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DEFORMATIONS OF RIEMANNIAN METRICS ON 3-DIMENSIONAL MANIFOLDS

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1. Introduction. Let M be a compact and orientable 3-dimensional manifold of class C^∞. Let $\mathscr{M}$ be the set of all Riemannian metrics g on M such that the total volume $\mathrm{Vol}(M, g) = 2\pi^2$. Since $\dim M = 3$, $g \in \mathscr{M}$ is an Einstein metric, if and only if (M, g) is a space of constant curvature. We consider a deviation function F on $\mathscr{M}$ from Einstein metric defined by

$$F(g) = \int_M [3R_{ij}R_{rs}g^{ir}g^{js} - S^2]dM \,, \tag{1.1}$$

where dM, (R_{ij}), and S denote the volume element, the Ricci curvature tensor, and the scalar curvature with respect to $g = (g_{ij})$. $F(g)$ is the integral of $3|G|^2 = 3G_{rs}G^{rs}$, where $G = (G_{rs}) = (R_{rs} - (S/3)g_{rs})$. $F(g) = 0$ holds if and only if g is an Einstein metric.

In the case of 3-dimensional C^∞-manifolds, the famous Poincaré problem is equivalent with the following: If M is a compact and simply connected 3-dimensional C^∞-manifold, does $\mathscr{M}$ contain an Einstein metric (i.e., constant curvature metric)?

Since $F(g)$ is non-negative, $g \in \mathscr{M}$ such that $F(g) = 0$ is a critical point of F. Problems are as follows:

(1) Can F have a critical point g such that $F(g) \neq 0$?

(2) If there is some critical point $g \in \mathscr{M}$ such that $F(g) \neq 0$, then what is typical property of (M, g)?

(3) If F has no critical point such that $F(g) \neq 0$, then, (3-a): is it possible to show existence of $g_0 \in \mathscr{M}$ such that $F(g_0) = 0$ by deformations? or (3-b): is there any counter example to (3-a)?

We calculate the necessary conditions for $g \in \mathscr{M}$ to be a critical point of F (cf. Proposition 2.1). Next we obtain the following theorems.

THEOREM A. *Let $g \in \mathscr{M}$ be a critical point of F. Assume that the scalar curvatures S of (M, g) is positive and g is not so much deviated from Einstein metric, i.e.,*

$$|G|^2 \leqq (1/114)S^2$$

holds on M. Then $G=0$ and g is itself an Einstein metric.

If we assume, furthermore, constancy of S, then the restriction on G is weakened. That is, we obtain

THEOREM B. *Let $g\in\mathscr{M}$ be a critical point of F. Assume that S is positive and constant, and that g is not so much deviated from Einstein metric, i.e., $|G|^2\leqq(1/26)S^2$ holds on M, or more generally*

$$F(g)\leqq(3/13)\pi^2S^2 .$$

Then $F(g)=0$ and g is an Einstein metric.

2. **Preliminaries.** Let $g(t)=(g(t)_{ij})$ be a (local) 1-parameter family of Riemannian metrics on a compact and orientable Riemannian manifold (M, g), $g=g(0)$. Since we are going to calculate only the first derivative of the variation $(F(t)=F(g(t)))$, we can put

$$g(t)_{ij}=g_{ij}+th_{ij}+[t^2] \tag{2.1}$$

in a local coordinate neighborhood $U(x^i)$, where $h=(h_{ij})$ is a symmetric $(0, 2)$-tensor field on M and $[t^2]$ denotes the terms of order $\geqq 2$ in t. Then it is classical that

$$(g(t)^{-1})^{jk}=g(t)^{jk}=g^{jk}-tg^{jr}h_{rs}g^{sk}+[t^2] , \tag{2.2}$$

$$\sqrt{|g(t)|}=\sqrt{|g|}+(1/2)t\sqrt{|g|}g^{rs}h_{rs}+[t^2] , \tag{2.3}$$

where $|g|$ and $|g(t)|$ denote the determinants of $g=(g_{ij})$ and $g(t)=(g(t)_{ij})$. The Christoffel's symbols are related by

$$\Gamma(t)^i_{jk}=\Gamma^i_{jk}+(1/2)t\,g^{ir}(\nabla_k h_{rj}+\nabla_j h_{rk}-\nabla_r h_{jk})+[t^2] , \tag{2.4}$$

where ∇ denotes the covariant derivation with respect to g. We define a $(1, 2)$-tensor field $W(t)$ for each t by

$$W(t)^i_{jk}=\Gamma(t)^i_{jk}-\Gamma^i_{jk} .$$

Then curvature tensors are related by

$$R(t)^i_{jkl}=R^i_{jkl}+\nabla_l W(t)^i_{jk}-\nabla_k W(t)^i_{jl}+[t^2] , \tag{2.5}$$

$$R(t)_{jk}=R_{jk}+\nabla_r W(t)^r_{jk}-\nabla_k W(t)^r_{jr}+[t^2] , \tag{2.6}$$

$$S(t)=S-tR^{rs}h_{rs}+t(\nabla^r\nabla^s h_{rs}-\nabla^r\nabla_r h^s_s)+[t^2] . \tag{2.7}$$

Now we assume $\dim M=3$ and consider the function $F(t)$:

$$F(t)=\int_M[3R(t)_{ij}R(t)_{rs}g(t)^{ir}g(t)^{js}-S^2(t)]\sqrt{|g(t)|}dx^1\wedge dx^2\wedge dx^3 , \tag{2.8}$$

where (x^1, x^2, x^3) is compatible with the fixed orientation of M. By a straightforward calculation using (2.2) $\sim$ (2.7), we get

$$\frac{dF(t)}{dt}\bigg|_0 = \int_M [h^{ij}L_{ij}]dM\,, \tag{2.9}$$

where we have put

$$\begin{aligned} L_{ij} = {} & [-(9/2)R_{rs}R^{rs} + (1/2)\nabla^r\nabla_r S + (5/2)S^2]g_{ij} \\ & + \nabla_i\nabla_j S + 12R_i^r R_{rj} - 3\nabla_r\nabla^r R_{ij} - 7S R_{ij}\,, \end{aligned} \tag{2.10}$$

and we have used $\nabla_r R_i^r = (1/2)\nabla_i S$ and

$$R^i_{jkl} = \delta^i_l R_{jk} - \delta^i_k R_{jl} + R^i_l g_{jk} - R^i_k g_{jl} - (1/2)S(\delta^i_l g_{jk} - \delta^i_k g_{jl})\,.$$

$d\,\mathrm{Vol}\,(M, g(t))/dt|_0 = 0$ holds, if and only if

$$\int_M [g^{rs}h_{rs}]dM = 0\,. \tag{2.11}$$

PROPOSITION 2.1. *Let (M, g) be a compact and orientable 3-dimensional Riemannian manifold. Assume that g is a critical point of F for any deformation $g(t)\,(g(0) = g)$ such that $d\,\mathrm{Vol}\,(M, g(t))/dt|_0 = 0$. Then there exists a constant C such that*

$$3R_{rs}R^{rs} - S^2 + \nabla_r\nabla^r S + 6C = 0 \tag{2.12}$$

and we have

$$\begin{aligned} & [-4R_{rs}R^{rs} + (2/3)\nabla_r\nabla^r S + (7/3)S^2]g_{ij} \\ & + \nabla_i\nabla_j S + 12R_i^r R_{rj} - 3\nabla_r\nabla^r R_{ij} - 7SR_{ij} = 0\,. \end{aligned} \tag{2.13}$$

The converse is also true.

PROOF. Let $h = (h_{ij})$ be an arbitrary symmetric (0, 2)-tensor field satisfying (2.11). By (2.9) and the classical Lagrange lemma we have $L_{ij} = Cg_{ij}$ for some constant C. Transvecting (2.10) with g^{ij} we obtain (2.12). Next we eliminate C from $L_{ij} = Cg_{ij}$ and (2.12) to get (2.13). The proof of the converse is easy.

REMARK 2.2. (2.12) is also written as

$$3|G|^2 - [\mathrm{Vol}\,(M, g)]^{-1}\int_M 3|G|^2 dM = \Delta S \tag{2.14}$$

or $3|G|^2 - [\mathrm{Vol}\,(M, g)]^{-1}F(g) = \Delta S$, where Δ denotes the Laplacian on (M, g).

REMARK 2.3. For a deformation $g(t) = g + th + [t^2]$, h satisfying (2.11), $\bar{g}(t) = [\mathrm{Vol}\,(M, g)/\mathrm{Vol}\,(M, g(t))]^{2/3}g(t)$ is the equivalent deformation of constant volume (i.e. $\bar{g}(t) \equiv g(t) \bmod t^2$, and $\mathrm{Vol}\,(M, \bar{g}(t)) = \mathrm{Vol}\,(M, g)$).

3. **Proof of Theorem A.** Let M be a compact and orientable 3-dimensional C^∞-manifold. Let $\mathscr{M}$ be the set of Riemannian metrics

explained in the introduction. If g is a critical point of F defined by (1.1), by Proposition 2.1 we have (2.12) and (2.13). We analyze these two identities. First we prove

PROPOSITION 3.1. *If $g \in \mathscr{M}$ is a critical point of F and if $S \neq 0$, then (M, g) is irreducible.*

PROOF. Assume that (M, g) is reducible. Let $U(x^i) = U_1(x^a) \times U_2(x^3)$ be a canonical coordinate neighborhood of the local product structure, where $(a, b, c) = (1, 2)$. Since $S \neq 0$, U_1 is not flat. By $^*g$ we denote the induced metric on U_1 from g, and the corresponding geometric objects are denoted by $(^*)$. Then (2.13) gives

$$(3.1) \qquad [-4{}^*R_{ab}{}^*R^{ab} + (2/3){}^*\nabla_c{}^*\nabla^c{}^*S + (7/3){}^*S^2]{}^*g_{ab} + {}^*\nabla_a{}^*\nabla_b{}^*S + 12{}^*R_a^c{}^*R_{cb} - 3{}^*\nabla_c{}^*\nabla^c{}^*R_{ab} - 7{}^*S{}^*R_{ab} = 0 .$$

Since $\dim U_1 = 2$, we have ${}^*R_{ab} = ({}^*S/2){}^*g_{ab}$ on U_1. Therefore (3.1) is written as

$$(3.2) \qquad -(1/6)({}^*S^2 + 5{}^*\nabla_c{}^*\nabla^c{}^*S){}^*g_{ab} + {}^*\nabla_a{}^*\nabla_b{}^*S = 0 .$$

Transvecting (3.2) with ${}^*g^{ab}$, we get

$$(3.3) \qquad -(1/3)[{}^*S^2 + 2{}^*\nabla_c{}^*\nabla^c{}^*S] = 0 .$$

(3.3) on U_1 implies

$$(3.4) \qquad S^2 + 2\nabla_r\nabla^r S = 0$$

on $U = U_1 \times U_2$. Integrating (3.4) on M we obtain $\int S^2 dM = 0$. This contradicts $S \neq 0$.

LEMMA 3.2. *Assume that $g \in \mathscr{M}$ is a critical point of F. Let $(\lambda_1 \geqq \lambda_2 \geqq \lambda_3)$ be continuous functions which are eigenvalues of the Ricci curvature tensor. If $(2/3)\lambda_1 \leqq \lambda_3$ holds on M, (M, g) is an Einstein space.*

PROOF. Transvecting (2.13) with $3R^{ij}$, we get

$$(3.5) \qquad f - 9R^{ij}\nabla_r\nabla^r R_{ij} + 2S\nabla_r\nabla^r S + 3R^{ij}\nabla_i\nabla_j S = 0 ,$$

where we have put

$$(3.6) \qquad f = -33SR_{rs}R^{rs} + 7S^3 + 36R_{rs}R^{st}R_t^r .$$

(3.5) is written as

$$f + 9\nabla_r R_{ij}\nabla^r R^{ij} - (7/2)\nabla_r S\nabla^r S - \nabla_r(9R^{ij}\nabla^r R_{ij} - 2S\nabla^r S - 3R^{rj}\nabla_j S) = 0 .$$

Integrating the last equation on M we get

$$(3.7)\qquad \int_M [2f + 18\nabla_r R_{ij}\nabla^r R^{ij} - 7\nabla_r S\nabla^r S]dM = 0 .$$

(3.7) is transformed to

$$(3.8)\qquad \int_M [2f - \nabla_r S\nabla^r S + 2H_{rij}H^{rij}]dM = 0$$

where we have put $H_{rij} = 3\nabla_r R_{ij} - \nabla_r S g_{ij}$. Multiplying (2.12) by S and integrating the result on M, we get

$$(3.9)\qquad \int_M [3SR_{rs}R^{rs} - S^3 - \nabla_r S\nabla^r S + 6CS]dM = 0 ,$$

where $6C = -[\mathrm{Vol}(M, g)]^{-1}F(g)$ and $6C$ is non-positive. Then eliminating $\nabla_r S\nabla^r S$ from (3.8) and (3.9), we obtain

$$(3.10)\qquad \int_M [(2f - 3SR_{rs}R^{rs} + S^3) - 6CS + 2H_{rij}H^{rij}]dM = 0 .$$

We show that $K = 2f - 3SR_{rs}R^{rs} + S^3 \geqq 0$. By $S = \sum \lambda_i$, $R_{rs}R^{rs} = \sum \lambda_i^2$ and (3.6) we get

$$(3.11)\qquad \begin{aligned} K = {} & 18 \sum \lambda_i^3 + 90\lambda_1\lambda_2\lambda_3 \\ & - 24[\lambda_1^2(\lambda_2 + \lambda_3) + \lambda_2^2(\lambda_1 + \lambda_3) + \lambda_3^2(\lambda_1 + \lambda_2)] . \end{aligned}$$

Let p be an arbitrary point of M. If $\lambda_1 = 0$ at p, then the assumption $(2/3)\lambda_1 \leqq \lambda_3$ implies $\lambda_3 = 0$ at p and hence $K = 0$ at p. $\lambda_1 < 0$ at p gives a contradiction. Hence we prove for the case $\lambda_1 > 0$ at p. We put $\lambda_1 = a$, $\lambda_2 = ax$ and $\lambda_3 = ay$. Put $P = K(p)/6a^3$. Then $P = P(x, y)$ and

$$P = 3(x^3 + y^3 + 1) + 15xy - 4[x + y + x^2 + y^2 + xy(x + y)] .$$

(i) For $x + y = k =$ constant, we have

$$P(x, k - x) = (13k - 23)(x - k/2)^2 - (1/4)(k - 2)^2(k - 3) .$$

This means that, for $23/13 < k < 3$, only ($x = 1$ and $k = 2$) satisfies $P(x, k - x) = 0$; otherwise $P(x, k - x) > 0$.

(ii) For $x = 1$, we have $P(1, y) = (y - 1)^2(3y - 2)$.

(iii) For $y = 2/3$, we have $P(x, 2/3) = (1/9)(x - 1)(27x^2 - 33x + 5)$. The solutions of $27x^2 - 33x + 5 = 0$ is $(11 \pm 7.8 \cdots)/18$.

(iv) $P(-1, 0) = 0$ and $P(0, -1) = 0$.

From these we see that $P(x, y) \geqq 0$ for (x, y) such that $1 \geqq x \geqq y \geqq 2/3$. Hence, we have $K(p) \geqq 0$ and $K \geqq 0$ on M. Since $(2/3)\lambda_1 \leqq \lambda_3$ shows also $S \geqq 0$, we get $-6CS \geqq 0$. Then (3.10) shows $K = 0$, $6CS = 0$ and $H_{rij} = 0$. By $g^{ir}H_{rij} = 0$, we get $3\nabla_r R^r_j - \nabla_j S = (3/2 - 1)\nabla_j S = 0$. Consequently, $\nabla_j S = 0$ and $S =$ constant.

If $S=0$ on M, $(2/3)\lambda_1 \leqq \lambda_3$ implies that all $\lambda_i = 0$ on M.

If $S \neq 0$, by $6CS = 0$ we get $C = 0$ and $F(g) = 0$. Therefore (M, g) is an Einstein space.

LEMMA 3.3. *If S is positive and if*

$$|G|^2 = R_{rs}R^{rs} - (1/3)S^2 \leqq (1/114)S^2$$

holds at p, then $(2/3)\lambda_1 \leqq \lambda_3$ holds at p.

PROOF. Let $\lambda_1 = a$, $\lambda_2 = ax$ and $\lambda_3 = ay$ at p as before. Then the assumption is $342R_{rs}R^{rs} \leqq 117S^2$ and is written as

$$225(1 + x^2 + y^2) \leqq 234(x + y + xy) .$$

The set $[(\bar{x}, \bar{y}): W(\bar{x}, \bar{y}) = 225(1 + \bar{x}^2 + \bar{y}^2) - 234(\bar{x} + \bar{y} + \overline{xy}) = 0]$ is an ellipse in $\boldsymbol{R} \times \boldsymbol{R}$, the axis being on the line $\bar{x} = \bar{y}$. We see that

(i) the ellipse is tangent to the line $\bar{y} = 2/3$ at $(13/15, 2/3)$,

(ii) $W(1, 1) < 0$.

Since the point (x, y) is inside of the ellipse, we get $y \geqq 2/3$.

PROOF OF THEOREM A. Since S is positive and $|G|^2 \leqq (1/114)S^2$ on M, we have $(2/3)\lambda_1 \leqq \lambda_3$ on M by Lemma 3.3. Next by Lemma 3.2, (M, g) is an Einstein space.

4. Proof of Theorem B.

LEMMA 4.1. *Assume that $g \in \mathscr{M}$ is a critical point of F and S is positive and constant. If continuous functions $(\lambda_1 \geqq \lambda_2 \geqq \lambda_3)$ which are eigenvalues of the Ricci curvature tensor satisfy $(2/5)\lambda_1 \leqq \lambda_3$ on M, then $\lambda_1 = \lambda_2 = \lambda_3$.*

PROOF. By (3.7) and $S =$ constant we get

$$\int_M [f + 9\nabla_r R_{st} \nabla^r R^{st}] dM = 0 . \tag{4.1}$$

We show that $f \geqq 0$. By (3.6) we get

$$\begin{aligned} f = {} & 10 \sum \lambda_i^3 + 42\lambda_1\lambda_2\lambda_3 \\ & - 12[\lambda_1^2(\lambda_2 + \lambda_3) + \lambda_2^2(\lambda_1 + \lambda_3) + \lambda_3^2(\lambda_1 + \lambda_2)] . \end{aligned} \tag{4.2}$$

At an arbitrary point p we put $\lambda_1 = a$, $\lambda_2 = ax$ and $\lambda_3 = ay$. Put $Q = f(p)/2a^3$, $Q = Q(x, y)$:

$$Q = 5(1 + x^3 + y^3) + 21xy - 6(x + y + x^2 + y^2 + xy(x + y)) .$$

It suffices to show that $Q \geqq 0$ in the range $(x, y) \in \boldsymbol{R} \times \boldsymbol{R}$ such that $1 \geqq x \geqq y \geqq 2/5$. This is seen by the following observation:

(i) For $x + y = k =$ constant, we get

$$Q(x, k-x) = 3(7k-11)(x-k/2)^2 - (1/4)(k-2)^2(k-5) .$$

This means that, for $11/7 < k < 5$, only ($x = 1$ and $k = 2$) satisfies $Q(x, k-x) = 0$; otherwise $Q(x, k-x) > 0$.

(ii) For $x = 1$, $Q(1, y) = (y-1)^2(5y-2)$.

(iii) For $y = 2/5$, $Q(x, 2/5) = (1/25)(x-1)(125x^2 - 85x - 49)$.

The solutions of $125x^2 - 85x - 49 = 0$ are $(17 \pm 35.6 \cdots)/50$.

(iv) $Q(-1, 0) = 0$ and $Q(0, -1) = 0$.

Therefore we obtain $f(p) \geqq 0$ and $f \geqq 0$ on M. Then (4.1) shows that $f = 0$ and $\nabla_r R_{st} = 0$ on M. By Proposition 3.1 (M, g) is an Einstein space.

LEMMA 4.2. *If S is positive and if*

$$|G|^2 = R_{rs}R^{rs} - (1/3)S^2 \leqq (1/26)S^2$$

holds at p, then $(2/5)\lambda_1 \leqq \lambda_3$ *holds at p.*

PROOF. The assumption is $78R_{rs}R^{rs} \leqq 29S^2$ and is written as

$$49(1 + x^2 + y^2) \leqq 58(x + y + xy) .$$

The set $[(\bar{x}, \bar{y}); V(\bar{x}, \bar{y}) = 49(1 + \bar{x}^2 + \bar{y}^2) - 58(\bar{x} + \bar{y} + \bar{x}\bar{y}) = 0]$ is an ellipse in $\boldsymbol{R} \times \boldsymbol{R}$, the axis being on the line $\bar{x} = \bar{y}$. We see that

(i) the ellipse is tangent to the line $\bar{y} = 2/5$ at $(29/35, 2/5)$,

(ii) $V(1, 1) < 0$.

From these we obtain $y \geqq 2/5$.

PROOF OF THEOREM B. Since S is positive and constant, (2.12) tells us that $R_{rs}R^{rs}$ is also constant. Hence, $|G|^2$ is constant. Therefore, $3\int |G|^2 dM = F(g) \leqq (3/13)\pi^2 S^2$ and $\mathrm{Vol}(M, g) = 2\pi^2$ imply $|G|^2 \leqq (1/26)S^2$. Then applying Lemma 4.2 and next Lemma 4.1, we see that (M, g) is an Einstein space.

REMARK 4.3. It is known that a Riemannian metric g is a critical point of a function $F_S(g) = \int S dM: \mathscr{M} \to \boldsymbol{R}$, if and only if g is an Einstein metric (cf. Hilbert [4], Nagano [5], Eliasson [3]). Analogous results were also obtained using other functions (cf. Berger [1], [2], Eliasson [3], etc.). In some sense these results have connections with the famous Poincaré conjecture. However they give only characterizations of Einstein metrics, and they could not give any directional method how to get Einstein metrics or how to deform a metric to an Einstein metric.

Our results have some directional property. That is, assume that M is simply connected and admits an Einstein metric g_0. Define a neighborhood U of g_0 in $\mathscr{M}$ by $U =$ (connected component of $V \cap W$), where $V = \{g \in \mathscr{M}; S > 4 \text{ on } M\}$ and $W = \{g \in \mathscr{M}; |G| < 16/114 \text{ on } M\}$. Notice that $S(g_0) = 6$ and $G(g_0) = 0$. Let g_1 be a given metric in $\mathscr{M}$. First one strikes a point g_α in U by deforming $g_1\,(g(t), \alpha \leqq t \leqq 1)$. This is possible, because $\mathscr{M}$ is contractible. Next g_α is deformed to g_0 $(g(t), 0 \leqq t \leqq \alpha)$ with the help of F in a theoretical sense that one can decrease the value $F(g) \to 0$ in U, because $S > 4$ and $|G| < 16/114$ imply $|G| < S^2/114$ and there is no critical point of F in U except g_0 by Theorem A.

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