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ON BOUNDED FUNCTIONS IN THE ABSTRACT HARDY SPACE THEORY II

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We continue in this paper the study of bounded functions in the abstract Hardy space theory in our former work [19]. The situation is as follows: Let (X, Σ, m) be a probability measure space and H a weak* closed subalgebra of the sup-norm algebra L^∞ of the bounded m-measurable functions, satisfying $1 \in H$ and $\int uvdm = \int udm \int vdm$ for any $u, v \in H$. H^p is the L^p closure of H $(0 < p < \infty)$. We have shown in [19] that for every non-constant $u \in H$ there corresponds a unique Carathéodory domain A such that $m\{x: u(x) \in \bar{A}\} = 1$, $\int udm \in A$ and $m\{x: |u(x) - a| < \varepsilon\} > 0$ for any $\varepsilon > 0$ and any $a \in \partial A$. It is then natural to ask: Is the spectrum of u contained in $\bar{A}$? If f is a continuous function on $\bar{A}$, holomorphic in A, does the composed function $f(u)$ lie again in H? Or, more generally, if f is in $H^p(A)$, does the appropriately defined composed function $f(u)$ lie in H^p? We answer to the third question in the paragraph 2. In the classical unit disc case these were studied by many mathematicians. (See for details Ryff [15] or Nordgren [11].) We consider in Section 2 the case where the essential range or the value carrier of u is contained in the unit disc and in Section 3 the case where the value carrier of u is contained in a more general domain, i.e., a Carathéodory domain. In the paragraph 3 the first and second questions are answered also affirmative. The second one is answered in somewhat different form: *Let D be a Carathéodory domain and $f(z)$ a continuous function on $\bar{D}$, holomorphic in D. Then for every $u \in H$ with $m\{x: u(x) \in \bar{D}\} = 1$ and $\int udm \in D$ it holds $f(u) \in H$ and $\Phi(f(u)) = f(\Phi(u))$ for all non-zero multiplicative linear functional Φ on H* (Theorem 4.2). Some problems related to it are also discussed there. One of them is, roughly speaking, the following: *Let D be an open set in the complex plane. Then, if a measurable function f on D operates on H, f is necessarily holomorphic in D* (Corollary 4.4). This is an analogous result to a familiar one in group algebras. Preliminaries and some remarks on conformal mapping are given in the next

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paragraph.

1. Preliminaries, notation and some remarks on conformal mapping.

1. Let D be an arbitrary simply connected domain in the complex plane with at least two boundary points. There is no difficulty in defining the space $H^\infty(D)$ of bounded holomorphic functions in D; it is a Banach algebra under the norm

$$||f||_\infty = \sup_{z \in D} |f(z)| .$$

For $0 < p < \infty$, a function f holomorphic in D is said to belong to the class $H^p(D)$ if the subharmonic function $|f(z)|^p$ has a harmonic majorant in D. The norm can be defined as

$$||f||_{H^p(D)} = ||f||_p = [u(z_0)]^{1/p} ,$$

where z_0 is some fixed point in D and u is the least harmonic majorant of $|f|^p$. It is easy to see that the space $H^p(D)$ is comformally invariant. That is, if $f \in H^p(D)$ and if $z = g(w)$ is a conformal mapping of a domain D^* onto D, then $f(g(w)) \in H^p(D^*)$. Furthermore, if the norm in $H^p(D^*)$ is defined in terms of the point $w_0 = g^{-1}(z_0)$, this correspondence $f \rightarrow f \circ g$ is an isometric isomorphism. If $1 \leqq p \leqq \infty$, $|| \ ||_p$ is a genuine norm, i.e., the triangle inequality holds:

$$||f + g||_p \leqq ||f||_p + ||g||_p .$$

This ceases to be true if $0 < p < 1$; in that case we have, however,

$$||f + g||_p^p \leqq ||f||_p^p + ||g||_p^p .$$

For the unit disc U the above definition of $H^p(D)$ coincides with the classical one, i.e., $f \in H^p(U)$ if and only if

$$\sup_{0<r<1}\left(\int |f(re^{i\theta})|^p d\theta\right)^{1/p} < \infty .$$

In that case each f in $H^p(U)$ has non-tangential boundary values a.e. on the unit circle T, which determine a well-defined element f^* of the space $L^p(T)$ with respect to the normalized Lebesgue measure L on T and the mapping $f \rightarrow f^*$ is an isometry of $H^p(U)$ onto a subspace of $L^p(T)$. We denote this space as $H^p(T)$. $H^\infty(T)$ is an example of our space H.

Next we recall the definition of the following familiar uniform algebras, which we shall use frequently. Let K be a compact set in the complex plane. The algebra $C(K)$ consists of the continuous functions on K, endowed with the supremum norm. The algebra $P(K)$ is the set of all uniform limits of polynomials in z. The algebra $R(K)$ is the set of all uniform limits of rational functions with poles off K. The algebra $A(K)$ is the set of all functions in $C(K)$, holomorphic in the interior of K.

Now let $\{G_n\}$ be a sequence of simply connected domains in the complex plane, each containing a fixed disc k with center z_0. Let E be the set of all points z with a neighborhood $N(z)$ contained in all the domains G_n starting from some value of n (depending on z). Obviously E is nonempty, since $k \subset E$. Moreover E is open, and hence E is the union of countably many disjoint domains, namaly the connected components of E. Of these components, let G_{z_0} be the one containing z_0. Then G_{z_0} is called the *kernel* of the sequence $\{G_n\}$ (relative to the point z_0). It is a fortiori simply connected.

DEFINITION 1.1. Let $\{G_n\}$ be a sequence of simply connected domains, with kernel G_{z_0} relative to the point z_0. Then $\{G_n\}$ is said to converge to G_{z_0} if every subsequence of $\{G_n\}$ has the same kernel G_{z_0} (relative to z_0) as $\{G_n\}$ itself. Otherwise, $\{G_n\}$ is said to diverge.

DEFINITION 1.2. By a Jordan domain, we mean a domain which is the interior of a closed Jordan curve.

DEFINITION 1.3. Let G be a bounded simply connected domain, and let G_∞ be the component of $(\bar{G})^c$ containing the point at infinity. Then G is said to be a Carathéodory domain if G and G_∞ have the same boundary.

In particular, every Jordan domain is a Carathéodory domain. A sequence of bounded simply connected domains $\{G_n\}$ is said to be *strictly decreasing* if $\bar{G}_{n+1} \subset G_n$ $(n = 1, 2, \cdots)$. If G is a Carathéodory domain, then there exists a strictly decreasing sequence of bounded simply connected domains with smooth boundaries, converging to G as its kernel (relative to any point $z_0 \in G$). Conversely, if G is a kernel of a strictly decreasing sequence of bounded simply connected domains $\{G_n\}$, then $\{G_n\}$ converges to G (relative to any point of G) and G is a Carathéodory domain.

We state first a sharpened result of a Carathéodory's theorem in the case of strictly decreasing sequence.

THEOREM 1.1. *Let $\{D_j\}$ be a strictly decreasing sequence of bounded simply connected domains and let D be a kernel of $\{D_j\}$. Fix a point $a \in D$. Let ϕ, ϕ_j be the conformal mappings of U onto D, D_j (respectively) such that $\phi(0) = \phi_j(0) = a$ and $\phi'(0)$, $\phi_j'(0) > 0$. Then we have*

$$\| \phi_j - \phi \|_{H^p(U)} \longrightarrow 0$$

as $j \longrightarrow \infty$, for every $0 < p < \infty$.

To prove this, we need the following lemma which can be proved with a slight modification of the proof of Lemma 9.1 of Gamelin [3, p. 35].

LEMMA 1.1. *Let K be a compact set of the complex plane. Let K_n be compact sets of the complex plane such that ∂K_n consists of a finite number of Jordan curves, $K_{n+1} \subset K_n$, and $\bigcap_{n=1}^{\infty} K_n = K$. Suppose u is a real-valued continuous function on a neighborhood of K_1 with piecewise continuous partial derivatives of first order and u_n is the harmonic extension of $u|_{\partial K_n}$ to the interior of K_n. If every $z \in \partial K$ satisfies Lebesgue's condition, then u_n converges to u uniformly on ∂K.*

Here a point $z \in \partial K$ is said to satisfy *Lebesgue's condition* if $\int_S (dr)/r = +\infty$, where S consists of all r, $0 < r < 1$, such that the circle of radius r and center z meets the complement of K.

PROOF OF THEOREM 1.1. We suppose first the boundaries ∂D_j are smooth. Put $K = \bigcap \bar{D}_j$. Then we note first that K is compact and K^c is connected, and hence every point of ∂K satisfies Lebesgue's condition. Clearly we have $\partial D \subset \partial K$. Let $u_j(z)$ be the least harmonic majorant of $|z|^p$ in D_j. Then we have $[u_j(a)]^{1/p} = ||z||_{H^p(D_j)} = ||\phi_j||_{H^p(U)} = ||\phi_j||_p$. Since $|z|^p$ is continuously differentiable on $C\backslash\{0\}$, $u_j\{z\}$ is also the harmonic extension of $|z|^p|_{\partial D_j}$ to D_j and so we can apply Lemma 1.1 to $|z|^p$. Thus we see that $u_j(z)$ converges to $|z|^p$ uniformly on ∂K and hence on ∂D. Hence $u_j(z)$ converges to a continuous function $u(z)$ uniformly on $\bar{D}$. This function is harmonic on D and satisfies $u|_{\partial D} = |z|^p|_{\partial D}$. One can thus easily deduce that u is also the least harmonic majorant of $|z|^p$ in D. In particular, $u_j(a)$ tends to $u(a)$, that is,

$$||\phi_j||_p \to ||\phi||_p \qquad (j \to \infty) .$$

Next we have already noted that D_j converges to D in the sense of Carathéodory. Hence in virtue of Carathéodory's theorem, $\phi_j(z)$ converges to $\phi(z)$ uniformly on compact sets of U. For $1 < p < \infty$ this implies that ϕ_j tends to ϕ weakly in $L^p(T)$. Since $||\phi_j||_p$ tends to $||\phi||_p$, we have thus $||\phi_j - \phi||_p \to 0$ by the well-known property of $L^p(1 < p < \infty)$. Hence we have $||\phi_j - \phi||_p \to 0$ for any $0 < p < \infty$. Next we shall prove in the general case. When ∂D_j are not smooth, we can choose a strictly decreasing sequence of bounded simply connected domains $\{G_j\}$ with smooth boundaries such that $\bar{D}_{j+1} \subset G_j \subset \bar{G}_j \subset D_j$ $(j = 1, 2, \cdots)$. Then $\{G_j\}$ also converges to D and we have $||z||_{H^p(G_{j-1})} \geqq ||z||_{H^p(D_j)} \geqq ||z||_{H^p(G_j)}$. By the above argument $||z||_{H^p(G_j)}$ tends to $||z||_{H^p(D)}$ and hence $||z||_{H^p(D_j)}$ tends to $||z||_{H^p(D)}$. Again by the above argument we have $||\phi_j - \phi||_p \to 0$ as $j \to \infty$. This completes the proof.

COROLLARY 1.1. *Let D, D_j, a be as in Theorem 1.1 and $0 < p < \infty$. Further suppose $f(z)$ is holomorphic on a neighborhood of $\bigcap \bar{D}_j$. Then for sufficiently large j $f(z) \in H^p(D_j)$ and $\|f\|_{H^p(D_j)}$ converges to $\|f\|_{H^p(D)}$.*

The proof follows along the same lines as in the proof of Theorem 1.1. Further, for the inverse conformal mappings we have an analogous result to Theorem 1.1.

THEOREM 1.2. *Let D be a bounded simply connected domain and $\{D_j\}$ a sequence of uniformly bounded simply connected domains, converging to D and satisfying $D \subset D_j$. Let a point $a \in D$ be fixed. Let g, g_j be conformal mappings of D, D_j onto U respectively such that $g(a) = g_j(a) = 0$ and $g'(a)$, $g'_j(a) > 0$. Then we have $g_j \to g$ in $H^p(D)$ for any $0 < p < \infty$.*

PROOF. Since $D \subset D_j$, we have first for any $0 < p < \infty$

$$(*) \qquad \|g_j\|_{H^p(D)} = \|g_j \circ g^{-1}\|_{H^p(U)} \leqq 1 = \|g \circ g^{-1}\|_{H^p(U)} = \|g\|_{H^p(D)} .$$

(i) Case $p > 1$. Since $g_j \to g$ uniformly on compact sets in D by Carathéodory's theorem, we have $g_j \circ g^{-1}(w) \to w$ for all $w \in U$ and hence $g_j \circ g^{-1}(w) \to w$ in the weak topology of $L^p(T)$. Hence we have

$$({*\atop *}) \qquad \liminf_{j\to\infty} \|g_j \circ g^{-1}\|_{H^p(U)} \geqq \|w\|_{H^p(U)} = 1 .$$

Combining $(*)$ and $({*\atop *})$ we see that $\|g_j \circ g^{-1}\|_{H^p(U)}$ converges to $\|w\|_{H^p(U)}$. Since $g_j \circ g^{-1}(w) \to w$ weakly in $L^p(T)$, we have again as in the proof of Theorem 1.1 that $g_j \circ g^{-1}(w) \to w$ in the strong topology of L^p, i.e., $\|g_j \circ g^{-1} - g \circ g^{-1}\|_{H^p(U)} = \|g_j - g\|_{H^p(D)} \to 0$ as $j \to \infty$.

(ii) Case $0 < p \leqq 1$. Since $\|f\|_{H^p(D)} \leqq \|f\|_{H^2(D)}$ for any $f \in H^\infty(D)$, we have from (i) that $g_j \to g$ in $H^p(D)$. This completes the proof.

In the same way we can show the following

COROLLARY 1.2. *Let D, D_j, a, g and g_j be as in Theorem 1.2 and $1 \leqq p < \infty$. Then, for any $f(z) \in H^p(U)$ it holds $f \circ g_j \to f \circ g$ in $H^p(D)$.*

PROOF. For $1 < p < \infty$ the proof follows along the same lines as in the proof of Theorem 1.2. For $p = 1$ that $f \circ g_j \circ g^{-1} \to f$ uniformly on compact sets implies that $f \circ g_j \circ g^{-1} \to f$ in the weak* topology as the subspace of the dual space of $C(T)$. Hence we have $\liminf_{j\to\infty} \|f \circ g_j \circ g^{-1}\|_{H^p(U)} \geqq \|f\|_{H^p(U)}$ and as before $\lim_{j\to\infty} \|f \circ g_j \circ g^{-1}\|_{H^p(U)} = \|f\|_{H^p(U)}$. Now, using the so-called pseudo-uniform convexity of $H^1(U)$ we have the desired conclusion.

Now let K be a fixed compact set in the complex plane with connected complement. In this case we have $R(K) = P(K)$ and hence $R(K)$ is a Dirichlet algebra on ∂K. For $z \in K^0$, m_z will denote the unique representing measure for z on ∂K, that is, m_z is the harmonic measure for z on ∂K.

Let D be a component of K^0. A point $z \in \partial D$ is said to be *accessible* from D if z is the endpoint of a continuous curve which has its other points in D.

Since K^c is connected, D is a bounded simply connected domain and there is a conformal mapping f of U onto D. Let S be the subset of T at which f has non-tangential boundary values. Then S is a Borel set, and f extends to be a Borel function on S. The extension of f to S will also be denoted by f. We note that every point of $f(S)$ is an accessible point of ∂D. Since K^c is connected, we have $f(z_1) \neq f(z_2)$ if $z_1 \neq z_2$, z_1, $z_2 \in S$.

Let μ_w be the harmonic measure on T for $w \in U$. Then we can formulate Lemma 4.3 in Gamelin [3, p. 149] as follows.

LEMMA 1.2. *Let K, D, f, μ_w and m_z be as above. Then there exists a Borel set $E \in T$ such that*

(i) *f has non-tangential boundary values at every point of E, i.e., $E \subset S$.*

(ii) *f is one to one on E, $f(E) \subset \partial D$ is a Borel set, and f^{-1} is a Borel function on $f(E)$.*

(iii) *μ_w is supported on E for all $w \in U$.*

(iv) *m_z is supported on $f(E)$ for all $z \in D$. In particular, m_z is supported on the set of points of ∂D accessible from D.*

(v) *For all bounded Borel function g on ∂D,*

$$\int g dm_z = \int g \circ f d\mu_w\,, \quad w \in U\,, \quad z = f(w)\,.$$

REMARK. Let h be another conformal mapping of U onto D. Then, if we replace $(f, S, T, E, f(E))$ in Lemma 1.2 by $(h, h^{-1}\circ f(S), h^{-1}\circ f(T), h^{-1}\circ f(E), f(E))$, Lemma 1.2 is still valid.

2. **Holomorphic functions of functions in H.** Our purpose in this paragraph is to generalize the classical results on composed functions of bounded holomorphic functions in the unit disc. We devide this paragraph into two sections.

2. Disc case.

By elementary calculation we obtain the following lemma.

LEMMA 2.1. *Let $u \in H$, $|u| \leqq 1$, $\neq e^{i\alpha}$(α: real), and $b = \int udm$. Then we have*

$$\int \frac{1-r^2|u|^2}{|e^{i\theta}-ru|^2}dm = \frac{1-r^2|b|^2}{|e^{i\theta}-rb|^2} \quad for \quad 0<r<1\,, \quad e^{i\theta}\in T\,.$$

Using this equality one can prove the following lemma, a generalization of the classical Löwner's lemma.

LEMMA 2.2 [18]. *Let u, b be as in Lemma 2.1. Then for any Lebesgue measurable set E on the unit circle T, we have*

$$\begin{aligned}\int_{|u(x)|<1} dm(x)\int_E \frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}d\theta &= \int_E d\theta \int_{|u(x)|<1} \frac{1-|u(x)|^2}{|e^{i\theta}-u(x)|^2}d\theta \\ &= \int_E \frac{1-|b|^2}{|e^{i\theta}-b|^2}d\theta - 2\pi m\{x: u(x)\in E\}\,.\end{aligned}$$

In particular, we have

$$m\{x: u(x)\in E\} \leqq \frac{1+|b|}{1-|b|}L(E)\,.$$

Further, if $|u|=1$, we have

$$\frac{1-|b|}{1+|b|}L(E) \leqq m\{x: u(x)\in E\} \leqq \frac{1+|b|}{1-|b|}L(E)\,.$$

Using this lemma we have the following result on composed functions.

THEOREM 2.1. *Let $1\leqq p<\infty$. Let $u\in H$, non-constant, $|u|=1$ and $b=\int udm$. Then for any $f\in L^p(T)$ the composed function $f(u(x))$ is well-defined and we have*

(i) $f(u(x)) = \lim_{r\to 1}\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}\frac{1-r^2}{|e^{i\theta}-ru(x)|^2}f(e^{i\theta})d\theta$ *a.e. and in L^p-means.*

(ii) $\displaystyle\int f(u)dm = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-|b|^2}{|e^{i\theta}-b|^2}f(e^{i\theta})d\theta\,.$

(iii) $\left(\dfrac{1-|b|}{1+|b|}\right)^{1/p}\|f\|_p \leqq \|f(u)\|_p \leqq \left(\dfrac{1+|b|}{1-|b|}\right)^{1/p}\|f\|_p\,.$

(iv) *The above constants in both sides are the best possible ones.*

PROOF. By Lemma 2.2 $f(u(x))$ is well-defined. (i) Let $f(r, e^{i\theta})$ be the Poisson integral of f, i.e., $f(r, e^{i\theta}) = 1/(2\pi)\int_{-\pi}^{\pi}(1-r^2)|e^{i\beta}-re^{i\theta}|^{-2}f(e^{i\beta})d\beta$. Then it is well-known that $f(r, e^{i\theta})$ tends to $f(e^{i\theta})$ almost everywhere as $r\to 1$. Hence again by Lemma 2.2 $f(r, u(x))$ tends to $f(u(x))$ a.e. as $r\to 1$. We write next the Poisson integral of $f(r, e^{i\beta})$ as $f(s, r, e^{i\theta})$. Then, since $f(s, r, e^{i\theta}) = f(sr, e^{i\theta})$ $(0<s, r<1)$, we have $f(r, u(x)) = f(s, r/s, u(x))$ if $r<s<1$. Hence it holds for $0<r, s<t<1$

$$f(r, u(x)) - f(s, u(x)) = f(t, r/t, u(x)) - f(t, s/t, u(x))$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-t^2}{|e^{i\theta}-tu(x)|^2}[f(r/t, e^{i\theta}) - f(s/t, e^{i\theta})]d\theta .$$

Therefore, we have by Cauchy-Schwarz inequality and by Lemma 2.1

$$\|f(r, u(x)) - f(s, u(x))\|_p \leqq \left(\frac{1+|b|}{1-|b|}\right)^{1/p}\|f(r/t, e^{i\theta}) - f(s/t, e^{i\theta})\|_p$$
$$\leqq \left(\frac{1+|b|}{1-|b|}\right)^{1/p}\|f(r, e^{i\theta}) - f(s, e^{i\theta})\|_p .$$

Since $f(r, e^{i\theta})$ tends to $f(e^{i\theta})$ in $L^p(T)$ as $r \to 1$, $f(r, u(x))$ also tends to $f(u(x))$ in $L^p(m)$. Thus we have proved (i). (ii) follows immediately from (i), and (iii) follows immediately from (ii). We shall show next (iv). That the constant $((1-|b|)/(1+|b|))^{1/p}$ is the best possible one in the left inequality can be shown for example as follows: Consider the functions $f_r(e^{i\theta}) = [(1-r^2)(b|b|^{-1} + re^{i\theta})^{-2}]^{1/p}$. Then by Lemma 2.1 we have

$$\|f_r(u(x))\|_p^p = \frac{1-r^2|b|^2}{|b|b|^{-1}+rb|^2} = \frac{1-r|b|}{1+r|b|} = \frac{1-r|b|}{1+r|b|}\|f_r(e^{i\theta})\|_p^p .$$

Letting $r \to 1$, we see that the constant is the best possible one. In the same way we see that the constant in the right side is also the best possible one. We have thus completed the proof.

REMARK 2.1. The equality (ii) itself can be derived as a special case from the more general result on composed functions by Mürmann [9]. But it seems to us that (i) is often useful.

REMARK 2.2. Under the situation of Theorem 2.1 it holds $\|f\|_p = \|f(u(x))\|_p$ for all $f \in H^p(T)$ for some $0 < p < \infty$ if and only if $\int u\,dm = 0$. In fact, we have $f(z) = (1 + \bar{b}z/|b|)^{2/p} \in H^p(U)$ and $\|f(u)\|_p^p = 2 + 2|b| > 2 = \|f\|_p^p$ if $b = \int u\,dm \neq 0$. This shows the only if-part. The if-part is immediate from (iii).

We reformulate (iii) in Theorem 2.1 ($p = 2$) as a result on l^2 sequences as follows.

COROLLARY 2.1. *Let b be a complex number with $|b| < 1$. Then we have for any $\{a_i\} \in l^2$*

$$\frac{1-|b|}{1+|b|}\left(\sum_{i=-\infty}^{\infty}|a_i|^2\right) \leqq \left|\sum_{i,j=-\infty}^{\infty} a_i\bar{a}_j b^{*|i-j|}\right| \leqq \frac{1+|b|}{1-|b|}\left(\sum_{i=-\infty}^{\infty}|a_i|^2\right),$$

where $b^* = b$ if $i \geqq j$ and $= \bar{b}$ if $i < j$.

We next consider the case $|u| \leqq 1$. We formulate as follows.

THEOREM 2.2. *Let $1 \leqq p < \infty$. Let $u \in H$, $|u| \leqq 1$, $\neq e^{i\alpha}$(α: real) and $b = \int udm$. Suppose further $f \in L^p(T)$ and let $f(r, e^{i\theta})$ be its Poisson integral. Then the composed function $f(|u(x)|, u(x)/|u(x)|)$ is well-defined and we have*

(i) $$f(|u(x)|, u(x)/|u(x)|) = \lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1 - r^2|u(x)|^2}{|e^{i\theta} - ru(x)|^2}f(e^{i\theta})d\theta$$

a.e. and in L^p-means.

(ii) $$\|f(|u|, u/|u|)\|_p \leqq \left(\frac{1 + |b|}{1 - |b|}\right)^{1/p}\|f\|_p .$$

(iii) *The constant $\left(\frac{1 + |b|}{1 - |b|}\right)^{1/p}$ is the best possible one.*

PROOF. The proof of (i) follows along the same lines as that of Theorem 2.1 (i). (ii) Set

$$f_r(u(x)) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1 - r^2|u(x)|^2}{|e^{i\theta} - ru(x)|^2}f(e^{i\theta})d\theta .$$

Then we have by Cauchy-Schwarz inequality

$$|f_r(u(x))| \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1 - r^2|u(x)|^2}{|e^{i\theta} - ru(x)|^2}|f(e^{i\theta})|^p d\theta\right)^{1/p}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1 - r^2|u(x)|^2}{|e^{i\theta} - ru(x)|^2}d\theta\right)^{1/p'} ,$$

where $1/p + 1/p' = 1$. Since the last term is equal to 1, we see in virtue of Lemma 2.1 and Fubini's theorem that

$$\|f_r(u)\|_p \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1 - r^2|b|^2}{|e^{i\theta} - rb|^2}|f(e^{i\theta})|^p\, d\theta\right)^{1/p} \leqq \left(\frac{1 + |b|}{1 - |b|}\right)^{1/p}\|f\|_p .$$

Letting $r \to 1$ we have (ii) by Fatou's lemma. (iii) Consider the functions $f_r(e^{i\theta}) = [(1 - r^2)/|b|b|^{-1} - re^{i\theta}|^2]^{1/p}$. Then we have $f_r(|u|, u/|u|) = [(1 - r^2|u|^2)/|b|b|^{-1} - ru|^2]^{1/p}$ and hence via Lemma 2.1

$$\|f_r(|u|, u/|u|)\|_p^p = \int\frac{1 - r^2|u|^2}{|b/|b| - ru|^2}dm = \frac{1 - r^2|b|^2}{|b/|b| - rb|^2} = \frac{1 + r|b|}{1 - r|b|}\|f_r\|_p^p .$$

Letting $r \to 1$, we see that the constant $((1 + |b|)/(1 - |b|))^{1/p}$ is the best possible one. This completes the proof.

Now restricting Theorem 2.2 to the spaces $H^p(T)$ or $H^p(U)$ we have

THEOREM 2.3. *Let* $0 < p \leqq \infty$. *Let* $u \in H$, $|u| \leqq 1$, $\neq e^{i\alpha}$(α: *real*) *and* $b = \int u dm$. *Then for any* $f(z) \in H^p(U)$ *we have*

(i) $f(u(x))$ *is well-defined and in* H^p.

(ii) $\|f(u)\|_p \leqq \left(\frac{1+|b|}{1-|b|}\right)^{1/p} \|f\|_p$.

(iii) *If* $1 \leqq p \leqq \infty$, $\int f(u)dm = f(b)$.

PROOF. (1) The case $1 \leqq p < \infty$. By Theorem 2.2 $f(u(x))$ is well-defined and $\|f(u)\|_p \leqq ((1+|b|)/(1-|b|))^{1/p}\|f\|_p$. Now clearly $f(ru) \in H$ and we have

$$f(ru) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2|u|^2}{|e^{i\theta}-ru|^2} f(e^{i\theta})d\theta .$$

Hence by Theorem 2.2 $f(ru)$ tends to $f(u)$ in L^p and so $f(u)$ is in H^p by definition and further

$$\int f(u)dm = \lim_{r\to 1}\int f(ru)dm = \lim_{r\to 1} f\left(r\int u dm\right) = f\left(\int u dm\right),$$

since $b = \int u dm \in U$ (this follows from $|u| \leqq 1$ and $u \neq e^{i\alpha}$ (α: real)). (2) The case $p = \infty$ is clear from (1). (3) The case $0 < p < 1$. We have by the classical inner-outer factorization theorem $f(z) = g(z)h^{1/p}(z)$ for some inner function $g(z)$ and some outer function $h(z) \in H^1(U)$. Then by Lemma 2.2 $f(u)$, $g(u)$ and $h(u)$ are well-defined and it holds $f(u) = g(u)h^{1/p}(u)$. Hence we see easily that $f(u) \in H^p$ and

$$\|f(u)\|_p^p \leqq \int |h(u)|\, dm \leqq \frac{1+|b|}{1-|b|}\|h\|_1 = \frac{1+|b|}{1-|b|}\|f\|_p^p .$$

This completes the proof.

REMARK 2.3. If $|u| = 1$, we have the same inequality as in Theorem 2.1 (iii) and the constants are the best, which is known in the classical case [11]. Indeed, since the positive valued function $f_r(e^{i\theta})$ in the proof of Theorem 2.1 (iv) is in $L^p(T)$ and $\log f_r$ is in $L^1(T)$, we have $f_r(e^{i\theta}) = |g_r(e^{i\theta})|$ for some $g_r \in H^p(U)$ as is well-known. The rest of the proof follows along the same lines as that of Theorem 2.1 (iv).

REMARK 2.4. If $\int u dm = 0$, the constant in (ii) in Theorem 2.3 is the best possible one. If $\int u dm \neq 0$, we do not know whether that constant is the best. But the following example shows that the best possible constant is greater than 1. Example: Let $b = \int udm \neq 0$ and a be a positive number satisfying $2a|b| \geqq 1$. Put $f(z) = (a + \bar{b}z/|b|)^{2/p}$. Then we have $f(z) \in H^p(U)$ and $||f(u)||_p^p = \int |a + |b| + \bar{b}u/|b| - |b||^2 dm = (a + |b|)^2 + \int |\bar{b}u/|b| - |b||^2 dm > a^2 + 2a|b| + |b|^2 > a^2 + 1 = ||f||_p^p$.

We note next that we have also a generalization of a theorem of Ryff with a slight modification of his proof. We state it without proof.

THEOREM 2.4. *Let $0 < p < \infty$. Let $u \in H$, $|u| \leqq 1$ and $\int udm = 0$. Then in order $||f(u)||_p = ||f||_p$ for some non-constant $f \in H^p(U)$ it is necessary and sufficient that $|u| = 1$.*

From the above theorem and Remarks 2.3 and 2.4 we can deduce the following.

COROLLARY 2.2. *Let $0 < p < \infty$. Let $u \in H$, $|u| \leqq 1$ and $u \neq e^{i\alpha}$ (α: real). Then in order $||f(u)||_p = ||f||_p$ for all $f \in H^p(U)$ it is necessary and sufficient that $|u| = 1$ and $\int udm = 0$.*

3. General case.

Let D be a Jordan domain and g a conformal mapping of D onto U. Then g can be extended continuously to $\bar{D}$ and if we denote this extended function also by g, g maps $\bar{D}$ onto $\bar{U}$ topologically. Hence if $u \in H$ and $m\{x: u(x) \in \bar{D}\} = 1$, the composed function $g(u(x))$ is well-defined. Further $g(u)$ lies in H. In fact, by Walsh's theorem or by Mergelyan's theorem, there is a sequence of polynomials $P_n(z)$ such that $P_n(z) \to g(z)$ uniformly on $\bar{D}$. Hence $P_n(u)$ tends to $g(u)$ in L^∞ norm. Since clearly $P_n(u) \in H$, we have $g(u) \in H$ and $\int g(u)dm = g\left(\int udm\right)$. Hence using Theorem 2.3 we can state the following lemma.

LEMMA 3.1. *Let D be a Jordan domain and let an $a \in D$ be fixed. Let $f(z) \in A(\bar{D})$. Then if $u \in H$ satisfies $m\{x: u(x) \in \bar{D}\} = 1$, the composed function $f(u)$ lies in H and we have for any $0 < p \leqq \infty$*

$$||f(u)||_p \leqq C^{1/p} ||f||_{H^p(D)},$$

and

$$\int f(u)dm = f\left(\int udm\right),$$

where C depends only on a and u.

PROOF. We may assume that u is not constant, since otherwise the conclusion is trivial. Now let $g(z)$ be the conformal mapping of D onto U satisfying $g(a) = 0$ and $g'(a) > 0$. Then we have by the above argument $g(u) \in H$ and $\int g(u)dm = g\left(\int udm\right)$. Let $b = \int udm$. Now as $f(u) = f \circ g^{-1}(g(u))$ and $f \circ g^{-1} \in A(\bar{U})$ we get by Theorem 2.3 $f(u) \in H$ and

$$\|f(u)\|_p \leqq \left(\frac{1 + |g(b)|}{1 - |g(b)|}\right)^{1/p} \|f \circ g^{-1}\|_{H^p(U)} = \left(\frac{1 + |g(b)|}{1 - |g(b)|}\right)^{1/p} \|f\|_{H^p(D)} .$$

Here $g(b)$ depends only on a and u. Next by Theorem 2.3 (iii) we have

$$\int f(u)dm = f \circ g^{-1}\left(\int g(u)dm\right) = f \circ g^{-1}\left(g\left(\int udm\right)\right) = f\left(\int udm\right) .$$

This completes the proof.

LEMMA 3.2. *Let D be a Carathéodory domain and $\hat{\bar{D}}$ the polynomial convex hull of the closure of D. Let an $a \in D$ be fixed and g a conformal mapping of D onto U satisfying $g(a) = 0$. Further suppose $u \in H$ with $m\{x: u(x) \in \bar{D}\} = 1$ and $\int udm \in D$. Then for any $f \in A(\hat{\bar{D}})$ it holds $f(u) \in H$, $\int f(u)dm = f\left(\int udm\right)$ and for every $0 < p \leqq \infty$*

$$\|f(u)\|_p \leqq \left(\frac{1 + |g(b)|}{1 - |g(b)|}\right)^{1/p} \|f\|_{H^p(D)} ,$$

where $b = \int udm$.

PROOF. Since $(\hat{\bar{D}})^c$ is connected, we have $A(\hat{\bar{D}}) = P(\hat{\bar{D}})$. Hence we may assume f is a polynomial. Then $f(u) \in H$ and $\int f(u)dm = f\left(\int udm\right)$ are obvious. Now there exists a strictly decreasing sequence of Jordan domains $\{D_j\}$ converging to D. Let g_j be the conformal mappings of D_j onto U such that $g_j(a) = 0$ and $g_j'(a) > 0$. Then by Lemma 3.1 we have

$$\|f(u)\|_p \leqq \left(\frac{1 + |g_j(b)|}{1 - |g_j(b)|}\right)^{1/p} \|f\|_{H^p(D_j)} .$$

Since $g_j(z)$ converges to $g(z)\,|g'(a)|/g'(a)$ by Carathéodory's theorem, and since $\|f\|_{H^p(D_j)}$ tends to $\|f\|_{H^p(D)}$ by Corollary 1.1, we obtain

$$\|f(u)\|_p \leqq \left(\frac{1 + |g(b)|}{1 - |g(b)|}\right)^{1/p} \|f\|_{H^p(D)} .$$

That completes the proof.

LEMMA 3.3. *Let D, a, g, u be as in Lemma 3.2. Further let $g'(a) > 0$ and $\{D_j\}$ be a strictly decreasing sequence of bounded simply connected* *domains converging to D and g_j the conformal mappings of D_j onto U such that $g_j(a) = 0$ and $g_j'(a) > 0$. Then $\{g_j(u)\}$ is a Cauchy sequence in L^p for every $1 \leqq p < \infty$. In particular, $g_j(u)$ converges to the same element of H for every choice of $\{D_j\}$. If we write this element as $g[u]$, it holds $\int g[u]dm = g\left(\int udm\right)$.*

PROOF. By Lemma 3.2 we have

$$\| g_j(u) - g_k(u) \|_p \leqq C^{1/p} \| g_j - g_k \|_{H^p(D)} ,$$

since $D_j \supset \hat{\bar{D}}$ and $g_j \in A(\bar{D}_j)$. Hence by Theorem 1.2 $\{g_j(u)\}$ is a Cauchy sequence in L^p for any $1 \leqq p < \infty$. Let us write its L^p limit as $g[u]$. Then, since $\| g_j(u) \|_\infty \leqq 1$ we have $g[u] \in H$ and $|g[u]| \leqq 1$. Clearly $g[u]$ is independent of the choice of $\{D_j\}$. Now we have

$$\int g[u]dm = \lim_{j\to\infty} \int g_j(u)dm = \lim_{j\to\infty} g_j\left(\int udm\right) = g\left(\int udm\right) ,$$

which completes the proof.

LEMMA 3.4. *Let D, a, g, u be as in Lemma 3.3. Further let ϕ be the inverse conformal mapping of U onto D satisfying $\phi(0) = a$ and $\phi'(0) > 0$. Then for the composed function $\phi(g[u])$ it holds*

$$\phi(g[u]) = u .$$

PROOF. Let $\{D_j\}$, g_j be as in Lemma 3.2. Then by Lemma 3.2 we get for $1 \leqq p < \infty$

$$(*) \qquad \begin{aligned} \| \phi(g_j(u)) - u \|_p &\leqq C^{1/p} \| \phi \circ g_j(z) - z \|_{H^p(D)} \\ &= C^{1/p} \| \phi \circ g_j - \phi \circ g \|_{H^p(D)} . \end{aligned}$$

By Corollary 1.2 $\phi \circ g_j$ converges to $\phi \circ g$ in $H^p(U)$. Hence by $(*)$ $\phi(g_j(u))$ converges to u in H^p and boundedly. On the other hand, since $g_j(u) \to g[u]$ in H^p, $g_j(u) \to g[u]$ a.e. by taking a subsequence if necessary. Hence $\phi(g_j(u))$ converges to $\phi(g[u])$ a.e.. Therefore we have $\phi(g[u]) = u$. That proves the lemma.

Now noting that $(\hat{\bar{D}})^c$ is connected and D is a component of $(\hat{\bar{D}})^0$ for a Carathéodory domain D, we are in the position to state the following theorem.

THEOREM 3.1. *Let D be a Carathéodory domain and an $a \in D$ be fixed. Let ϕ be the conformal mapping of U onto D satisfying $\phi(0) = a$ and $\phi'(0) > 0$ and g its inverse conformal mapping of D onto U. Further let E be a set defined in Lemma 1.2 for ϕ. Then if $u \in H$ is not constant, $m\{x: u(x) \in \bar{D}\} = 1$ and $b = \int udm \in D$, we have*

(i) $m\{x: u(x) \in D \cup \phi(E)\} = 1$.

(ii) *For every m_a (the harmonic measure with respect to a)-measurable set G on ∂D it holds*

$$m\{x: u(x) \in G\} \leqq \int_G \frac{1 - |g(b)|^2}{|g(z) - g(b)|^2} dm_a(z) \leqq \frac{1 + |g(b)|}{1 - |g(b)|} m_a(G) .$$

In particular, if $m\{x: u(x) \in \partial D\} = 1$, it holds

$$m\{x: u(x) \in G\} = \int_G \frac{1 - |g(b)|^2}{|g(z) - g(b)|^2} dm_a(z) = \frac{1}{2\pi} \int_{g(G \cap \phi(E))} \frac{1 - |g(b)|^2}{|e^{i\theta} - g(b)|^2} d\theta$$

and

$$\frac{1 - |g(b)|}{1 + |g(b)|} m_a(G) \leqq m\{x: u(x) \in G\} \leqq \frac{1 + |g(b)|}{1 - |g(b)|} m_a(G) .$$

(iii) *The composed function $g(u)$ is well-defined and coincides with the function $g[u]$ in Lemma 3.3, and hence lies in H.*

(iv) *If h is another conformal mapping of D onto U, h is extendable to $D \cup \phi(E)$ in the sense of Lemma 1.2 and the composed function $h(u)$ also lies in H and it holds $h(u) = h \circ \phi(g(u))$ and $\int h(u) dm = h\left(\int u dm\right)$.*

We remark first that $g(u)$ is independent of the choice of set E.

PROOF. Since $L(T \backslash E) = 0$, we have by Lemma 2.2

$$m\{x: g[u](x) \in U \cup E\} = 1 .$$

Since $\phi(g[u]) = u$ by Lemma 3.4, we get thus

$$m\{x: u(x) \in D \cup \phi(E)\} = 1 .$$

Now for any harmonically measurable set $G \subset \partial D$ we have

$$m\{x: u(x) \in G\} = m\{x: u(x) \in G \cap \phi(E)\} = m\{x: g[u](x) \in g(G \cap \phi(E))\} .$$

Hence by Lemma 2.2 we have setting $F = g(G \cap \phi(E))$

$$m\{x: u(x) \in G\} \leqq \frac{1}{2\pi} \int_F \frac{1 - |g(b)|^2}{|e^{i\theta} - g(b)|^2} d\theta \leqq \frac{1 + |g(b)|}{1 - |g(b)|} L(F) .$$

In particular, if $m\{x: u(x) \in \partial D\} = 1$, we have

$$m\{x: u(x) \in G\} = \frac{1}{2\pi} \int_F \frac{1 - |g(b)|^2}{|e^{i\theta} - g(b)|^2} d\theta .$$

Since the function $(1 - |g(b)|^2)/|g(z) - g(b)|^2$ is a bounded Borel function on $\phi(E)$, we have by Lemma 1.2 (v)

$$\frac{1}{2\pi} \int_F \frac{1 - |g(b)|^2}{|e^{i\theta} - g(b)|^2} d\theta = \int_{G \cap \phi(E)} \frac{1 - |g(b)|^2}{|g(z) - g(b)|^2} dm_a(z) = \int_G \frac{1 - |g(b)|^2}{|g(z) - g(b)|^2} dm_a(z)$$

for a Borel set G and hence for any measurable set G. We have also

$$m_a(G) = m_a(G \cap \phi(E)) = \int_{G \cap \phi(E)} dm_a(z) = \frac{1}{2\pi}\int_F d\theta = L(F) .$$

Next by (ii) the composed function $g \circ u$ is defined almost everywhere, and it holds $m\{x: g(u(x)) \in E \cup U\} = 1$. By Lemmas 1.2 (i) and 3.4 we have $\phi(g[u]) = \phi(g(u))$. Since ϕ is one to one on $E \cup U$, we have $g[u] = g(u)$ a.e.. The proof of (iv) is clear from the remark to Lemma 1.2. This establishes the theorem.

Now we can sharpen Lemma 3.2.

THEOREM 3.2. *Let D be a Carathéodory domain. Then if $u \in H$ with $m\{x: u(x) \in \bar{D}\} = 1$ is non-constant and $\int udm \in D$, and if $f \in A(\bar{D})$, the composed function $f(u)$ is in H and it holds $\int f(u)dm = f\left(\int udm\right)$. Further if we fix a point $a \in D$ and g is a conformal mapping of D onto U satisfying $g(a) = 0$, then we have for any $0 < p \leqq \infty$*

$$\|f(u)\|_p \leqq \left(\frac{1 + |g(b)|}{1 - |g(b)|}\right)^{1/p} \|f\|_{H^p(D)} .$$

PROOF. We suppose first $g'(a) > 0$. Let $\int udm = b$. Then by Theorem 3.1 we have $f(u(x)) = f \circ g^{-1}(g(u(x)))$ a.e.. Since $g(u) \in H$, $|g(u)| \leqq 1$ and $f \circ g^{-1} \in H^\infty(U)$, we have by Theorem 2.3 $f(u) \in H$ and for $0 < p \leqq \infty$

$$\|f(u)\|_p \leqq \left(\frac{1 + |g(b)|}{1 - |g(b)|}\right)^{1/p} \|f \circ g^{-1}\|_{H^p(U)}$$

which gives the desired inequality for this special g, since $\|f \circ g^{-1}\|_{H^p(U)} = \|f\|_{H^p(D)}$. Now if g_1 is a conformal mapping of D onto U satisfying $g_1(a) = 0$, then we have $g_1 = e^{i\alpha}g$ for some real α and hence we have the desired inequality. This completes the proof.

From this theorem one can easily deduce that the spectrum of u is contained in $\bar{D}$. But on the consequences of this type we shall discuss in the next section. Another consequence is the following.

THEOREM 3.3. *Let $0 < p \leqq \infty$. Let D be a Carathéodory domain and an $a \in D$ be fixed. Let g be the conformal mapping of D onto U satisfying $g(a) = 0$ and $g'(a) > 0$. Further suppose $u \in H$ is non-constant, $m\{x: u(x) \in \bar{D}\} = 1$ and $b = \int udm \in D$. Define τ as a linear mapping from $H^p(D)$ into H^p by: $\tau(f) = f \circ g^{-1}(g(u))$. Then τ is a bounded linear operator from $H^p(D)$ into H^p, more precisely we have*

$$\| \tau(f) \|_p \leqq \left(\frac{1 + |g(b)|}{1 - |g(b)|} \right)^{1/p} \| f \|_{H^p(D)} .$$

If in particular $p = \infty$, this is an algebraic homomorphism from $H^\infty(D)$ into H.

The proof is clear.

3. Spectrum and operating functions. In this paragraph we shall investigate the spectrum of a $u \in H$ and then give an anologous result on operating functions to the case of group algebras. In our case, however, if K is compact in $\boldsymbol{C}$, $K^0 = \varnothing$ and K^c is connected, and if $u \in H$ satisfies $m\{x: u(x) \in K\} = 1$, then u is necessarily constant by Corollary 1 in [19]. Hence we consider the case when the domain of an operating function is open or a compact set with non-empty interior.

4. On the spectrum of a non-constant $u \in H$ we have the following result by combining Theorem A in [19] and Theorem 3.2.

THEOREM 4.1. *Let $u \in H$ be non-constant. Then there is a unique Carathéodory domain A such that $m\{x: u(x) \in \bar{A}\} = 1$, $\int udm \in A$, and for any $\varepsilon > 0$ and $a \in \partial A$ it holds $m\{x: |u(x) - a| < \varepsilon\} > 0$. Further the spectrum $\sigma(u)$ of u is contained in $\bar{A}$ and every point of ∂A belongs to $\sigma(u)$.*

PROOF. The first assertion is a version of Theorem A. It holds further $\partial \hat{\bar{A}} = \partial A$, since A is a Carathéodory domain. Hence we get $(\hat{\bar{A}})^0 \backslash A = \hat{\bar{A}} \backslash \bar{A}$. Let $a \in \boldsymbol{C} \backslash \hat{\bar{A}}$. Then the function $(z - a)^{-1}$ is holomorphic on a neighborhood of $\hat{\bar{A}}$. Hence by Theorem 3.2 we obtain $(u - a)^{-1} \in H$. This means that a is a point of the resolvent, and hence it follows $\sigma(u) \subset \hat{\bar{A}}$. The last assertion follows from the first one. This establishes the theorem.

In the same way we can show the following

COROLLARY 4.1. *Let D be a Carathéodory domain and $u \in H$ with $m\{x: u(x) \in \bar{D}\} = 1$. Then if $\int udm \in \bar{D}$, the spectrum of u is contained in $\hat{\bar{D}}$.*

The case $\int udm \in D$ and u is non-constant is clear from the proof of the above theorem. In the other case u must be constant by Lemma 1 of [19], and the assertion is obvious.

REMARK 4.1. From the assumption $m\{x: u(x) \in \bar{D}\} = 1$ we can conclude only that the spectrum is contained in $\hat{\bar{D}}$. In the above corollary the condition $\int u dm \in \bar{D}$ is not superfluous. For instance, let D be the cornucopia, which is a ribbon winding the outside of the unit circle and accumulating on that circle. Then D is a Carathéodory domain. Let H be $H^\infty(T)$ and $u(z)$ the identity function. Then $L\{u(e^{i\theta}) \in \bar{D}\} = 1$ and $\int u dL = u(0) = 0$, and that $\sigma(u) = \{|z| \leqq 1\}$.

Next we give a definition.

DEFINITION 4.1. Let $\mathfrak{M} = \mathfrak{M}(H)$ be the maximal ideal space of H. Let D be a set in the complex plane and $f(z)$ an everywhere defined function on D. $f(z)$ is said to operate on H (with respect to D) if for every $u \in H$ with $m\{x: u(x) \in D\} = 1$ the composed function $f(u)$ belongs to H and $\Phi(f(u)) = f(\Phi(u))$ for any $\Phi \in \mathfrak{M}$. We say that $f(z)$ operates conditionally on H (with respect to D) if for every $u \in H$ with $m\{x: u(x) \in D\} = 1$ and $\int u dm \in D$ the composed function $f(u)$ belongs to H and $\Phi(f(u)) = f(\Phi(u))$ for any $\Phi \in \mathfrak{M}$.

We are now able to state another consequence of Theorem 3.2.

COROLLARY 4.2. *Let D be a Carathéodory domain. Then every $f(z) \in A(\bar{D})$ operates conditionally on H.*

PROOF. Since D is a Carathéodory domain, $D_\infty = (\hat{\bar{D}})^c$ is simply connected, and $P(\hat{\bar{D}}) = R(\hat{\bar{D}})$ is a Dirichlet algebra, and so $R(\bar{D})$ is also a Dirichlet algebra on ∂D, since $R(\bar{D}) \supset R(\hat{\bar{D}})|_{\bar{D}}$. Hence we have $R(\bar{D}) = A(\bar{D})$. Now let $\Phi \in \mathfrak{M}$, $f(z) \in A(\bar{D})$ and $u \in H$ with $m\{x: u(x) \in \bar{D}\} = 1$ and $\int u dm \in D$. Then there is a sequence of rational functions $\{f_n(z)\}$ with poles off $\bar{D}$ converging to $f(z)$ uniformly on $\bar{D}$. For every rational function $h(z)$ with poles off $\bar{D}$ we have by a well-known theorem $h(u) \in H$ and $\Phi(h(u)) = h(\Phi(u))$, since the spectrum of u is contained in $\bar{D}$ by Corollary 4.1. Again by Theorem 3.2 we get $f(u) \in H$. Therefore we have

$$\Phi(f(u)) = \lim_{n\to\infty} \Phi(f_n(u)) = \lim_{n\to\infty} f_n(\Phi(u)) = f(\Phi(u)) .$$

If u is constant, then we have trivially $f(u) \in H$ and $\Phi(f(u)) = f(\Phi(u))$. If u is not constant, we have $\int u dm \in D$ by Lemma 1 in [19]. Hence the proof is completed.

We shall next state a similar result to the above as a lemma which we shall use later.

LEMMA 4.1. *Let K be a compact set in the complex plane whose complement is connected and whose interior K^0 is non-empty. Let D be a component of K^0. If $f(z)$ is bounded and holomorphic in D, then it holds $f(u) \in H$ and $\int f(u)dm = f\left(\int udm\right)$ for all $u \in H$ with $m\{x: u(x) \in D\} = 1$.*

PROOF. Let $u \in H$ with $m\{x: u(x) \in D\} = 1$. Then by Lemma 2 of our former work [19] we have $\int udm \in D$. Now by Farrell-Rubel-Shields theorem (Gamelin [3, p. 154]) there exists a sequence of polynomials $\{P_n(z)\}$ such that $P_n(z) \to f(z)$ for all $z \in D$ and $|P_n(z)| \leqq \sup_{\zeta \in D} |f(\zeta)|$ for all $z \in D$. Since clearly $P_n(u) \in H$ and H is weak* closed, we have $f(u) \in H$ and also $\int f(u)dm = f\left(\int udm\right)$. This completes the proof.

When D is a bounded simply connected domain, the boundedness of an operating function will be shown.

LEMMA 4.2. *Suppose $H \neq C$. Let D be a bounded simply connected domain. Then if $f(z)$ is holomorphic on D and $f(u) \in H$ for all $u \in H$ with $m\{x:u(x) \in D\} = 1$ and $\int udm \in D$, it follows that $f(z)$ is bounded on D.*

PROOF. Combining Theorem 3.3 and Theorem 4.1, for every Carathéodory domain G there exists a $u \in H$ such that $m\{x: u(x) \in \bar{G}\} = 1$, $\int udm \in G$, and for any $\varepsilon > 0$ and any $a \in \partial G$ it holds $m\{x: |u(x) - a| < \varepsilon\} > 0$. Now let g be a conformal mapping of U onto D. We assume that the conclusion is false, i.e., $f(z)$ is unbounded. Then $f \circ g(z)$ is also unbounded in U. Hence there is a point $a \in \partial U$ and a Jordan curve $J = J_1 + J_2$ such that J_1, J_2 are Jordan arcs with common endpoint a and

$$(*) \qquad \limsup_{\substack{z \to a \\ z \in J_1}} |f \circ g(z)| = \infty \,.$$

Let G be the Jordan domain bounded by J and u a corresponding $u \in H$ pointed out above. Since g is bounded and holomorphic on U, $g(u)$ is in H by Lemma 4.1 and further $m\{x: g(u(x)) \in D\} = 1$ and $\int g(u)dm = g\left(\int udm\right) \in D$. Hence we have $f(g(u)) \in H$ by the assumption. On the other hand, by the third property of u and $(*)$ the function $f(g(u))$ is not bounded, which contradicts $f(g(u)) \in H$. That proves the lemma.

When there exists a non-constant $u \in H$ with $|u| = 1$ we can show a converse of Lemma 4.1.

THEOREM 4.2. *Let D be an open set in the complex plane, $f(z)$ an everywhere defined locally integrable function on D and suppose there* *exists a non-constant* $u \in H$ *with* $|u| = 1$. *Then, if* $f(v) \in H$ *for all* $v \in H$ *with* $m\{x: v(x) \in D\} = 1$ *and* $\int v dm \in D$, *it follows that* $f(z)$ *is Lebesgue-almost everywhere equal to a function holomorphic on* D.

PROOF. Considering the function $\left(u - \int u dm\right)\Big/\left(1 - u\int \bar{u} dm\right)$, we may assume $\int u dm = 0$. Hence by Lemma 2.2 we have for any Lebesgue measurable set E on the circle T (1) $m\{x: u(x) \in E\} = L(E)$. By assumption $f(z)$ is locally integrable. Now let Q be a rectangle with its sides parallel to the axes such that its closure lies in D and

$$\int_{\partial Q} |f(z)|\,|dz| < \infty \,.$$

Further let g be a conformal mapping of U onto Q. Then, since ∂Q is a rectifiable curve, we have $g'(z) \in H^1(U)$ and hence by Theorem 2.3 $g'(u) \in H^1$. Next we have the following equality.

$$\int_{\partial Q} f(z)dz = \int_{\partial U} f(g(w))g'(w)dw = i\int_0^{2\pi} f(g(e^{i\theta}))g'(e^{i\theta})e^{i\theta}d\theta \,.$$

Combining this with (1) we obtain further

$$\int_{\partial Q} f(z)dz = 2\pi i \int f(g(u))g'(u)u dm = 2\pi i \int f(g(u))g'(u)dm \int u dm = 0 \,,$$

since $g(u) \in H$, $g'(u) \in H^1$ and hence by assumption $f(g(u)) \in H$. Hence by a generalization of Morera's theorem (Royden [12]), $f(z)$ is almost everywhere equal to a function holomorphic in D. That proves the theorem.

The next corollary is then trivial.

COROLLARY 4.3. *Let* D *be an open set in the complex plane,* $f(z)$ *a continuous function on* D *and suppose there exists a non-constant* $u \in H$ *with* $|u| = 1$. *Then, if* $f(v) \in H$ *for all* $v \in H$ *with* $m\{x: v(x) \in D\} = 1$ *and* $\int v dm \in D$, *it follows that* $f(z)$ *is holomorphic on* D.

Under the following additional assumption on f the local boundedness of f follows and one can show the continuity of f.

COROLLARY 4.4. *Let* D *be an open set in the complex plane,* $f(z)$ *an everywhere defined measurable function on* D *and suppose there exists a non-constant* $u \in H$ *with* $|u| = 1$. *Then, if* $f(v) \in H$ *and* $\int f(v)dm = f\left(\int v dm\right)$ *for all* $v \in H$ *with* $m\{x: v(x) \in D\} = 1$ *and* $\int v dm \in D$, *it follows that* $f(z)$ *is holomorphic on* D.

PROOF. We may assume $\int udm = 0$. We have only to show that $f(z)$ is continuous on D. Now fix a point $z_0 \in D$ arbitrarily and let $2R$ be the supremum of radii r such that the discs $C(z_0; r)$ with radii r and centers z_0 are contained in D. For every $a \in U$ we set $u_a = (u + a)/(1 + \bar{a}u)$. Then by Lemma 2.2 we get for every Lebesgue measurable set E on the unit circle

$$(*) \qquad \frac{1 - |a|}{1 + |a|} L(E) \leqq m\{x: u_a(x) \in E\} \leqq \frac{1 + |a|}{1 - |a|} L(E) ,$$

since $u_a \in H$, $|u_a| = 1$ and $\int u_a dm = a$. Since $m\{x: z_0 + Ru_a(x) \in D\} = m\{x: |u_a(x)| = 1\} = 1$ and $\int (z_0 + Ru_a)dm = z_0 + Ra \in D$, we have by assumption $f(z_0 + Ru_a) \in H$ and

$$\int f(z_0 + Ru_a)dm = f\left(\int (z_0 + Ru_a)dm\right) = f(z_0 + Ra) .$$

Hence we have

$$|f(z_0 + Ra)| \leqq \|f(z_0 + Ru_a)\|_\infty .$$

By $(*)$ we see that $\|f(z_0 + Ru_a)\|_\infty = \|f(z_0 + Ru)\|_\infty$. Hence $f(z)$ is bounded on $C(z_0; R)$. Since f is measurable on D, f is integrable on $C(z_0; R)$, and so by Fubini's theorem $f(z_0 + re^{i\theta})$ is $d\theta$-integrable for almost all $0 < r \leqq R$. For such r we have as in the proof of Theorem 4.2

$$\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = \int f(z_0 + ru(x))dm(x) = f\left(\int (z_0 + ru)dm\right) = f(z_0) .$$

By integrating this equality with respect to rdr we have

$$f(z_0) = \frac{1}{\pi s^2} \int_{C(z_0; s)} f(z_0 + re^{i\theta})rdrd\theta$$

for all $0 < s < R$. The continuity of f then follows immediately from this expression. This completes the proof.

Combining Corollary 4.2 with Corollary 4.3 we have

THEOREM 4.3. *Suppose there exists a non-constant $u \in H$ with $|u| = 1$. Let D be a Carathéodory domain. Then a continuous function $f(z)$ on $\bar{D}$ operates conditionally on H if and only if $f(z)$ is holomorphic on D.*

Combining Lemma 4.1 with Lemma 4.2 and Corollary 4.4 we have

THEOREM 4.4. *Suppose there exists a non-constant $u \in H$ with $|u| = 1$. Let D be a Carathéodory domain. Then an everywhere defined measurable* *function* $f(z)$ *on* D *satisfies* $f(v) \in H$ *and* $\int f(v)dm = f\left(\int vdm\right)$ *for all* $v \in H$ *with* $m\{x: v(x) \in D\} = 1$ *and* $\int vdm \in D$, *if and only if* $f(z)$ *is bounded and holomorphic on* D.

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