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DENSITIES WITHOUT EVANS SOLUTIONS

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Consider a 2-form $P(z)dxdy$ on an open Riemann surface R such that the coefficients $P(z)$ are nonnegative locally Hölder continuous functions of local parameters $z = x + iy$ on R. Such a 2-form $P(z)dxdy$ which is not identically zero will be referred to as a *density* on R. If the integral $\int_R P(z)dxdy$ is finite, then the density $P(z)dxdy$ is said to be finite. An *Evans solution* $u(z)$ of the elliptic equation

$$(1) \qquad \Delta u(z) = P(z)u(z) \qquad (\text{i.e., } d*du(z) = u(z)P(z)dxdy)$$

on R is a function $u(z)$ of class C^2 satisfying (1) on R such that

$$(2) \qquad \lim_{z \to a_\infty} u(z) = \infty$$

where a_∞ is the Alexandroff ideal boundary point of R, i.e., $\inf_{R-K} u$ tend to infinity as compact subsets K exhaust R. It has been a conjecture that for any density P on R, or at least for finite density P on R, the existence of an Evans solution of (1) on R is equivalent to $(R, P) \in O_B$, i.e., the only bounded solution of (1) on R is the constant zero. The purpose of this paper is to show that *this conjecture is false* by proving the following

THEOREM. *There always exists a finite density $P(z)dxdy$ on an arbitrarily given open Riemann surface R such that every nonnegative solution of* (1) *on R has the zero infimum.*

Actually we will prove a bit more: Let R be an open Riemann surface, $\{z_n\}$ a sequence of distinct points in R not accumulating in R, N an open subset of R containing $\{z_n\}$, $\{\alpha_n\}$ a sequence of positive numbers converging to zero, and η a positive number. For an arbitrary such system $(R, \{z_n\}, N; \{\alpha_n\}, \eta)$ there exists a density $P(z)dxdy$ of class C^∞ with the following properties: The support of $P(z)dxdy$ is contained in N, i.e., $P(z)dxdy \equiv 0$ on $R - N$; $\int_R P(z)dxdy \leqq \eta$; $\{u(z_n)\} \ll \{\alpha_n\}$ for any nonnegative solution u of (1) on R, i.e., $u(z_n) < \alpha_n$ for every large n, and in particular $\inf_R u = 0$. The existence proof of such a P will be given in nos. 1-2. That the last of the above properties is also valid if

u is replaced by any nonnegative solution of (1) on R outside a compact set is shown in no. 3. The relation to the existence question of Evans solutions will be discussed in no. 4.

1. Let Ω be a regular subregion of an open Riemann surface R and $P(z)dxdy$ be a density on R. We denote by P_f^Ω for an $f \in C(\partial\Omega)$ the continuous function on $\bar{\Omega}$ such that $P_f^\Omega \mid \partial\Omega = f$ and P_f^Ω is a solution of (1) on Ω. We also use the standard notation H_f^Ω for P_f^Ω with $P \equiv 0$ on $\bar{\Omega}$. Fix an arbitrary point p in R and an arbitrary parametric disk $U: |z| < 1$ about p so that p is identified with $z = 0$ in U.

LEMMA. *For any pair* (ε, η) *of positive numbers and any concentric parametric disk* $V: |z| < \rho$ *in* U *with* $0 < \rho \leqq e^{-4\pi/\eta}$ *there exists a density* $P(z)dxdy$ *on* R *whose support is contained in* V *such that*

$$|P_f^V(p)| \leqq \varepsilon \cdot \left| \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) d\theta \right| \tag{3}$$

for every f *in* $C(\partial V)$ *and*

$$\int_R P(z)dxdy \leqq \eta \, . \tag{4}$$

PROOF. Since $\rho \in (0, e^{-4\pi/\eta}]$, $\rho \in (0, 1)$ and $4\pi/\log\sigma \leqq \eta$ with $\sigma = 1/\rho$. Take a positive number a so small that $a \log\sigma < \varepsilon/2$. Consider a continuous function $\varphi(\tau)$ on $[-\rho, \rho]$ given by

$$\varphi(\tau) = \begin{cases} 2/(\tau \log|\tau|)^2(1 - a\log|\tau|) & (\tau \neq 0)\,; \\ \infty & (\tau = 0) \end{cases}$$

which is positive and symmetric on $[-\rho, \rho]$. Choose an increasing sequence $\{\varphi_n(\tau)\}$ of nonnegative symmetric C^∞ functions φ_n on $[-\rho, \rho]$ with compact support in $(-\rho, \rho)$ such that $\lim_n \varphi_n(\tau) = \varphi(\tau)$ on $(-\rho, \rho)$. Let $P_n(z)dxdy$ be the density on R such that $P_n(z)dxdy \equiv 0$ on $R - V$ and $P_n(z) = \varphi_n(|z|)$ on V, which is then of class C^∞. Set $u_n(z) = (P_n)_1^V(z)$. By the Green formula we see that u_n satisfies the integral equation

$$u_n(z) = 1 - \frac{1}{2\pi} \int_V G(z, \zeta) u_n(\zeta) P_n(\zeta) d\xi d\eta \qquad (\zeta = \xi + i\eta) \tag{5}$$

where $G(z, \zeta)$ is the harmonic Green's function $\log(|\rho^2 - \bar{\zeta}z|/\rho|z - \zeta|)$ on V. Since $P_n(z) \leqq P_{n+1}(z)$, the comparison principle implies that $u_n(z) \geqq u_{n+1}(z)$ on $\bar{V}$. Therefore

$$u(z) = \lim_{n\to\infty} u_n(z)$$

exists on $\bar{V}$. Observe that

$$u_n(\zeta)P_n(\zeta) \leqq (|\zeta| \log |\zeta|)^{-2} \cdot \left(-\frac{a}{2} \log |\zeta|\right)^{-1}$$

and the function on the right is integrable on V. Hence the Lebesgue dominated convergence theorem applied to (5) as $n \to \infty$ yields

$$(6) \qquad u(z) = 1 - \frac{1}{2\pi} \int_V G(z, \zeta) u(\zeta) \varphi(|\zeta|) d\xi d\eta.$$

This identity shows that $u(z)$ is a bounded solution of

$$(7) \qquad \Delta u(z) = \varphi(|z|) u(z)$$

on $0 < |z| < \rho$ with continuous boundary values 1 on $|z| = \rho$. On the other hand, by a direct computation, we see that

$$v(z) = \left(\left(\log \frac{1}{|z|}\right)^{-1} + a\right) \Big/ \left((\log \sigma)^{-1} + a\right)$$

is a bounded solution of (7) on $0 < |z| < \rho$ with continuous boundary values 1 on $|z| = \rho$. Observe that $v(z)$ is continuously extendable to V by setting $v(0) = a/((\log \sigma)^{-1} + a)$. Therefore, $u(z) - v(z)$ is a bounded solution of (7) on $0 < |z| < \rho$ with continuous boundary values zero on $|z| = \rho$, and thus $u(z) - v(z) \equiv 0$ on $0 < |z| < \rho$. This can be seen by many ways. For example, observe that $|u(z) - v(z)|$ is subharmonic. Then $-m^{-1} \log |z| - |u(z) - v(z)|$ is a superharmonic function on $0 < |z| < \rho$ with nonnegative boundary values at $|z| = 0$ and ρ. Thus

$$|u(z) - v(z)| < -\frac{1}{m} \log |z|$$

on $0 < |z| < \rho$ for every $m = 1, 2, \cdots$, and we arrive at the desired conclusion.

Since $P_n(z) = P_n(|z|)$ on V and the boundary function 1 is also rotation free, we have $u_n(z) = u_n(|z|)$ on V. The maximum principle yields that $u_n(\tau)$ is an increasing function on $[0, \rho]$ and the same is true of $v(\tau)$. Since $v(\tau) = \lim u_n(\tau)$ on $(0, \rho]$ decreasingly, the Dini theorem implies the uniformness of the convergence on $[\rho', \rho]$ for every $\rho' \in (0, \rho)$. This shows that $v(0) = \lim u_n(0)$. Fix a k such that

$$(8) \qquad u_k(0) \leqq v(0) + \frac{\varepsilon}{2}.$$

We now maintain that $P(z)dxdy = P_k(z)dxdy$ is a required density. Let $K(z, \zeta)$ be the Green's function of (1) with this particular $P = P_k$ on V. Again by $P(z) = P(|z|)$ on V and the rotation invariantness of V, we see that $K(0, \zeta) = K(0, |\zeta|)$ for every $\zeta \in \bar{V}$. Therefore,

$$\rho\cdot\left[\frac{\partial}{\partial r}K(0, re^{i\theta})\right]_{r=\rho} = \rho\cdot\left[\frac{\partial}{\partial r}K(0, r)\right]_{r=\rho}$$

is a negative constant $-2\pi A$ $(A>0)$ on $[0, 2\pi]$, and thus

$$^*dK(0, \rho e^{i\theta}) = -2\pi A d\theta .$$

For any $f\in C(\partial V)$ we then have

$$P_f^V(0) = -\frac{1}{2\pi}\int_{\partial V} f(\zeta)^* d_\zeta K(0, \zeta) = A\cdot\int_0^{2\pi} f(\rho e^{i\theta})d\theta . \tag{9}$$

In particular on putting $f\equiv 1$ in (9) we have $u_k(0)=2\pi A$. By (8) and by the choice of a we see that

$$A \leqq \frac{1}{2\pi}\left(v(0)+\frac{\varepsilon}{2}\right) \leqq \frac{1}{2\pi}\left(a\log\sigma+\frac{\varepsilon}{2}\right) \leqq \frac{\varepsilon}{2\pi} .$$

This with (9) yields (3). To show the validity of (4) we compute as follows:

$$\int_R P(z)dxdy = \int_V \varphi_k(|z|)dxdy \leqq \int_V \varphi(r)rdrd\theta$$
$$\leqq 2\pi\int_0^\rho \frac{2}{(r\log r)^2}rdr = \frac{4\pi}{\log\sigma} \leqq \eta .$$

2. Fix an arbitrary system $(R, \{z_n\}, N; \{\alpha_n\}, \eta)$ as described in the introduction. Take a sequence $\{U_n\}$ of parametric disks on R such that $\bar{U}_n\subset N$, $\bar{U}_n\cap\bar{U}_m=\emptyset$ $(n\neq m)$, and the center of U_n is z_n $(n=1, 2, \cdots)$. Let $\{\eta_n\}$ be a sequence of positive numbers such that

$$\eta = \sum_{n=1}^\infty \eta_n .$$

We denote by V_n the concentric parametric disk $|z|<\rho_n=e^{-4\pi/\eta_n}$ of U_n $(n=1, 2, \cdots)$. Take the *harmonic* Green's function $G(z, \zeta)$ on

$$S = R - \bigcup_{n=1}^\infty \bar{V}_n .$$

Observe that the inner normal derivative $(\partial/\partial n_\zeta)G(z, \zeta)$ at any $\zeta\in\partial V_n$ is strictly positive for any $z\in S$ and any $n=1, 2\cdots$, and

$$^*d_\zeta G(z, \zeta) = -\frac{\partial}{\partial n_\zeta}G(z, \zeta)\rho_n d\theta \tag{10}$$

for each $\zeta=\rho_n e^{i\theta}\in\partial V_n$ $(n=1, 2, \cdots)$. Fix an arbitrary point $z_0\in S$ and set

$$m_n = \min_{\zeta\in\partial V_n}\rho_n\cdot\frac{\partial}{\partial n_\zeta}G(z_0, \zeta) > 0 , \qquad \varepsilon_n = \alpha_n\cdot m_n \tag{11}$$

for each $n = 1, 2, \cdots$. Fix a density $P_n(z)dxdy$ on R given as in Lemma in no. 1 determined by (ε_n, η_n) and $V_n\colon |z| < \rho_n$ $(n = 1, 2, \cdots)$. Since $P_n(z)dxdy$ $(n = 1, 2, \cdots)$ have disjoint compact supports in R, we can define the density

$$P(z)dxdy = \sum_{n=1}^{\infty} P_n(z)dxdy \tag{12}$$

on R which is of class C^∞ and by (4)

$$\int_R P(z)dxdy = \sum_{n=1}^{\infty} \int_R P_n(z)dxdy \leqq \sum_{n=1}^{\infty} \eta_n = \eta .$$

Clearly we have

$$\text{supp. } P(z)dxdy = \bigcup_{n=1}^{\infty} \text{supp. } P_n(z)dxdy \subset \bigcup_{n=1}^{\infty} V_n \subset N .$$

We then have the following

LEMMA. *For any nonnegative solution u of* (1) *on R with P given by* (12) *the following inequality*

$$u(z_n) < \alpha_n \tag{13}$$

is valid for every large n and in particular

$$\lim_{n\to\infty} u(z_n) = 0 , \qquad \inf_{z \in R} u(z) = 0 . \tag{14}$$

PROOF. Let $\{R_n\}_1^\infty$ be an exhaustion of R with regular subregions such that $z_0 \in R_1$, $R_n \supset \bigcup_{j=1}^{n} \bar{V}_j$, and $R - \bar{R}_n \supset \bigcup_{j=n+1}^{\infty} V_j$, and let $u_{n,k}$ $(n < k)$ be the boundary function for the region $S_k = R_k - \bigcup_{j=1}^{k} \bar{V}_j$ such that

$$u_{n,k} = \begin{cases} u & \text{on} \quad \bigcup_{j=1}^{n} \partial V_j ; \\ 0 & \text{on} \quad (\partial R_k) \cup \bigcup_{j=n+1}^{k} \partial V_j . \end{cases}$$

Since $P(z)dxdy = 0$ on S_k, $u(z)$ is harmonic on S_k. Therefore, the maximum principle yields

$$H_{u_{n,k}}^{S_k}(z) \leqq u(z) \qquad (n = 1, 2, \cdots; k = n+1, n+2, \cdots) \tag{15}$$

for every $z \in S_k$ and in particular for $z = z_0$. Let $G_k(z, \zeta)$ be the harmonic Green's function on S_k. Then

$$\begin{aligned} H_{u_{n,k}}^{S_k}(z) &= -\frac{1}{2\pi} \int_{\partial S_k} u_{n,k}(\zeta) * d_\zeta G_k(z, \zeta) \\ &= -\frac{1}{2\pi} \sum_{j=1}^{n} \int_{\partial V_j} u(\zeta) * d_\zeta G_k(z, \zeta) . \end{aligned}$$

Therefore by (15)

$$-\sum_{j=1}^{n}\int_{\partial V_j} u(\zeta)*d_\zeta G_k(z_0,\zeta) \leqq 2\pi u(z_0)\,. \tag{16}$$

Since $\{G_k(z_0,\zeta)\}_{k=n+1}^{\infty}$ converges increasingly to $G(z_0,\zeta)$ for every $\zeta\in\bar{S}$, on letting $k\to\infty$ in (16) we deduce

$$-\sum_{j=1}^{n}\int_{\partial V_j} u(\zeta)*d_\zeta G(z_0,\zeta) \leqq 2\pi u(z_0)$$

and again by letting $n\to\infty$ we obtain

$$-\sum_{j=1}^{\infty}\int_{\partial V_j} u(\zeta)*d_\zeta G(z_0,\zeta) \leqq 2\pi u(z_0)\,. \tag{17}$$

On the other hand, since $u(\zeta)\geqq 0$ and the line element $ds_\zeta=\rho_j d\theta$ on ∂V_j, we deduce by (10) and (11) that

$$-\int_{\partial V_j} u(\zeta)*d_\zeta G(z_0,\zeta) = \int_{\partial V_j} u(\zeta)\frac{\partial}{\partial n_\zeta}G(z_0,\zeta)ds_\zeta$$
$$= \int_0^{2\pi} u(\rho_j e^{i\theta})\left[\frac{\partial}{\partial n_\zeta}G(z_0,\zeta)\right]_{\zeta=\rho_j e^{i\theta}}\rho_j d\theta \geqq m_j\int_0^{2\pi} u(\rho_j e^{i\theta})d\theta\,.$$

Therefore if we set

$$a_j = m_j\int_0^{2\pi} u(\rho_j e^{i\theta})d\theta > 0\,,$$

then by (17) we have

$$\sum_{j=1}^{\infty} a_j \leqq 2\pi u(z_0)$$

and in particular

$$\lim_{j\to\infty} a_j = 0\,. \tag{18}$$

Observe that $P(z)=P_j(z)$ on V_j and hence $u(z_j)=(P_j)_u^{V_j}(z_j)$. By (3),

$$u(z_j) \leqq \varepsilon_j\int_0^{2\pi} u(\rho_j e^{i\theta})d\theta = \varepsilon_j m_j^{-1}a_j = a_j\alpha_j\,. \tag{19}$$

Thus by (18) we see the validity of (13) for sufficiently large n.

3. We remark that the lemma in no. 2 is also valid for any non-negative solution $v(z)$ of (1) on $R-X$ where X is a compact subset of R. Let $\{R_n\}_0^\infty$ be an exhaustion of R with regular subregions such that $R_0\supset X$. For any $f\in C(\partial R_0)$ set

$$(Lf)(z) = \lim_{n\to\infty} P_{f^*}^{R_n-\bar{R}_0}(z) \tag{20}$$

on $R - \bar{R}_0$, where $f^* = f$ on ∂R_0 and $f^* = 0$ on ∂R_n. The existence of the limit in (20) is clear for $f \geqq 0$, and the general case follows from this. The equation

$$L(u_1 - v) = u_1 - v \tag{21}$$

on $R - \bar{R}_0$ always possesses a solution u_1 which is a soultion of (1) on R ([6, p. 403]). Since (1) has a positive solution on R by Myrberg's theorem [2], we can find a positive solution u_2 of (1) on R such that $u_2 > v - u_1$ on ∂R_0. Then by the maximum principle and (20) we see that $u_2 > L(v - u_1) = v - u_1$ on $R - R_0$. Therefore, $u \equiv u_1 + u_2 > v$ on $R - R_0$ and by (13)

$$v(z_n) < u(z_n) < \alpha_n$$

for every sufficiently large n, for u is a positive solution of (1) on R.

4. We denote by O_B the class of every pair (R, P) of a Riemann surface R and a density P on R such that the equation (1) does not admit any bounded solution on R except for the constant zero. We also denote by O_G the class of pairs (R, P) such that R is parabolic, i.e., R has the harmonic null boundary. The Brelot [1]-Ozawa [4]-Royden [5] theorem asserts that

$$O_G < O_B \qquad (\textit{strict inclusion}) \tag{22}$$

and

$$O_G \cap \mathscr{F} = O_B \cap \mathscr{F} \tag{23}$$

where $\mathscr{F} = \left\{(R, P); \int_R P(z)dxdy < \infty\right\}$. We denote by $\mathscr{E}$ the class of pairs (R, P) such that (1) has an Evans solution on R. We have

$$\mathscr{E} \subset O_B\,, \quad \mathscr{E} \cap \mathscr{F} \subset O_G \cap \mathscr{F} = O_B \cap \mathscr{F}. \tag{24}$$

In fact, let $(R, P) \in \mathscr{E}$ and u be an Evans solution of (1) on R. Let v be any bounded soultion of (1) on R. Then for any $m = 1, 2, \cdots$

$$\lim_{z \to a_\infty} \left(\frac{1}{m}u(z) \pm v(z)\right) = \infty$$

and the maximum principle yields $(1/m)u(z) \pm v(z) > 0$ on R, i.e.,

$$|v(z)| \leqq \frac{1}{m}u(z)$$

on R for every m. Thus $v \equiv 0$ and $(R, P) \in O_B$.

It has been suspected that the inclusions in (24) are improper (see

[3, p. 92]). However, our theorem stated in the introduction negates this conjecture. Let R be any parabolic Riemann surface. For example let R be a compact surface less a point. Take a density $P(z)dxdy$ on R as described in the theorem. Since $(R, P) \in O_G \cap \mathscr{F}$, $(R, P) \in O_B \cap \mathscr{F}$. If there existed an Evans solution $u(z)$ of (1) on R, then $u(z) > 0$ on R and $\inf_R u = 0$, a contradiction. Thus

$$\mathscr{E} < O_B, \quad \mathscr{E} \cap \mathscr{F} < O_G \cap \mathscr{F} = O_B \cap \mathscr{F}. \tag{25}$$

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