

Combining (9), (10) and (11) we get

$$A(\beta)=\frac{1-s}{(1+s)^{\alpha+\beta+2}}\sum_{n=0}^{\infty}\frac{\Gamma(2n+\alpha+\beta+2)}{2^{2n}n!\Gamma(n+\beta+1)}\left(\frac{r^2}{k^2}\right)^n .$$

Now we rewrite the series in the righthand side using the function $I_\nu(\zeta)$ introduced by Bessel which is defined by

$$(12)\qquad I_\nu(\zeta)=\left(\frac{\zeta}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{(\zeta/2)^{2n}}{n!\Gamma(\nu+n+1)},\quad \zeta\neq\text{negative real number}.$$

$I_\nu(\zeta)$ has the integral representation

$$(13)\qquad I_\nu(\zeta)=\frac{1}{\pi}\int_0^\pi e^{\zeta\cos\tau}\cos\nu\tau d\tau-\frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-\zeta\cosh u-\nu u}du,$$
$$\Re\nu>-\frac{1}{2},\qquad \Re\zeta>0 .$$

From definition of Γ-function and (12), it follows that

$$(14)\qquad \sum_{n=0}^{\infty}\frac{\Gamma(2n+\alpha+\beta+2)}{2^{2n}n!\Gamma(n+\beta+1)}\left(\frac{r^2}{k^2}\right)^n=\int_0^\infty\left(\frac{2k}{r}\right)^\beta I_\beta\left(\frac{r}{k}t\right)t^{\alpha+1}e^{-t}dt .$$

By (13),

$$I_\beta\left(\frac{r}{k}t\right) = \frac{1}{\pi}\int_0^\pi e^{t(r/k)\cos\tau}\cos\beta\tau d\tau\ , \tag{15}$$

for $t, r > 0$ and $\beta = 0, 1, 2, \cdots$. From (14) and (15), it follows that

$$A(\beta) = \frac{1-s}{\pi(1+s)^{\alpha+2}}\left\{\frac{2k}{(1+s)r}\right\}^\beta \int_0^\infty\int_0^\pi e^{t(r/k)\cos\tau}t^{\alpha+1}e^{-t}\cos\beta\tau d\tau dt\ . \tag{16}$$

Combining (8) and (16) we get

$$\begin{aligned} \boldsymbol{P}_s^{(\alpha)}(z) &= \frac{1-s}{\pi\Gamma(\alpha+2)(1+s)^{\alpha+2}}\int_0^\infty\int_0^\pi 2\Re\left[\frac{1}{2}\right. \\ &\quad \left. + \sum_{\beta=1}^\infty \left\{\frac{2skz}{(1+s)r}\right\}^\beta \cos\beta\tau\right] e^{t(r/k)\cos\tau}t^{\alpha+1}e^{-t}d\tau dt\ . \end{aligned}$$

But,

$$\begin{aligned} &2\Re\left[\frac{1}{2} + \sum_{\beta=1}^\infty\left\{\frac{2skz}{(1+s)r}\right\}^\beta\cos\beta\tau\right] \\ &= 1 + 2\sum_{\beta=1}^\infty s^{\beta/2}\cos\beta\theta\cos\beta\tau \\ &= 1 + \sum_{\beta=1}^\infty s^{\beta/2}(\cos\beta(\theta+\tau) + \cos\beta(\theta-\tau)) \\ &= P_{\sqrt{s}}(\theta+\tau) + P_{\sqrt{s}}(\theta-\tau)\ , \end{aligned}$$

and so by a change of variable it is clear that $\boldsymbol{P}_s^{(\alpha)}(z)$ has the integral representation described in the Theorem 5. The proof of the Theorem 5 is complete.

COROLLARY 6. *If $f \in L^p(D, m_\alpha)$, $p \geqq 1$, then the Poisson integral $\boldsymbol{P}_s^{(\alpha)} * f$ converges to f in the norm.*

In fact, if f is a polynomial of $R_{m,n}^{(\alpha)}$, it is obvious. Since polynomials of $R_{m,n}^{(\alpha)}$ is dense in $C(D)$, the Corollary holds for any $f \in L^p(D, m_\alpha)$, $p \geqq 1$.

4. A theorem of F. and M. Riesz type. In this section, we will give a theorem of F. and M. Riesz type using Theorem 5.

Let $\mu \in M_\alpha(D)$. Then

$$\mu \sim \sum_{n=0}^\infty\left\{\sum_{\beta=0}^\infty h_{n+\beta,n}\hat{\mu}(n+\beta, n)R_{n+\beta,n}^{(\alpha)}(z) + \sum_{\beta=1}^\infty h_{n,n+\beta}\hat{\mu}(n, n+\beta)R_{n,n+\beta}^{(\alpha)}(z)\right\}\ .$$

From (4) and (7),

$$h_{n+\beta,n} = O(n^{2\alpha+1} + n^\alpha\beta^{\alpha+1}) \text{ as } n \to \infty \text{ or } \beta \to \infty\ ,$$

and

$$|R_{n+\beta,n}^{(\alpha)}(z)| \leqq C\frac{\Gamma(n+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(\beta+1)}r^{\beta},$$

where the constant C depends only on α. Therefore we have

$$h_{n+\beta,n}|R_{n+\beta,n}^{(\alpha)}(z)| = O(\beta^{n+\alpha+1}r^{\beta}) \quad \text{as} \quad \beta \to \infty .$$

Since $h_{n+\beta,n}|R_{n+\beta,n}^{(\alpha)}(z)| = h_{n,n+\beta}|R_{n,n+\beta}^{(\alpha)}(z)|$, both series

$$\sum_{\beta=0}^{\infty} h_{n+\beta,n}\hat{\mu}(n+\beta, n)R_{n+\beta,n}^{(\alpha)}(z) \quad \text{and} \quad \sum_{\beta=1}^{\infty} h_{n,n+\beta}\hat{\mu}(n, n+\beta)R_{n,n+\beta}^{(\alpha)}(z)$$

converge uniformly in the wide sense on the interior of D for $n = 0, 1, 2, \cdots$.

THEOREM 7. *Let $\alpha > 0$ and μ be an element in $M_{\alpha}(D)$. Suppose there exists an integer N such that*

$$\hat{\mu}(m, n) = 0 \quad \textit{for all} \quad m \wedge n > N . \tag{17}$$

Then μ is absolutely continuous with respect to m_{α}, that is, in L_{α}^{1}.

PROOF. Suppose that μ is an element in $M_{\alpha}(D)$ satisfying (17). Then we have

$$\mu \sim \sum_{n=0}^{N} \left\{\sum_{\beta=0}^{\infty} h_{n+\beta,n}\hat{\mu}(n+\beta, n)R_{n+\beta,n}^{(\alpha)}(z) + \sum_{\beta=1}^{\infty} h_{n,n+\beta}\hat{\mu}(n, n+\beta)R_{n,n+\beta}^{(\alpha)}(z)\right\} .$$

Therefore there exists a continuous function $f(z)$ such that

$$\begin{aligned} (18) \quad & f(z) \\ & = \sum_{n=0}^{N} \left\{\sum_{\beta=0}^{\infty} h_{n+\beta,n}\hat{\mu}(n+\beta, n)R_{n+\beta,n}^{(\alpha)}(z) + \sum_{\beta=1}^{\infty} h_{n,n+\beta}\hat{\mu}(n, n+\beta)R_{n,n+\beta}^{(\alpha)}(z)\right\} \end{aligned}$$

on the interior of D. By Fatou's lemma, we get

$$\begin{aligned} \int_D |f(z)|\, dm_{\alpha} &= \int_D \liminf_{s\to 1} |\boldsymbol{P}_s^{(\alpha)} * \mu(z)|\, dm_{\alpha}(z) \\ &\leqq \liminf_{s\to 1} \int_D |\boldsymbol{P}_s^{(\alpha)} * \mu(z)|\, dm_{\alpha}(z) , \end{aligned}$$

and by Theorem 5, $\|\boldsymbol{P}_s^{(\alpha)} * \mu\| \leqq \|\mu\|$. Therefore we have

$$\int_D |f(z)|\, dm_{\alpha}(z) \leqq \|\mu\| .$$

It is clear that the coefficients of f coincide with those of μ since the series (18) converges uniformly in the wide sence on D^{0} and the system $\{R_{m,n}^{(\alpha)}\}$ is orthogonal. Therefore, we get $f = \mu$ which completes the proof.

REMARK. If μ is an analytic measure on $\boldsymbol{T}$, then $\hat{\mu}(m, n) = 0$ for $m < n$ and μ is singular with respect to m_{α}. So that our formulation will be natural in a sence.

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Added in proof, 28 January 1976: We have learned after submitting this paper that G. B. Folland gives a spherical harmonic expansion of the Poisson-Szegö kernel for the ball, Proc. Amer. Math. Soc. 47 (1975). One would obtain Theorem 7 using his expansion formula.

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