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FUNCTIONS OF L^p-MULTIPLIERS II

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1. **Introduction.** Let G be a locally compact abelian group and Γ be the dual to G. Let $1 \leqq p \leqq \infty$. A function ϕ on Γ is called L^p-multiplier if for every f in $L^p(G)$ there exists a function g in $L^p(G)$ such that $\phi\hat{f} = \hat{g}$, where $\hat{f}$ denotes the Fourier transform of f. In this case g will be denoted by $T_\phi f$. The set of all L^p-multipliers will be written by $M_p(\Gamma)$ and the norm of ϕ in $M_p(\Gamma)$ is defined by

$$||\phi||_{M_p(\Gamma)} = \sup \{|| T_\phi f||_{L^p(G)}; ||f||_{L^p(G)} \leqq 1\} .$$

$M_p(\Gamma)$ is a unitary commutative Banach algebra with the product of pointwise multiplication. In the previous paper [3] we have proved the following: *Let Γ be a locally compact non-compact abelian group. Assume $1 \leqq p < 2$ and Φ is a function in $[-1, 1]$. Then $\Phi(\phi) \in M_p(\Gamma)$ for all ϕ in $M_1(\Gamma)$ whose range is contained in $[-1, 1]$, if and only if Φ is extended to an entire function.*

This theorem does not hold if Γ is compact, which is due to Wiener-Lévy theorem. In this paper we restrict our attension to the case when $G = Z$, the integer group. The dual to Z will be denoted by T or $[0, 1)$. Put $m_p(T) = M_p(T) \cap C(T)$, where $C(T)$ is the set of all continuous functions on T. $m_p(T)$ is a closed subalgebra of $M_p(T)$.

Our main object is to prove the following

THEOREM 1. *Assume $1 < q \leqq p < 2$ and Φ is a function in $[-1, 1]$. Then $\Phi(\phi) \in M_p(T)$ for all ϕ in $M_q(T)$ whose range is contained in $[-1, 1]$, if and only if Φ is extended to an entire function.*

THEOREM 2. *Let $1 < p < 2$ and O be any non-empty open set in the real line R. Then there exists a function ϕ in $M_p(R)$ such that $\phi \geqq 1$ in O and $1/\phi$ restricted in O is not contained in the restriction of $M_p(R)$ in O.*

2. **The multiplier** $\exp i\theta(\xi)$. In the following we put $m_j = 2^{2^j}$, $j = 0, 1, 2, \cdots$. Define a function in T by $\theta(\xi) = m_{j+1}\xi$ for $\xi \in [m_j^{-1}/2, m_j^{-1}]$ and $= 0$ outside $\cup [m_j^{-1}/2, m_j^{-1}]$.

THEOREM 3. $\exp 2\pi it\theta(\xi) \in M_q(T)$ *for every $1 < q < \infty$ and the norm* *is uniformly bounded in* $1 \leqq |t| \leqq 2$.

Let k be a function in R such that $\hat{k}(\xi) = \int_{-\infty}^{\infty} k(x) e^{2\pi i \xi x} dx \in C^\infty(R)$, the support of $\hat{k}$ is contained in $(1/4, 5/4)$, $\hat{k}(\xi) = 1$ in $(1/2, 1)$ and $\int_{-\infty}^{\infty} \xi \hat{k}(\xi) d\xi = 0$. Then we have

$$k(x) = O(x^{-3}), \quad k'(x) = O(x^{-3}) \text{ as } x \to \infty \tag{2.1}$$

and

$$k(x) = O(1), \quad k'(x) = O(x) \quad \text{as } x \to 0 . \tag{2.2}$$

For non-negative integer s not of the form 2^j define

$$k_s(x) = \int_{-\infty}^{\infty} \hat{k}(2^s \xi) \exp 2\pi i x \xi d\xi$$

and

$$k_{2^j}(x) = \int_{-\infty}^{\infty} \hat{k}(m_j \xi) \exp 2\pi i (x - m_{j+1} t) \xi d\xi .$$

LEMMA 1. *We have*

$$\sum_{|n| > 2^M + 2} \left(\sum_{s=0}^{\infty} | k_s(n-m) - k_s(n) |^2 \right)^{1/2} < c$$

for all $1 \leqq |t| \leqq 2$ *and* $|m| < 2^M$, $M = 1, 2, 3, \cdots$, *where* c *is a constant not depending on* t *and* M[1].

PROOF. To simplify the notations we put $k_{2^j} = k_j^*$ and prove that

$$\sum_{|n| > 2^M + 2} \left(\sum_{j=0}^{\infty} | k_j^*(n-m) - k_j^*(n) |^2 \right)^{1/2} < c \tag{2.3}$$

for $t = 1$.

Since $k_j^*(x) = m_j^{-1} k(m_j^{-1} x - m_j)$, we get, by (2.1) and (2.2),

$$| k_j^*(n-m) - k_j^*(n) | \leqq \begin{cases} c m_j^{-3} | n - \eta m - m_{j+1} | \, | m | \\ c m_j | n - \eta m - m_{j+1} |^{-3} | m | \end{cases} \tag{2.4}$$

where $0 \leqq \eta \leqq 1$ and

$$| k_j^*(n) | \leqq \begin{cases} c m_j^{-1} \\ c m_j^2 | n - m_{j+1} |^{-3} . \end{cases} \tag{2.5}$$

Let N be a smallest integer such that $2^{M+1} < m_{N+1}$. Put

$$I(N) = \{ n ; 2^{M+2} < |n| \leqq m_{N+2} \}$$

and

$$I(k) = \{ n ; m_{k+1} < |n| \leqq m_{k+2} \} \quad \text{for} \quad k > N .$$

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[1] c will be different in each occasion.

Suppose that $n \in I(k)$. Then $|n - \eta m - m_{j+1}| \leqq |n|/2$ for $0 \leqq j < k$ and $> m_{j+2}/2$ for $j > k + 1$. Note that

$$\left(\sum_{j=0}^{\infty} |k_j^*(n - m) - k_j^*(n)|^2\right)^{1/2} \leqq \sum_{j=0}^{\infty} |k_j^*(n - m) - k_j^*(n)| \,. \tag{2.6}$$

Thus if $n \in I(k)$, by (2.4) and (2.5) the left hand side of (2.6) is bounded by

$$c\sum_{j=0}^{k-1} \frac{m_j}{n^3} |m| + c\sum_{j=k}^{k+1} \min\left(m_j^{-3}|n-\eta m - m_{j+1}|, \frac{m_j}{|n - \eta m - m_{j+1}|^3}\right)|m| + c\sum_{j=k+2}^{\infty} \frac{1}{m_j^4} \,. \tag{2.7}$$

Therefore

$$\begin{aligned} &\sum_{n \in I(k)} \left(\sum_{j=0}^{\infty} |k_j^*(n - m) - k_j^*(n)|^2\right)^{1/2} \\ &\leqq c\sum_{j=0}^{k-1} \frac{m_j}{m_{k+2}^2} |m| + c\sum_{j=k}^{k+1} (m_j^{-3}\cdot m_j^2 + m_j\cdot m_j^{-2})\,|m| + \sum_{j=k+2}^{\infty} \frac{m_{k+2}}{m_j^4} \\ &\leqq c\,|m|\,m_{k+1}^{-1} + c\,|m|\,m_k^{-1} + cm_{k+2}^{-3} \leqq c\,|m|\,m_k^{-1} \end{aligned} \tag{2.8}$$

for $k > N$. If $k = N$, we replace the second term of (2.7) by $c\sum_{j=N}^{N+1}\sum_{\eta=0}^{1}$ min $(m_j^{-1}, m_j^2|n - m_{j+1}|^{-3})$. Then we get, by the same way,

$$\sum_{n \in I(N)} \left(\sum_{j=0}^{\infty} |k_j^*(n - m) - k_j^*(n)|^2\right)^{1/2} \leqq c\,|m|\,m_{N+1}^{-1} + c + cm_{N+1}^{-3} < c \,. \tag{2.9}$$

Therefore the left hand side of (2.3) is bounded by $c + c\sum_{k>N} |m|\, m_k^{-1} \leqq c + c\,|m|\,m_{N+1}^{-1} < c$.

To prove our lemma it remains to show

$$\sum_{|m|>2^{M+2}} \left(\sum_s{}' |k_s(n - m) - k_s(n)|^2\right)^{1/2} < c \,, \qquad |m| < 2^M \,, \tag{2.10}$$

where the summation $\sum'$ runs over all s not of the form 2^j. This is proved by the similar way to the above and it will be simpler. Actually (2.10) is given in S. Igari [2], so that we omit the proof.

PROOF OF THEOREM 3. We use the following two facts whose proof is given, for example, in S. Igari [2].

Let H be the Hilbert space of square summable sequences on non-negative integers. Assume $1<q<\infty$. For H-valued $L^q(Z)$-function $f=\{f_j\}$ define $Tf = \{T_jf\}$ by $(T_jf)^\wedge(\xi) = \chi_{I_j}(\xi)\hat{f}_j(\xi)$, where $\hat{f}_j(\xi) = \sum_{n=-\infty}^{\infty} f_j(n)e^{2\pi i n\xi}$ and χ_{I_j} is the characteristic function of the interval I_j in T. Then we have

$$\|Tf\|_{L^q(Z,H)} \leqq A_q \|f\|_{L^q(Z,H)} \,, \tag{2.11}$$

where A_q is a constant which does not depend on the choice of $\{I_j\}$ and f.

For $f\in L^q(Z)$ define $\Delta(f)=\{\Delta_j(f)\}$ by

$$(\Delta_j f)^\wedge(\xi)=\chi_{[2^{-j-1},2^{-j}]}(\xi)\hat{f}(\xi)\,.$$

Then

$$A'_q\,\|f\|_{L^q(Z)}\leqq\|\Delta(f)\|_{L^q(Z,H)}\leqq A''_q\,\|f\|_{L^q(Z)}\,, \tag{2.12}$$

where constant A'_q and A''_q depend only on q.

Put $K=(k_0,k_1,k_2,\cdots)$. Then the mapping $f\to K*f$ of $L^q(Z)$ to $L^q(Z,H)$ is bounded, by the argument of [2] with Lemma 1, that is,

$$\|K*f\|_{L^q(Z,H)}\leqq c_q\,\|f\|_{L^q(Z)}\,. \tag{2.13}$$

Apply (2.11) and then (2.10) to (2.13). Then we get

$$\begin{aligned}\|T_{\exp 2\pi it\theta}f\|_{L^q(Z)}&\leqq A'^{-1}_q\,\|\Delta(T_{\exp 2\pi it\theta}f)\|_{L^q(Z,H)}\\&\leqq A'^{-1}_q A_q\,\|K*f\|_{L^q(Z,H)}\leqq A'^{-1}_q A_q c_q\,\|f\|_{L^q(Z)}\,,\end{aligned}$$

which implies Theorem 3.

3. Proof of Theorem 1. We prove the sufficiency. The necessity is obvious. Let $1<q\leqq p<2$. Remark that $M_1(T)=A_1(T)\subset M_q(T)$ and $M_p(T)\subset A_p(T)$, where $A_p(T)$ is the set of Fourier transforms of functions in $L^p(Z)$. Thus by the theorem of W. Rudin [7], Φ is extended to an analytic function in a neighborhood of $[-1,1]$. We may assume that $\Phi(0)=0$ and Φ is periodic with period 1 considering $\Phi(\sin 2\pi x)$ and $\Phi(\varepsilon\sin 2\pi x)$, $0<\varepsilon<1$.

LEMMA 2. (1) (K. de Leeuw [1]) *Let $1\leqq r\leqq 2$ and $\phi\in M_r(T)$. If $\tilde{\phi}$ is the periodic extension of ϕ, then*

$$\|\phi\|_{M_r(T)}=\|\tilde{\phi}\|_{M_r(R)}\,. \tag{3.1}$$

(2) ([1] and S. Igari [3]) *If $\psi\in M_r(R)$ and ψ is regulated, then*

$$\|\psi\|_{M_r(R)}\geqq\|\psi(\varepsilon n)\|_{M_r(Z)} \tag{3.2}$$

for every $\varepsilon>0$. If, furthermore, ψ is continuous almost everywhere,

$$\|\psi\|_{M_r(R)}=\lim_{\varepsilon\to 0}\|\psi(\varepsilon n)\|_{M_r(Z)}\,. \tag{3.3}$$

LEMMA 3 (J.-P. Kahane and W. Rudin [5]). *For a given sequence $\{n_j\}$ of positive integers, there exist $\{\nu_j\}$ and $\{\mu_j\}$ of positive integers satisfying:*

$$m_{i_j}/2\nu_j<-n_j+\mu_j<n_j+\mu_j<m_{i_j}/\nu_j<m_{i_{j+1}}/2\nu_{j+1} \tag{3.4}$$

$j=1,2,3,\cdots$ for some $0<i_1<i_2<\cdots$. Thus the sets

$$S_j = \{m = \nu_j(n + \mu_j);\ |n| \leqq n_j\}$$

are mutually disjoint.

For every continuous function g in T such that $\operatorname{supp} \hat{g} \subset \bigcup S_j$, *we have*

$$\|g\|_\infty \leqq \sum_{j=1}^{\infty} \Big\| \sum_{m \in S_j} \hat{g}(m) e^{2\pi i m x} \Big\|_\infty \leqq 2 \|g\|_\infty . \tag{3.5}$$

LEMMA 4. *For every $s > 1$ there is a constant c_s such that*

$$\|\Phi(\phi)\|_{M_p(T)} < c_s$$

for every real valued function ϕ in $M_1(T)$ satisfying $\|\phi\|_{M_1(T)} < s$.

PROOF. Fix $s > 1$. If the lemma were false for s, there exists a sequence $\{\phi_j\}$ in $M_1(T)$ such that

$$\|\phi_j\|_{M_1(T)} < s\ , \quad \text{range of } \phi \subset R \text{ and } \quad \|\Phi(\phi_j)\|_{M_p(T)} > j$$

for $j = 1, 2, 3, \cdots$.

Let $\tilde{\phi}_j$ be the periodic extention of ϕ_j with period 1. Then by Lemma 2, there is $\varepsilon_j > 0$ such that

$$\|\tilde{\phi}_j(\varepsilon_j n)\|_{M_1(Z)} < s \quad \text{and} \quad \|\Phi(\tilde{\phi}_j(\varepsilon_j n))\|_{M_p(Z)} > j\ .$$

Let $V_a(\xi) = 2\Delta_{2a}(\xi) - \Delta_a(\xi)$, where $\Delta_a(\xi) = \max(1 - |\xi|/a, 0)$. Then $\|V_a\|_{M_1(R)} \leqq 3$ for all $a > 0$. Thus if $a_j > 0$ is sufficiently large and $\psi_j(n) = V_{a_j}(n)\tilde{\phi}_j(\varepsilon_j n)$, then

$$\|\psi_j\|_{M_1(Z)} < 3s \quad \text{and} \quad \|\Phi(\psi_j)\|_{M_p(Z)} > j\ . \tag{3.6}$$

Pick n_j so that $2a_j < n_j$. Choose ν_j and μ_j, and define S_j by Lemma 3. Put $X = \{f \in C(T); \operatorname{supp} \hat{f} \subset \bigcup_{j=1}^{\infty} S_j\}$. Then X is a closed subspace of $C(T)$. If $Tf = \sum_{j=1}^{\infty} \sum_{m \in S_j} \psi_j(n)\hat{f}(m)$, $m = \nu_j(n + \mu_j)$,

$$|Tf| \leqq \sum_{j=1}^{\infty} \Big\| \sum_{m \in S_j} \psi_j(n)\hat{f}(m) e^{2\pi i m \xi} \Big\|_\infty \leqq 6s \|f\|_\infty$$

applying Lemma 3. Since T is extended to a bounded linear functional on $C(T)$, there is a bounded Borel measure μ on T such that

$$Tf = \int_0^1 f d\bar{\mu}$$

for $f \in X$. In particular $\hat{\mu}(m) = \psi_j(n)$, $m = \nu_j(n + \mu_j)$.

Put $\phi(\xi) = \Re e \hat{\bar{\mu}}(\xi) e^{-2\pi i \xi}$. Since $\phi(\theta(\xi)) = \Re e \int_0^1 e^{-2\pi i \theta(\xi)(x+1)} d\bar{\mu}(x)$,

$$\|\phi(\theta)\|_{M_q(T)} \leqq \int_0^1 \sup_{1 \leqq t \leqq 2} \|e^{-2\pi i t \theta}\|_{M_q(T)} |d\mu|(x) < \infty\ .$$

Now put $(\Phi\circ\phi\circ\tilde{\theta})^*(\xi)=[\Phi\circ\phi\circ\tilde{\theta}(\xi+0)+\Phi\circ\phi\circ\tilde{\theta}(\xi-0)]/2$ if ξ is not integer and $=\Phi(\Re e\hat{\mu}(0))$ otherwise, where $f\circ g$ denotes the composition function $f(g(\cdot))$. Then $(\Phi\circ\phi\circ\tilde{\theta})^*$ is regulated. In fact put $u_j(\xi)=m_j^{-1}\chi_j(\xi)$, where χ_j is the characteristic function of the interval $(-1/2m_j, 1/2m_j)$. Then it is not hard to prove that $u_j*(\Phi\circ\phi\circ\tilde{\theta})^*(\xi)\to(\Phi\circ\phi\circ\tilde{\theta})^*(\xi)$ for every ξ. Thus by the assumption and Lemma 2,

$$
\begin{aligned}
(3.7)\qquad &\|(\Phi\circ\phi\circ\tilde{\theta})^*(a(n+b))\|_{M_p(Z)}\leqq\|(\Phi\circ\phi\circ\tilde{\theta})(a(\xi+b))\|_{M_p(R)}\\
&=\|(\Phi\circ\phi\circ\tilde{\theta})(\xi)\|_{M_p(R)}=\|\Phi\circ\phi\circ\theta\|_{M_p(T)}<\infty
\end{aligned}
$$

for every a and b.

Choose a and b so that $a=\nu_j m_{i_{j+1}}^{-1}$ and $b=\mu_j$. Then $\theta(a(n+b))=\nu_j(n+\mu_j)$ for $|n|<n_j$. Remark that $\theta(a(\xi+b))$ has no point of discontinuity in $|\xi|\leqq n_j$. Thus $\phi\circ\theta(a(n+b))=\psi_j(n)$ for $|n|<n_j$. Thus by (3.6) and M. Riesz theorem the left hand side of (3.7) is arbitrarily large. The contradiction implies the lemma.

LEMMA 5 ([3], cf. [7]). *If $p\neq 2$,*

$$\sup\{\|e^{i\psi}\|_{M_p(T)};\ \|\psi\|_{M_1(T)}<s,\ \text{range of }\psi\subset R\}>Ae^{Bs},$$

where A and B are constants independent on s.

PROOF OF THEOREM 1. If $\hat{\Phi}(n)$ is the n-th Fourier coefficient of Φ, then

$$\hat{\Phi}(n)e^{2\pi in\phi}=\int_0^1\Phi(x+\phi)e^{-2\pi inx}dx\ .$$

Taking supremum over real valued ϕ such that $\|\phi\|_{M_1(T)}<s$, we get $|\hat{\Phi}(n)|Ae^{2\pi B|n|s}\leqq c_{s+1}$, which proves the theorem.

4. Corollaries of Theorem 1. By the well-known argument we get the following corollaries of Theorem 1 (cf. S. Igari [3] or Y. Katznelson [6]).

COROLLARY 1. *Assume $p\neq 1, 2$. Then there exists ϕ in $M_p(T)$ such that $\phi\geqq 1$ on T but $1/\phi\notin M_p(T)$.*

COROLLARY 2. *If $p\neq 1, 2$, then the Banach algebra $M_p(T)$ is asymmetric and not regular.*

PROOF OF THEOREM 2. We may assume that $O=(0, 1)$. Take ϕ possessing the properties of Corollary 1. Then the periodic extension $\tilde{\phi}$ of ϕ satisfies the conditions, in fact, otherwise, $1/\tilde{\phi}\cdot\chi_{[0,1)}\in M_p(R)$. Thus $1/\tilde{\phi}\in M_p(T)$ by a theorem of M. Jodeit, Jr. [4].

5. Remarks. (1) In Theorem 1 we cannot replace $M_q(T)$ by $m_q(T)$, if $q<p<2$. In fact if $\phi\in m_q(T)$

$$\|\phi - F_j * \phi\|_{M_p(T)} \leqq \|\phi - F_j * \phi\|_{M_q(T)}^{1-\theta} \|\phi - F_j * \phi\|_\infty^\theta$$

where $1/p = (1 - \theta)/q + \theta/2$ and F_j is the Fejér kernel. Since the first term of the right hand side is bounded by $(2\|\phi\|_{M_q(T)})^{1-\theta}$ and the second term tends to zero as $n \to \infty$, ϕ is approximated by polynomials. Let h be a non-trivial homomorphism on $M_p(T)$. Then there is a point t in T such that $h(\psi) = \psi(t)$ for all polynomials ψ. Thus the range of Gelfand transform of ϕ on the maximal ideal space of $M_p(T)$ coinsides with $\phi(T)$.

Therefore for the continuous multipliers the possibility of Theorem 1 comes into question only if $p = q$.

(2) Corollary 1 does not hold for $p = 1$ and 2. The former case is due to N. Wiener and the latter to O. Toeplitz.

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