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ON THE k-th NULLITY SPACE OF THE RIEMANNIAN CURVATURE TENSOR

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Introduction. The nullity space of the curvature tensor in a Riemannian manifold was first introduced by S. S. Chern and N. H. Kuiper [3] and then generalized to any curvature-like tensor by A. Gray [6]. The nullity distribution is an interesting object, and it has been studied by many geometers with applications in the imbedding problem ([3]-[11]). The distribution is differentiable and involutive. Its maximal integral manifolds are totally geodesic, and complete provided that the ambient manifold is complete.

In this paper we define the k-th nullity space in Riemannian manifolds which includes Chern-Kuiper's as the 0-th nullity space. The defining equations contains the successive covariant derivatives of the Riemannian curvature tensor up to the k-th order.

We fix the notations in §1. The k-th nullity space is introduced in §2 and we discuss the differentiability of its distribution. §3 is devoted itself for some lemmas. It is shown in §4 that the distribution is integrable and the maximal integral manifolds are totally geodesic. The stable nullity distribution is introduced. In the last section an example of the 1-st (but not 0-th) nullity distribution is given.

A similar discussion for the relative nullity will appear in a forthcoming paper [12].

1. Preliminaries. Let M be an n dimensional C^∞ Riemannian manifold, and $\langle\ ,\ \rangle$ its Riemannian metric. We denote by $T_p(M)$, $\mathscr{F}(M)$ and $\mathscr{X}(M)$ the tangent space of M at the point p, the algebra of C^∞ differentiable functions and the algebra of vector fields on M, respectively. The Riemannian curvature tensor R of M is a tensor field of type (1,3) which gives an endomorphism $R(X, Y)$ of $\mathscr{X}(M)$ for $X, Y \in \mathscr{X}(M)$ by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\,,$$

where ∇_X denotes the Riemannian connection.

As is well known, R satisfies the following identities:

$$R(X, Y) = -R(Y, X)\,,$$

$$\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle = \langle R(Z, W)X, Y\rangle ,$$
$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 ,$$
$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0 .$$

For any tensor field K of type (r, s) the covariant differential ∇K of K is defined by

$$(\nabla K)(W; X_1, \cdots, X_s) = (\nabla_W K)(X_1, \cdots, X_s), \qquad X_i,\ W \in T_p(M) .$$

∇K is a tensor field of type $(r, s+1)$. The k-th covariant differential $\nabla^k K$ is defined inductively to be $\nabla(\nabla^{k-1}K)$: For simplicity, we use the notation

$$(\nabla^k K)(W_{k,\cdots,1}; X_1, \cdots, X_s)$$

or

$$(\nabla^k K)(W_{k,\cdots,i+1}; W_i; W_{i-1,\cdots,1}; X_1, \cdots, X_s)$$

instead of

$$\begin{aligned}&(\nabla^k K)(W_k; \cdots; W_1; X_1, \cdots, X_s)\\ &\quad = (\nabla_{W_k}(\nabla^{k-1}K))(W_{k-1}; \cdots; W_1; X_1, \cdots, X_s) ,\end{aligned}$$

where $\nabla^0 K$ means K.

Henceforth let us agree with the following conventions unless otherwise stated:

$U, V, W, W_1, \cdots, W_k$ mean any vectors (or vector fields) ;

X, Y are specified whenever they appear .

2. The k-th nullity space of R. The nullity space at p in the sense of Chern-Kuiper is the subspace of $T_p(M)$ defined as

$$\mathcal{N}_p^{(0)} = \{X \in T_p(M) \mid R(U, V)X = 0\} .$$

Generalizing this space we give the following

DEFINITION. For any point p of M and a non-negative integer k, $\mathcal{N}_p^{(k)}$ is the subspace of $T_p(M)$ given by

$$\mathcal{N}_p^{(k)} = \{X \in T_p(M) \mid (\nabla^h R)(W_{h,\cdots,1}; U, V)X = 0 \text{ for } 0 \leqq h \leqq k\} .$$

We call $\mathcal{N}_p^{(k)}$ the k-th *nullity space* of R at p, and its dimension the k-th *nullity* of R at p.

The series $\mathcal{N}_p^{(k)}$ $(k = 0, 1, \cdots)$ of subspaces of $T_p(M)$ clearly satisfies

$$\mathcal{N}_p^{(0)} \supset \mathcal{N}_p^{(1)} \supset \cdots \supset \mathcal{N}_p^{(k)} \supset \cdots .$$

We denote by $\mathcal{N}^{(k)}$ the distribution which assigns $\mathcal{N}_p^{(k)}$ to p.

THEOREM 1. *If* $\mu^{(k)} = \dim \mathscr{N}^{(k)}$ *is constant on* M, *the distribution* $\mathscr{N}^{(k)}$ *is differentiable.*

PROOF. Let $\mathscr{S}_p$ be the subspace of $T_p(M)$ spanned by vectors of the form

$$(\nabla^h R)(W_{h,\dots,1}; U, V)W, \qquad 0 \leqq h \leqq k .$$

Then we have $\mathscr{N}_p^{(k)} = \mathscr{S}_p^{\perp}$, the orthogonal complements of $\mathscr{S}_p$. For, $X \in \mathscr{N}_p^{(k)}$ is equivalent to $X \in \mathscr{S}_p^{\perp}$ by virtue of the identity

$$\langle (\nabla^h R)(W_{h,\dots,1}; U, V)W, X\rangle = -\langle (\nabla^h R)(W_{h,\dots,1}; U, V)X, W\rangle .$$

The rest of the proof is similar to that of Rosenthal [11, p. 470, Th. 2.1].

3. Propositions. We shall prepare some lemmas which will be useful in the next section.

PROPOSITION 1. $X \in \mathscr{N}^{(k)}$ *implies* $\nabla_W X \in \mathscr{N}^{(k-1)}$ *for* $1 \leqq k$.

PROOF. It is easy to see the following identity to be valid for $0 \leqq h$:

$$(\,*\,) \qquad \begin{aligned}&(\nabla^h R)(W_{h,\dots,1}; U, V)\nabla_W X \\ &= \nabla_W((\nabla^h R)(W_{h,\dots,1}; U, V)X) - (\nabla^{h+1}R)(W; W_{h,\dots,1}; U, V)X \\ &\quad - \sum_{i=1}^{h} (\nabla^h R)(W_{h,\dots,i+1}; \nabla_W W_i; W_{i-1,\dots,1}; U, V)X \\ &\quad - (\nabla^h R)(W_{h,\dots,1}; \nabla_W U, V)X - (\nabla^h R)(W_{h,\dots,1}; U, \nabla_W V)X .\end{aligned}$$

As the right hand members all vanish under the assumption $X \in \mathscr{N}^{(k)}$ and $h \leqq k-1$, the proof is completed.

PROPOSITION 2. *For* $X, Y \in \mathscr{N}^{(k)}$ *and* $0 \leqq h \leqq k$, *the following equation holds good:*

$$(\nabla^{h+1}R)(X; W_{h,\dots,1}; U, V)Y = 0 .$$

PROOF. For any $X, Y \in \mathscr{X}(M)$ and $1 \leqq h$, we have

$$\begin{aligned}&(\nabla^{h+1}R)(X; W_{h,\dots,1}; U, V)Y \\ &= (\nabla_X \nabla_{W_h} \nabla^{h-1}R)(W_{h-1,\dots,1}; U, V)Y - (\nabla^h R)(\nabla_X W_h; W_{h-1,\dots,1}; U, V)Y \\ &= ((\nabla_{W_h}\nabla_X + \nabla_{[X,W_h]} + R(X, W_h))\nabla^{h-1}R)(W_{h-1,\dots,1}; U, V)Y \\ &\quad - (\nabla^h R)(\nabla_X W_h; W_{h-1,\dots,1}; U, V)Y \\ &= \nabla_{W_h}((\nabla^h R)(X; W_{h-1,\dots,1}; U, V)Y) - (\nabla^h R)(\nabla_{W_h}X; W_{h-1,\dots,1}; U, V)Y \\ &\quad - \sum_{i=1}^{h-1} (\nabla^h R)(X; W_{h-1,\dots,i+1}; \nabla_{W_h} W_i; W_{i-1,\dots,1}; U, V)Y \\ &\quad - (\nabla^h R)(X; W_{h-1,\dots,1}; \nabla_{W_h} U, V)Y - (\nabla^h R)(X; W_{h-1,\dots,1}; U, \nabla_{W_h} V)Y \\ &\quad - (\nabla^h R)(X; W_{h-1,\dots,1}; U, V)\nabla_{W_h} Y + (R(X, W_h)\nabla^{h-1}R)(W_{h-1,\dots,1}; U, V)Y .\end{aligned}$$

Hence if $X, Y \in \mathscr{N}^{(k)}$ and $1 \leqq h \leqq k$, it follows that

$$(**) \qquad (\nabla^{h+1}R)(X; W_{h,\cdots,1}; U, V)Y = -(\nabla^{h}R)(X; W_{h-1,\cdots,1}; U, V)\nabla_{W_h}Y .$$

On the other hand, we know that X and $\nabla_{W_h}Y \in \mathscr{N}^{(k-1)}$ by taking account of Proposition 1. Thus the right hand side of (**) becomes

$$(\nabla^{h-1}R)(X; W_{h-2,\cdots,1}; U, V)\nabla_{W_{h-1}}\nabla_{W_h}Y .$$

Repeating this process and denoting $Y_h = \nabla_{W_1} \cdots \nabla_{W_h}Y$ we have

$$(***) \qquad \begin{aligned}(\nabla^{h+1}R)(X; W_{h,\cdots,1}; U, V)Y &= (-1)^h(\nabla_X R)(U, V)Y_h \\ &= (-1)^{h+1}((\nabla_U R)(V, X)Y_h + (\nabla_V R)(X, U)Y_h) .\end{aligned}$$

It must be noticed that (***) is true even for $h = 0$. Thus the case of $1 \leqq k$ is proved because the right hand members of (***) vanish by $X \in \mathscr{N}^{(k)}$. For the case $k = h = 0$, the proof follows from (***) taking account of

$$\begin{aligned}&(\nabla_U R)(V, X)Y \\ &= \nabla_U(R(V, X)Y) - R(\nabla_U V, X)Y - R(V, \nabla_U X)Y - R(V, X)\nabla_U Y .\end{aligned}$$

4. **Theorems.** In this section we study at first the integrability of the distribution $\mathscr{N}^{(k)}$ and generalize well known theorems for $\mathscr{N}^{(0)}$ to $\mathscr{N}^{(k)}$. Next the stable nullity distribution is introduced.

THEOREM 2. *If $\mu^{(k)}$ is constant on M, then the distribution $\mathscr{N}^{(k)}$ is involutive, and each maximal integral manifold of $\mathscr{N}^{(k)}$ is totally geodesic.*

PROOF. Let $X, Y \in \mathscr{N}^{(k)}$ and $0 \leqq h \leqq k$. Operating ∇_X to

$$(\nabla^h R)(W_{h,\cdots,1}; U, V)Y = 0$$

we have by virtue of (*)

$$(\nabla^{h+1}R)(X; W_{h,\cdots,1}; U, V)Y + (\nabla^h R)(W_{h,\cdots,1}; U, V)\nabla_X Y = 0 .$$

This equation and Proposition 2 lead us to

$$(\nabla^h R)(W_{h,\cdots,1}; U, V)\nabla_X Y = 0$$

which implies $\nabla_X Y \in \mathscr{N}^{(k)}$. Hence it follows that $\mathscr{N}^{(k)}$ is involutive.

Consider a maximal integral manifold L of $\mathscr{N}^{(k)}$. The second fundamental form α of L in M is defined by

$$\alpha(X, Y) = \nabla_X Y - \bar{\nabla}_X Y , \qquad X, Y \in \mathscr{X}(L) ,$$

where $\bar{\nabla}_X$ is the induced Riemannian connection on L. $\bar{\nabla}_X Y$ being nothing but the orthogonal projection of $\nabla_X Y$ to $\mathscr{X}(L)$, we have $\bar{\nabla}_X Y = \nabla_X Y$ and hence α vanishes identically. Thus L is totally geodesic. q.e.d.

Now, we assume that there is an integer $k \geqq 1$ such that $\mu^{(k)}$ is constant on M and $\mathscr{N}^{(k-1)} = \mathscr{N}^{(k)}$. Then, as $X \in \mathscr{N}^{(k)}$ implies $\nabla_W X \in \mathscr{N}^{(k-1)}$ by Proposition 1, we have

$$\begin{aligned} 0 &= (\nabla^k R)(W_{k,\cdots,1};\ U,\ V)\nabla_W X \\ &= (\nabla^{k+1} R)(W;\ W_{k,\cdots,1};\ U,\ V)X \end{aligned}$$

by virtue of (*). Thus $X \in \mathscr{N}^{(k+1)}$ follows.

Consequently we get the distribution

$$\mathscr{N}^{(k-1)} = \mathscr{N}^{(k)} = \mathscr{N}^{(k+1)} = \cdots$$

which will be called the *stable* nullity distribution.

From the above argument we get

THEOREM 3. *The stable nullity distribution is parallel.*

Conversely, suppose that the distribution $\mathscr{N}^{(k)}$ is parallel for an integer $k \geqq 0$, then we have $\nabla_W X \in \mathscr{N}^{(k)}$ for any $X \in \mathscr{N}^{(k)}$. Thus we see

$$(\nabla^{k+1} R)(W;\ W_{k,\cdots,1};\ U,\ V)X = 0$$

by virtue of (*). This implies $X \in \mathscr{N}^{(k+1)}$ and hence the distribution $\mathscr{N}^{(k)}$ is stable. Thus we have the following

THEOREM 4. *If the distribution $\mathscr{N}^{(k)}$ is parallel for an integer $k \geqq 0$, then it is stable.*

5. **An example.** We shall give an example of 0-th and 1-st nullity spaces. Let E be the half plane in the Euclidean n-space defined by $x^1 > 0$, where $(x^1, \cdots, x^n)$ is an orthogonal coordinate system of this Euclidean space. Let M be a Riemannian manifold of constant curvature $K(\neq 1)$. The warped product $\tilde{M} = E \times {}_f M$ [2] is the product manifold $E \times M$ with the Riemannian structure such that

$$\| U \|^2 = \| \pi_* U \|^2 + (f^2 \cdot \pi) \| \eta_* U \|^2 , \qquad U \in \mathscr{X}(\tilde{M}) ,$$

where $\pi \colon \tilde{M} \to E$, $\eta \colon \tilde{M} \to M$ are the projections and f is a positive function on E. Now we set $f = x^1$. Then, by Bishop-O'Neill [2], pp. 23–25 and Tanno [13], pp. 68–70, the Riemannian curvature tensor $\tilde{R}$ of $\tilde{M}$ satisfies the following relations:

$$\tilde{R}(U,\ V)\partial/\partial x^i = 0 , \qquad i = 1, 2, \cdots, n ,$$

$$\tilde{R}(X,\ Y)Z = (K-1)(\langle Y,\ Z\rangle X - \langle X,\ Z\rangle Y) ,$$

$$(\tilde{\nabla}_X \tilde{R})(Y,\ Z)\partial/\partial x^1 = -\frac{K-1}{x^1}(\langle Z,\ X\rangle Y - \langle Y,\ X\rangle Z) ,$$

$$(\tilde{\nabla}_U \tilde{R})(V,\ W)\partial/\partial x^i = 0 , \qquad i = 2, 3, \cdots, n ,$$

where $U, V, W \in \mathscr{X}(\tilde{M})$, $X, Y, Z \in \mathscr{X}(M)$ and $\langle\ ,\ \rangle$ is the Riemannian metric on M, and $\tilde{\nabla}_U$ is the Riemannian connection on $\tilde{M}$. Hence

$$\mathscr{N}_p^{(0)} = \text{the subspace spanned by } (\partial/\partial x^i)_p,\ i = 1, 2, \cdots, n\,,$$

$$\mathscr{N}_p^{(1)} = \text{the subspace spanned by } (\partial/\partial x^i)_p,\ i = 2, 3, \cdots, n$$

are the 0-th and 1-st nullity spaces at $p \in \tilde{M}$ respectively. As the distribution $\mathscr{N}^{(1)}$ is parallel, it is stable by virtue of Theorem 4. The metric of this type has appeared in Yano-Sasaki [14].

BIBLIOGRAPHY

[1] K. ABE, A characterization of totally geodesic submanifolds in S^N and CP^N by an inequality, Tôhoku Math. J., 23 (1971), 219-244.

[2] R. L. BISHOP AND B. O'NEILL, Manifold of negative curvature, Trans. Amer. Math. Soc., 145 (1969), 1-49.

[3] S. S. CHERN AND N. H. KUIPER, Some theorems on the isometric imbedding of compact Riemann manifolds in Euclidean space, Ann. of Math., 56 (1952), 422-430.

[4] Y. H. CLIFTON AND R. MALTZ, The K-nullity spaces of the curvature operator, Michigan Math. J., 17 (1970), 85-89.

[5] D. FERUS, On the completeness of nullity foliations, Michigan Math. J., 18 (1971), 61-64.

[6] A. GRAY, Spaces of constancy of curvature operators, Proc. Amer. Math. Soc., 17 (1966), 897-902.

[7] A. GRAY, Some relations between curvature and characteristic classes, Math. Ann., 184 (1970), 257-267.

[8] S. KOBAYASHI AND K. NOMIZU, Foundations of differential geometry, Vol. I and II, Wiley-Interscience, New York, 1963 and 1969.

[9] R. MALTZ, The nullity spaces of the curvature operator, Top. et Géom. Diff., 8 (1966), 1-20.

[10] T. ÔTSUKI, Isometric imbedding of Riemann manifolds in a Riemann manifold, J. Math. Soc. Japan, 6 (1954), 221-234.

[11] A. ROSENTHAL, Riemannian manifolds of constant nullity, Michigan Math. J., 14 (1967), 469-480.

[12] S. TACHIBANA AND M. SEKIZAWA, On the k-th relative nullity space of a Riemannian manifold in E^N, Tamkang J. Math., 5 (1974), 133-138.

[13] S. TANNO, A class of Riemannian manifolds satisfying $R(X, Y) \cdot R = 0$, Nagoya Math. J., 42 (1971), 67-77.

[14] K. YANO ET S. SASAKI, Sur la structure des espaces de Riemann dont le groupe d'holonomie fixe un plan à un nombre quelconque de dimensions, Proc. Japan Acad., 24 (1948), 7-13.

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