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ON ARTIN L-FUNCTIONS

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Let k be an algebraic number field of finite degree. Let K be a Galois extension of k of finite degree. Let G be the Galois group of this extension. Let χ be a character of G. Then Artin L-function $L(s, \chi)$ is defined. For some groups G, $L(s, \chi)$ is known to be an entire function for every non-trivial irreducible character χ [2, p. 225]. These cases can be proved through Blichfeldt's theorem [3, p. 348] reducing to abelian cases, i.e., Hecke L-functions. This theorem can be applied for other groups, i.e., for supersolvable groups. A group G is called supersolvable if G has normal subgroups $H_0, H_1, \cdots, H_r$ such that $G = H_0 \supset H_1 \supset \cdots \supset H_r = \{e\}$ and every H_{i-1}/H_i is cyclic [4].

THEOREM 1. *If the Galois group G is supersolvable, $L(s, \chi)$ is entire for every non-trivial irreducible character χ.*

PROOF.[1] If G is abelian, $L(s, \chi)$ is a Hecke L-function which is entire. So we assume that G is not abelian and we will prove by induction on the order of G. Let χ be the character of a representation module (G, V). If there exists a non-trivial normal subgroup N which operates trivially on V, χ is a character of G/N. As G/N is also supersolvable, $L(s, \chi)$ is entire by induction. Now we assume that there exists no such normal subgroup. Then G is a subgroup of $GL(V)$. Let C be the center of G. As G/C is also supersolvable, there exists a normal subgroup A of G such that A/C is cyclic and $A \neq C$. Then A is abelian because C is in the center of A. Now Blichfeldt's theorem shows that there exists a proper subgroup H of G such that $\chi = \varphi^G$ for some character φ of H, where φ^G means the induced character of G. It is easy to see that φ is non-trivial and irreducible. As $L(s, \chi) = L(s, \varphi)$ and as H is also supersolvable, our assertion is proved by induction.

REMARK. Professor M. Ishida kindly suggested this proof when G is nilpotent. We note that every finite nilpotent group is supersolvable.

[1] This proof shows that $L(s, \chi)$ is entire for every χ if the Galois group is an M-group. Hence Theorem 1 is a special case of Huppert's Theorem [5, p. 580]. We also note that every M-group is solvable [5, p. 581].

We now give an example of a finite solvable group on which Blichfeldt's theorem cannot be applied. In fact, the following example is of the smallest possible order. Let G be a finite group generated by σ, τ and ρ whose relations are as follows:

$$\sigma^4 = \rho^3 = 1 , \qquad \sigma^2 = \tau^2 , \qquad \sigma\tau\sigma^{-1} = \tau^{-1} ,$$
$$\rho\sigma\rho^{-1} = \tau , \qquad \rho\tau\rho^{-1} = \tau\sigma .$$

Then σ and τ generate the commutator subgroup G' which is isomorphic to the quaternion group. As $(G: G') = 3$, the order of G is 24. Now G can be represented as subgroups of $GL(2, C)$, where C is the complex numbers. In fact, if we put

$$\sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \rho = \alpha\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix},$$

we see easily that above relations hold. In the above, α is one of the following values:

$$\alpha = \frac{-1+i}{2}, \frac{-1+i}{2}\omega \quad \text{or} \quad \frac{-1+i}{2}\omega^2 ,$$

where ω is a primitive cube root of unity. Above representations give three 2-dimensional characters χ_1, χ_2 and χ_3 which are different with one another, as the values of the characters at ρ are different. As G has no subgroup of index 2, any 2-dimensional character cannot be induced from a proper subgroup. Hence Blichfeldt's theorem cannot be applied in this case. We see that G has seven conjugate classes which are represented by 1, σ^2, σ, ρ, ρ^2, $\rho\sigma^2$ and $\rho^2\sigma^2$. Hence every character of G is determined by values at these elements. We see that

$$\chi_i(1) = 2, \chi_i(\sigma^2) = -2 \quad \text{and} \quad \chi_i(\sigma) = 0$$

for every i, and

$$\chi_1(\rho) = \chi_1(\rho^2) = -1 , \quad \chi_1(\rho\sigma^2) = \chi_1(\rho^2\sigma^2) = 1 ,$$
$$\chi_2(\rho) = -\omega , \quad \chi_2(\rho^2) = -\omega^2 , \quad \chi_2(\rho\sigma^2) = \omega , \quad \chi_2(\rho^2\sigma^2) = \omega^2 ,$$

and $\chi_3 = \bar{\chi}_2$ is the complex conjugate of χ_2. Let H be the subgroup of G generated by $\rho\sigma^2$. Let φ and ψ be one-dimensional characters of H such that $\varphi(\rho\sigma^2) = -\omega$ and $\psi(\rho\sigma^2) = -1$. Let φ^G and ψ^G be induced characters of G. As we can take 1, σ, τ, $\sigma\tau$ as representatives of G/H, we see that

$$\varphi^G(1) = 4 , \quad \varphi^G(\sigma^2) = -4 , \quad \varphi^G(\sigma) = 0$$

and

$$\varphi^G(\mu) = \varphi(\mu) \quad \text{if} \quad \mu \in H - \{1, \sigma^2\} ,$$

and the same for ψ.

THEOREM 2. *Let K and k be algebraic number fields of finite degrees. We assume that K is a Galois extension of k with Galois group G defined above. Then*

$$L(s, \chi_1)^2 = L(s, \varphi)L(s, \bar{\varphi})/L(s, \psi)$$
$$L(s, \chi_2)^2 = L(s, \bar{\varphi})L(s, \psi)/L(s, \varphi)$$

and

$$L(s, \chi_3)^2 = L(s, \varphi)L(s, \psi)/L(s, \bar{\varphi})$$

hold.

PROOF. It is easy to see that

$$2\chi_1 = \varphi^G + \bar{\varphi}^G - \psi^G$$
$$2\chi_2 = \bar{\varphi}^G + \psi^G - \varphi^G$$

and

$$2\chi_3 = \varphi^G + \psi^G - \bar{\varphi}^G .$$

This shows above equalities.

REMARK. Let λ be a one-dimensional character of the subgroup generated by σ such that $\lambda(\sigma) = i$. Then

$$\chi_1 = \varphi^G + \bar{\varphi}^G - \lambda^G ,$$
$$\chi_2 = \lambda^G - \varphi^G$$

and

$$\chi_3 = \lambda^G - \bar{\varphi}^G$$

hold.

We now assume that k is the field of the rational numbers. Let F be the intermediate field of K/k corresponding to H. Then $L(s, \varphi)$, $L(s, \bar{\varphi})$ and $L(s, \psi)$ are different L-functions corresponding to an abelian extension K/F. Moreover they are multiplicatively independent because $L(s, \chi_i)$ are multiplicatively independent [1]. If we can show that different L-functions of an abelian extension have independent distributions of zero points, $L(s, \chi_i)$ have poles by Theorem 2. Following example seems to show that it is not absurd to think so, though Artin's conjecture asserts that $L(s, \chi_i)$ has no pole.

EXAMPLE. Let F be an algebraic number field of finite degree. Let

$L(s, \varphi)$ and $L(s, \psi)$ be different Hecke L-functions over F. Let F_φ and F_ψ be cyclic extensions of F corresponding to φ and ψ, respectively. If the numbers of real places ramified at F_φ and F_ψ are the same, and if the conductors of φ and ψ are the same, $L(s, \varphi)/L(s, \psi)$ has poles.

PROOF. Let

$$\Phi(s, \varphi) = A(\mathfrak{f}_\varphi)^s \Gamma\Big(\frac{s+1}{2}\Big)^\nu \Gamma\Big(\frac{s}{2}\Big)^{r_1-\nu} \Gamma(s)^{r_2} L(s, \varphi)$$

as usual, where $\mathfrak{f}_\varphi$ is the conductor of φ and ν is the number of real places ramified at F_φ/F. Above assumptions show that $L(s, \varphi)/L(s, \psi) = \Phi(s, \varphi)/\Phi(s, \psi)$. First we consider the case $L(s, \psi) = L(s, \bar\varphi)$. If $L(s, \varphi)/L(s, \bar\varphi)$ has a zero point ρ, $\bar\rho$ is a pole of this function. Hence if $L(s, \varphi)/L(s, \bar\varphi)$ has no pole, the zero points of $L(s, \varphi)$ and $L(s, \bar\varphi)$ are the same counting the multiplicities. It is known [6] that

$$\Phi(s, \varphi) = ae^{bs} \prod_\rho \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho}$$

for constants a and b depending on φ, where ρ runs over the zeros of $L(s, \varphi)$ such that $0 < \Re\rho < 1$. Therefore

$$L(s, \varphi)/L(s, \bar\varphi) = \Phi(s, \varphi)/\Phi(s, \bar\varphi) = ae^{bs}/\bar a e^{\bar b s}\,.$$

Let $a_1 = a/\bar a$ and let $b_1 = b - \bar b$. Let $s = re^{i\theta}$ for some θ such that $-\pi/2 < \theta < \pi/2$. The left hand side of the above equation goes to 1 when r goes to infinity. If $b_1 \neq 0$, the right hand side goes to zero or to infinity for suitable θ. Hence it must be $b_1 = 0$. Then a_1 must be 1, as the left hand side goes to 1 when r goes to infinity. This shows $L(s, \varphi) = L(s, \bar\varphi)$ which is a contradiction. Now let $L(s, \psi) \neq L(s, \bar\varphi)$. We put $M(s) = L(s, \varphi)L(s, \bar\varphi)/L(s, \psi)L(s, \bar\psi)$. If $M(s)$ has a pole ρ, ρ or $\bar\rho$ is a pole of $L(s, \varphi)/L(s, \psi)$. We assume that $M(s)$ has no pole. Now

$$\begin{aligned}
\Phi(s, \varphi)\Phi(s, \bar\varphi) &= a\bar a e^{(b+\bar b)s} \prod_\rho \Big(1 - \frac{s}{\rho}\Big)\Big(1 - \frac{s}{\bar\rho}\Big) e^{(1/\rho + 1/\bar\rho)s} \\
&= a\bar a e^{(b+\bar b)s} \prod_\rho \Big(1 - \frac{s}{\rho}\Big)\Big(1 - \frac{s}{\bar\rho}\Big) \prod_\rho e^{(1/\rho + 1/\bar\rho)s} \\
&= a\bar a \prod_\rho \Big(1 - \frac{s}{\rho}\Big)\Big(1 - \frac{s}{\bar\rho}\Big) \exp\Big\{s\Big(b + \bar b + \sum_\rho \Big(\frac{1}{\rho} + \frac{1}{\bar\rho}\Big)\Big)\Big\}\,,
\end{aligned}$$

as the products converge absolutely. Now Landau shows [6]

$$b + \bar b + \sum_\rho \Big(\frac{1}{\rho} + \frac{1}{\bar\rho}\Big) = 0\,.$$

Hence

$$\Phi(s, \varphi)\Phi(s, \bar{\varphi}) = a\bar{a} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{\bar{\rho}}\right).$$

As we assume that $M(s) = \Phi(s, \varphi)\Phi(s, \bar{\varphi})/\Phi(s, \psi)\Phi(s, \bar{\psi})$ has no pole, it must be

$$M(s) = c \prod_{\rho}{}' \left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{\bar{\rho}}\right)$$

for some constant c, where $\prod'$ means the product over the zero points of $M(s)$. As $0 < \Re\rho < 1$,

$$\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{\bar{\rho}}\right) > 1$$

for real $s > 2$, and $(1 - s/\rho)(1 - s/\bar{\rho})$ goes to infinity as s goes to infinity. Hence if $M(s)$ has at least one zero point, the absolute value of the right hand side goes to infinity, which is a contradiction because the left hand side goes to 1. Hence it must be $M(s) = 1$. If $L(s, \varphi)/L(s, \psi)$ has a zero point ρ, it is a pole of $L(s, \bar{\varphi})/L(s, \bar{\psi})$. Then $\bar{\rho}$ is a pole of $L(s, \varphi)/L(s, \psi)$ which contradicts to our assumption. Hence the zero points of $L(s, \varphi)$ must coincide to those of $L(s, \psi)$. Then it must be

$$L(s, \varphi)/L(s, \psi) = \Phi(s, \varphi)/\Phi(s, \psi) = a_1 e^{b_1 s}/a_2 e^{b_2 s}$$

for some constants a_1, a_2, b_1 and b_2. But this shows $L(s, \varphi) = L(s, \psi)$ as in the case $\psi = \bar{\varphi}$.

Above theorem shows that it is difficult to know whether $L(s, \chi)$ is entire or not, even if the Galois group is solvable. Following theorem is in contrast with this.

THEOREM 3. *Let k be an algebraic number field of finite degree. Let F be an algebraic extension of k of finite degree. Let K be the normal closure of this extension, i.e., the smallest Galois extension of k containing F. If the Galois group $G = G(K/k)$ is solvable, $\zeta_F(s)/\zeta_k(s)$ is an entire function.*

PROOF. If there exists an intermediate field E of F/k, entireness of $\zeta_F(s)/\zeta_k(s)$ follows from entireness of $\zeta_F(s)/\zeta_E(s)$ and $\zeta_E(s)/\zeta_k(s)$. So we may assume that F/k has no intermediate field. Let H be the subgroup of G corresponding to F. H contains no non-trivial normal subgroup of G because K is the normal closure of F/k. Hence G can be considered as a permutation group of $\Omega = G/H$. As H itself is the stabilizer of $H \in \Omega$ and as H is a maximal subgroup of G by our assumption, G is a primitive permutation group of Ω [5, p. 147]. As G is solvable, Galois' theorem

[5, p. 159] shows that there exists an abelian normal subgroup N such that

$$G = HN \quad \text{and} \quad H \cap N = (e) .$$

Let χ_0 and φ_0 be trivial characters of G and H, respectively. If we put

$$\varphi_0^G = \chi_0 + \chi_1 + \cdots + \chi_r ,$$

χ_i, $i \geqq 1$, are non-trivial. As $\zeta_F(s)/\zeta_k(s) = L(s, \varphi_0^G)/L(s, \chi_0) = \prod_{i=1}^r L(s, \chi_i)$, it suffices to show that every $L(s, \chi_i)$, $i \geqq 1$, is entire. First we show that every χ_i, $i \geqq 1$, is non-trivial over N. Let χ be an irreducible character of G which is trivial over N. Then χ can be considered as an irreducible character of G/N. As $H \cong G/N$, $\chi | H$ is irreducible, where $\chi | H$ means the restriction of χ to H. As

$$(\varphi_0^G, \chi)_G = (\varphi_0, \chi | H)_H ,$$

χ appears as a component of φ_0^G if and only if $\chi | H = \varphi_0$. But $\chi | H = \varphi_0$ means $\chi = \chi_0$. Now let χ be an irreducible character of G which is non-trivial over N. Let (G, V) be a representation module of G whose character is equal to χ. Then there exists a non-trivial irreducible character λ of N which appears as a component of $\chi | N$. That is, the subspace W of V defined by

$$W = \{w \in V | nw = \lambda(n)w \text{ for every } n \in N\}$$

is not trivial. Let G_1 be the subgroup of G which consists of the elements g_1 of G such that $g_1 W = W$. Let φ be the character of the representation module (G_1, W). Then φ is irreducible and it is known [3, § 50] that $\chi = \varphi^G$. Let N_1 be the kernel of λ. Then G_1 is contained in the normalizer $N_G(N_1)$ of N_1 in G. It holds that

$$\lambda(n)g_1 w = ng_1 w = g_1(g_1^{-1}ng_1)w = \lambda(g_1^{-1}ng_1)g_1 w$$

for any $n \in N$, $g_1 \in G_1$ and $w \in W$. This shows that G_1/N_1 is contained in the centralizer of N/N_1 in $N_G(N_1)/N_1$. It is also easy to show that every element in the centralizer of N/N_1 is contained in G_1/N_1. That is, G_1/N_1 is the centralizer of N/N_1 in $N_G(N_1)/N_1$. Especially, G_1 depends only on λ, not on χ. As G_1 contains N, there exists a subgroup H_1 of H such that $G_1 = H_1 N$. Hence G_1/N_1 is isomorphic to a direct product of H_1 and N/N_1, because N/N_1 is in the center of G_1/N_1. Let $\psi_0, \psi_1, \cdots, \psi_s$ be the irreducible characters of H_1, where ψ_0 is a trivial character. Let $\psi_i \otimes \lambda$ be a character of G_1 defined by $\psi_i \otimes \lambda(h_1 n) = \psi_i(h_1)\lambda(n)$ for every $h_1 \in H$ and $n \in N$. Then it is a character of G_1/N_1, and it is irreducible because $G_1/N_1 \cong H_1 \times N/N_1$. And φ defined above is one of the $\psi_i \otimes \lambda$. As $(\psi_i \otimes \lambda)^G | N$ contains λ as an irreducible component, above argument shows $(\psi_i \otimes \lambda)^G$ is an irreducible character of G. It holds that

$$\varphi_0^G(n) = 0 \quad \text{if} \quad n \in N - (e)$$

and

$$\varphi_0^G(e) = (N: 1) .$$

Hence

$$(\varphi_0^G, \lambda^G)_G = (\varphi_0^G \mid N, \lambda)_N = 1 .$$

Therefore there exists only one component of λ^G which appears as a component of φ_0^G. It is easy to see that $\lambda^{G_1} = \psi_{\text{reg}} \otimes \lambda$, where ψ_{reg} is the character of the regular representation of H_1. Then there exist only one $(\psi_i \otimes \lambda)^G$ which appears in φ_0^G. Let $h_i \in H$, $i = 1, \cdots, t$, be the representatives of G/G_1. Then it holds for any $h \in H$ that

$$\begin{aligned}(\psi_0 \otimes \lambda)^G(h) &= \sum_i \psi_0 \otimes \lambda(h_i^{-1}hh_i) \\ &= \text{the number of } h_i \text{ such that } h_i^{-1}hh_i \in H_1 .\end{aligned}$$

Hence every $(\psi_0 \otimes \lambda)^G(h) \geqq 0$ and $(\psi_0 \otimes \lambda)^G(e) = (H: H_1) > 0$. It then holds

$$\begin{aligned}(\varphi_0^G, (\psi_0 \otimes \lambda)^G)_G &= (\varphi_0, (\psi_0 \otimes \lambda)^G \mid H)_H \\ &= (H: 1)^{-1} \sum_{h \in H} (\psi_0 \otimes \lambda)^G(h) > 0 .\end{aligned}$$

Therefore $(\psi_0 \otimes \lambda)^G$ is the component of λ^G which appears in φ_0^G. We have shown that every component of $\varphi_0^G - \chi_0$ is of the form $(\psi_0 \otimes \lambda)^G$. As $\psi_0 \otimes \lambda$ is a non-trivial one-dimensional character of G_1, $L(s, \psi_0 \otimes \lambda)$ is a Hecke L-function which is entire. This shows that $\zeta_F(s)/\zeta_k(s)$ is entire.

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