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ON THE MEAN CURVATURE FOR ANTI-HOLOMORPHIC p-PLANE IN KÄHLERIAN SPACES

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Introduction. Let M^n be an n dimensional Riemannian spaces, and denote by $\rho(X, Y)$ the sectional curvature of a 2-plane spanned by vectors X and Y. For a q-plane π at a point P, we take an orthonormal base $\{e_\lambda\}$ of tangent space $T_P(M)$ such that $e_1, \cdots, e_q$ span π. Such a base is called an adapted base for π. S. Tachibana [1][1)] has defined the mean curvature $\rho(\pi)$ for π by

$$\rho(\pi) = \frac{1}{q(n-q)} \sum_{a=q+1}^{n} \sum_{i=1}^{q} \rho(e_i, e_a) ,$$

which is well-defined, i.e., independent of the choice of adapted bases for π. He has obtained the following.

THEOREM I. (S. Tachibana [1]). *In an $n(>2)$ dimensional Riemannian space M^n, if the mean curvature for q-plane is independent of the q-plane at each point, then*

(i) *M^n is an Einstein space, for $q = 1$ or $n-1$,*

(ii) *M^n is of constant curvature, for $n-1 > q > 1$ and $2q \neq n$,*

(iii) *M^n is conformally flat, for $n-1 > q > 1$ and $2q = n$.*

The converse is also true.

Taking holomorphic $2p$-planes instead of q-planes, an analogous result in Kählerian spaces is also known.

THEOREM II. (S. Tachibana [2], S. Tanno [3]). *In a Kählerian space $K^{2m}(m \geqq 2)$, if the mean curvature for $2p$-plane is independent of the holomorphic $2p$-plane at each point, then*

(i) *K^{2m} is of constant holomorphic curvature, for $1 < p < m-1$ and $2p \neq m$,*

(ii) *the Bochner curvature tensor of K^{2m} vanishes identically, for $1 < p < m-1$ and $2p = m$.*

The converse is also true.

The purpose of this paper is to prove an analogous theorem in

1) The number in brackets refers to Bibliography at the end of the paper.

Kählerian space taking anti-holomorphic p-plane in place of holomorphic $2p$-plane in the above theorem.

1. Preliminaries. Consider a Kählerian space K^{2m} of complex dimension $m(\geqq 2)$. Let $\langle\ ,\ \rangle$ and J be the inner product and the almost complex structure, then it holds that

$$(1.1)\qquad \langle X, Y\rangle = \langle JX, JY\rangle\ , \quad JJX = -X\ , \quad \nabla J = 0\ ,$$

where X and Y denote vector fields on K^{2m} (or tangent vectors at a point) and ∇ Levi-Civita connection. By R, R_1, and S we denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively. Then they satisfy for any vectors X, Y and Z,

$$(1.2)\qquad R(X, Y)Z = R(JX, JY)Z\ ,$$

$$(1.3)\qquad R_1(X, Y) = R_1(JX, JY)\ .$$

For a J-base $\{e_\lambda, Je_\lambda = e_{\lambda^*}\}$[2] and the sectional curvature $\rho(e_\lambda, e_\mu) = -\langle R(e_\lambda, e_\mu)e_\lambda, e_\mu\rangle$, we have

$$(1.4)\qquad \rho(e_\lambda, e_\mu) = \rho(e_{\lambda^*}, e_{\mu^*})\ , \quad \rho(e_{\lambda^*}, e_\mu) = \rho(e_\lambda, e_{\mu^*})\ .$$

If an orthonormal pair $\{X, Y\}$ at P satisfies

$$\langle X, JY\rangle = 0\ ,$$

then such a pair will be called an anti-holomorphic orthonormal pair. In [5] and [6], the following lemma has been proved.

LEMMA 1.1. *In a Kählerian space, the following three Propositions $A \sim C$ are equivalent to one another.*

A. $$\rho(X, Y) = \rho(X, JY)$$

holds good for any anti-holomorphic orthonormal pair $\{X, Y\}$.

B. $$\rho(X, Y) = \frac{1}{8}\{H(X) + H(Y)\}$$

holds good for any anti-holomorphic orthonormal pair $\{X, Y\}$, where $H(X) = \rho(X, JX)$, viz. the holomorphic sectional curvature for X.

C. The Bochner curvature tensor of K^{2m} vanishes.

2. The mean curvature for anti-holomorphic p-plane. Consider a p-plane π at a point P of a Kählerian space K^{2m}. If we find p vectors $X_1, \cdots, X_p$ such that $X_1, \cdots, X_p$ span π and $JX_1, \cdots, JX_p$ are perpendicular to π, then π is called anti-holomorphic. If π is an anti-holomorphic p-plane, then there exists a J-base $\{e_\lambda, e_{\lambda^*}\}$ of $T_P(K^{2m})$ such that $e_1, \cdots, e_p$

2) As the notations we follow S. Tachibana [2]. $\lambda, \mu = 1, 2, \cdots, m$.

span π. Such a J-base will be called an adapted J-base for π. Hereafter π will always mean an anti-holomorphic p-plane.

The mean curvature $\rho(\pi)$ for π is

$$\rho(\pi) = \frac{1}{p(2m-p)} \sum_{i=1}^{p} \left[\sum_{j=1}^{p} \rho(e_i, e_{j^*}) + \sum_{a=p+1}^{m} \{\rho(e_i, e_a) + \rho(e_i, e_{a^*})\} \right]. \tag{2.1}$$

LEMMA 2.1. *If $m \geqq p \geqq 2$ and if the mean curvature for p-plane is independent of the anti-holomorphic p-plane at P, then Proposition A in Lemma* 1.1 *holds good.*

PROOF. Consider an anti-holomorphic p-plane π at P and adapted J-base $\{e_\lambda, e_{\lambda^*}\}$ for π. Let π' be the anti-holomorphic p-plane spanned by $e_{1^*}, e_2, \cdots, e_p$. Then the mean curvature $\rho(\pi')$ is given by

$$\begin{aligned} \rho(\pi') = \frac{1}{p(2m-p)} \Big[& \sum_{j=2}^{p} \rho(e_{1^*}, e_{j^*}) + \rho(e_{1^*}, e_1) + \sum_{i=2}^{p} \rho(e_i, e_1) \\ & + \sum_{i=2}^{p} \sum_{j=2}^{p} \rho(e_i, e_{j^*}) + \sum_{a=p+1}^{m} \sum_{i=1}^{p} \{\rho(e_i, e_a) + \rho(e_i, e_{a^*})\} \Big]. \end{aligned} \tag{2.2}$$

By the assumption we have $\rho(\pi) = \rho(\pi')$, and hence

$$\sum_{j=2}^{p} \rho(e_1, e_{j^*}) = \sum_{j=2}^{p} \rho(e_1, e_j) , \tag{2.3}$$

taking account of (1.4). Similarly we have

$$\rho(e_2, e_{1^*}) + \sum_{j=3}^{p} \rho(e_2, e_{j^*}) = \rho(e_2, e_1) + \sum_{j=3}^{p} \rho(e_2, e_j) . \tag{2.4}$$

In the case $p = 2$, by (2.3) we have

$$\rho(e_1, e_{2^*}) = \rho(e_1, e_2) . \tag{2.5}$$

When $p \geqq 3$, we consider p-plane π'' which is spanned by $e_{1^*}, e_{2^*}, e_3, \cdots, e_p$. The similar process for π'' instead of π' leads us to

$$\sum_{j=3}^{p} \rho(e_1, e_{j^*}) + \sum_{j=3}^{p} \rho(e_2, e_{j^*}) = \sum_{j=3}^{p} \rho(e_1, e_j) + \sum_{j=3}^{p} \rho(e_2, e_j) . \tag{2.6}$$

Taking account of (2.3), (2.4) and (2.6), we see that

$$\rho(e_1, e_{2^*}) = \rho(e_1, e_2) . \tag{2.7}$$

Then (2.5) and (2.7) show that Proposition A holds good. q.e.d.

LEMMA 2.2. *If $m > p \geqq 2$, and if the mean curvature for p-plane is independent of the anti-holomorphic p-plane at each point, then K^{2m} is of constant holomorphic curvature.*

PROOF. By virtue of Lemma 2.1 and Lemma 1.1, it follows that

$$\rho(e_\lambda, e_\mu) = \rho(e_\lambda, e_{\mu^*}) = \frac{1}{8}\{H(e_\lambda) + H(e_\mu)\}\ , \quad \lambda \neq \mu\ . \tag{2.8}$$

Hence

$$\sum_{j=1}^{p}\sum_{i=1}^{p}\rho(e_i, e_{j^*}) = \frac{p+3}{4}\sum_{i=1}^{p}H(e_i)\ , \tag{2.9}$$

$$\sum_{i=1}^{p}\sum_{a=p+1}^{m}\{\rho(e_i, e_a) + \rho(e_i, e_{a^*})\} = \frac{m-2p}{4}\sum_{i=1}^{p}H(e_i) + \frac{p}{4}\sum_{\lambda=1}^{m}H(e_\lambda)\ . \tag{2.10}$$

By assumption $\rho(\pi)$ being independent of π, we put $\rho = \rho(\pi)$. Then substituting (2.9) and (2.10) into (2.1), we get

$$p(2m-p)\rho = \frac{m-p+3}{4}\sum_{i=1}^{p}H(e_i) + \frac{p}{4}\sum_{\lambda=1}^{m}H(e_\lambda)\ . \tag{2.11}$$

Taking account of $m > p$, we consider p-plane π' which is spanned by $e_2, e_3, \cdots, e_{p+1}$. The similar process for π' instead of π leads us to

$$p(2m-p)\rho = \frac{m-p+3}{4}\sum_{i=2}^{p+1}H(e_i) + \frac{p}{4}\sum_{\lambda=1}^{m}H(e_\lambda)\ . \tag{2.12}$$

By (2.11) and (2.12) we obtain,

$$H(e_1) = H(e_{p+1})\ .$$

Similarly it follows that

$$H(e_1) = H(e_2) = \cdots = H(e_m)\ . \tag{2.13}$$

For any unit vector X at a point P, there exists a J-base $\{e_\lambda, e_{\lambda^*}\}$ such that $X = e_1$. Then we get from (2.12) and (2.13)

$$H(X) = \frac{4(2m-p)}{2m+3-p}\rho\ ,$$

which means that $H(X)$ is independent of X. q.e.d.

3. A theorem analogous to Theorem I and II. By virtue of Lemma 1.1, Lemma 2.1 and Lemma 2.2, we get the following theorem including the trivial case where $p = 1$. Its converse part is obtained by straightforward calculation.

THEOREM. *In a Kählerian space $K^{2m}(m \geqq 2)$, if the mean curvature for p-plane is independent of the anti-holomorphic p-plane at each point, then*

(i) *K^{2m} is an Einstein space, for $p = 1$,*

(ii) *K^{2m} is of constant holomorphic curvature, for $m > p \geqq 2$,*

(iii) *The Bochner curvature tensor of* K^{2m} *vanishes identically, for* $m = p \geqq 2$.

The converse is also true.

REMARK. In the case (iii), we obtain by straight-forward calculation

$$\rho(\pi) = -\frac{m+3}{4m(m+1)} S .$$

Thus $\rho(\pi)$ is independent of the point P if and only if the scalar curvature S is constant.

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