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A LOCAL ERGODIC THEOREM FOR SEMI-GROUP ON L_p

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1. **Introduction.** The purpose of this note is to prove a local ergodic theorem for a strongly continuous one-parameter semi-group (semi-group) of positive bounded linear operators on $L_p(X)$ $(1 \leqq p < \infty)$ and for a semi-group of linear contractions on $L_1(X)$ which are also linear contractions on $L_\infty(X)$. A local ergodic theorem for a semi-group of positive linear contractions on $L_1(X)$ was conjectured by U. Krengel and was proved by U. Krengel [5] and D. Ornstein [8] independently. M. Akcoglu-R. Chacon [1] and T. Terrell [9] gave different treatments of the theorem. The author generalized the theorem and proved a local ergodic theorem for a semi-group of positive bounded operators on $L_1(X)$ [6]. We shall generalize the theorem and prove a local ergodic theorem for a semi-group of positive bounded linear operators on $L_p(X)$ $(1 \leqq p < \infty)$ (Theorem 1). A local ergodic theorem for a semi-group (T_t) $(t \geqq 0)$ of linear contractions on $L_1(X)$ which are also linear contractions on $L_\infty(X)$ was proved by D. Ornstein [8]. This theorem asserts that we have

$$(*) \qquad \lim_{\alpha \to 0} \frac{1}{\alpha} \int_0^\alpha (T_t f)(x) dt = f(x) \text{ a.e.}$$

for any $f \in L_1(X)$. T. Terrell proved the theorem in an n-parameter case [9]. We shall prove $(*)$ for any $f \in L_p(X)$ $(1 \leqq p < \infty)$ under the same conditions as D. Ornstein (Theorem 2). The author proved that we have $(*)$ for any $f \in L_1(X)$, provided that (T_t) $(t \geqq 0)$ is a semi-group of linear contractions on $L_1(X)$ [7]. It may be one of the most interesting open problems in ergodic theorems whether we have $(*)$ for any $f \in L_p(X)$, provided that (T_t) is a semi-group of bounded linear operators on $L_p(X)$ $(1 \leqq p < \infty)$.

2. **Definitions and the theorems.** Let (X, B, m) be a σ-finite measure space and $L_p(X) = L_p(X, B, m)$ $(1 \leqq p \leqq \infty)$ the Banach space of complex-valued measurable functions on X such that $\| f \|^p = \int | f(x) |^p dm < \infty$ $(1 \leqq p < \infty)$ or ess. $\sup_x | f(x) | < \infty$ $(p = \infty)$. Let (T_t) $(t \geqq 0)$ be a strongly continuous one-parameter semi-group (semi-group) of bounded linear operators on $L_p(X)$. This means that

(A) T_t is a bounded linear operator on $L_p(X)$ for any $t \geqq 0$,

(B) $T_{t+s}f = T_t \circ T_s f$ for any $t, s \geqq 0$ and $f \in L_p(X)$,

and

(C) $\lim_{t\to 0} \| T_t f - f \| = 0$ for any $f \in L_p(X)$.

Then there exist constants M, β such that $\| T_t \| \leqq Me^{\beta t}$ [11]. (If we can take $M = 1$, $\beta = 0$, then (T_t) is said to be a semi-group of linear contractions on $L_p(X)$.) Let $f \in L_p(X)$ and $(I, \mathscr{L}, \lambda)$ the Lebesgue measure spaces on the interval $I = [0, a]$ $(0 < a < \infty)$. Then there exists a function $g(t, x)$ such that [4, 10],

(1) $g(t, x)$ is $\mathscr{L} \times B$-measurable on $I \times X$,

(2) for a fixed $t \in I$, $g(t, x) = (T_t f)(x)$ for a.a. x,

(3) if $g'(t, x)$ satisfies (1) and (2), then for almost all x,

$$g'(t, x) \text{ is integrable on } I$$

and

$$\int_0^\alpha g'(t, x)dt = \int_0^\alpha g(t, x)dt \quad \text{for any} \quad \alpha \in I ,$$

and

$$(4) \qquad \int_0^\alpha g(t, x)dt = \operatorname*{s-lim}_{n\to\infty} \frac{1}{[n\alpha]} \sum_{k=0}^{[n\alpha]} (T_{1/n}^k f)(x) \quad \text{for a.a. } x .$$

We define the integral $\int_0^\alpha (T_t f)(x)dt$ $(0 \leqq \alpha < a < \infty)$ by $\int_0^\alpha g(t, x)dt$.

Let T be a bounded linear operator on $L_p(X)$ $(1 \leqq p < \infty)$. T is said to be positive if it satisfies (D).

(D) if $f \geqq 0$ and $f \in L_p(X)$, then $Tf \geqq 0$.

We shall prove the following.

THEOREM 1 (A local ergodic theorem). *Let (T_t) $(t \geqq 0)$ be a semi-group of positive bounded linear operators on $L_p(X)$ $(1 \leqq p < \infty)$. Then we have*

$$\lim_{\alpha\to 0} \frac{1}{\alpha} \int_0^\alpha (T_t f)(x)dt = f(x) \text{ a.e. for any } f \in L_p(X) .$$

COROLLARY. *Under the same conditions as Theorem* 1, *we have*

$$\lim_{\alpha\to 0} \frac{\int_0^\alpha (T_t f)(x)dt}{\int_0^\alpha (T_t g)(x)dt} = \frac{f(x)}{g(x)} \quad a.e.$$

for any $f, g \in L_p(X)$ on $\{x\colon g(x) \neq 0\}$.

U. Krengel [5] and D. Ornstein [8] proved the conclusion of Theorem

1 under assumptions that $p=1$ and $\|T_t\| \leqq 1$ $(t \geqq 0)$. Different treatments were given by M. Akcoglu-R. Chacon [1] and T. Terrell [9] in this case. The author proved the conclusion of Theorem 1 only assuming that $p=1$ [6].

Secondly we shall prove the following.

THEOREM 2. *Let* (T_t) $(t \geqq 0)$ *be a semi-group of linear contractions on* $L_1(X)$. *If* (T_t) $(t \geqq 0)$ *satisfies the following condition* (E), *then we have*

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_0^\alpha (T_t f)(x) dt = f(x) \ \textit{a.e. for any } f \in L_p(X) .$$

(E) $\text{ess.}\sup_x |(T_t f)(x)| \leqq \text{ess.}\sup_x |f(x)|$ *for any* $t \geqq 0$ *and* $f \in L_1(X) \cap L_\infty(X)$.

D. Ornstein proved the conclusion of Theorem 2 for any $f \in L_1(X)$ [8]. T. Terrell proved the same conclusion as D. Ornstein in an n-parameter case [9]. The author proved the conclusion of Theorem 2 for any $f \in L_1(X)$ without the condition (E) [7].

3. The proof of Theorem 1.

LEMMA 1. *Let* (T_t) $(t \geqq 0)$ *be a semi-group of bounded linear operators on* $L_p(X)$ $(1 \leqq p < \infty)$ *and* $f \in L_p(X)$. *Then for a.a. s with respect to the Lebesgue measure on the half real line we have*

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_0^\alpha (T_{t+s} f)(x) dt = (T_s f)(x) \ \textit{for a.a. } x .$$

PROOF. A little modification of the proof of Lemma 2 in U. Krengel [5] is valid for the proof of Lemma 1.

LEMMA 2 (A maximal ergodic lemma). *Let* (T_t) $(t \geqq 0)$ *be a semi-group of positive bounded linear operators on* $L_p(X)$ $(1 \leqq p < \infty)$ *and* $f \in L_p(X)$. *If*

$$\limsup_{\alpha \to 0} \frac{1}{\alpha} \int_0^\alpha (T_t f)(x) dt > 0 \ \textit{on } E ,$$

then we have

$$\int_E (f^-(x))^p dm \leqq \int_X (f^+(x))^p dm .$$

PROOF. Lemma 2 was proved by the author in case of $p=1$ [6]. We shall prove Lemma 2 by modifying the proof of Lemma 2 in [6]. Let $f \in L_p(X)$ and ε $(0 < \varepsilon < 1)$ an arbitrary positive number. By the strong continuity of (T_t), there exists a positive number δ such that

$$(5)\qquad \| T_t f^- \chi_E - f^- \chi_E \| \leqq \varepsilon \| f^- \chi_E \| \quad \text{and} \quad \| T_t f^+ \| \leqq (1+\varepsilon) \| f^+ \| \quad \text{for any } t\ (0 \leqq t \leqq \delta)$$

and

$$(6)\qquad \sup_{0 \leqq t \leqq \delta} \| T_t \| = K < \infty .$$

(We denote the characteristic function of a set G by χ_G.) Let us choose a positive number η $(0 < 2\eta < \delta)$ such that

$$(7)\qquad \frac{4\eta K}{\delta - 2\eta} \| f^- \| < \varepsilon .$$

There exists a positive integer l $(l > 2/\delta)$ and a subset F of E with properties

$$(8)\qquad \sup_{0 \leqq j \leqq [l\eta]} \sum_{i=0}^{j} (T_{1/l}^i f)(x) > 0 \quad \text{on} \quad F$$

and

$$(9)\qquad K^p \int_{E-F} (f^-(x))^p dm < \varepsilon^p ,$$

where $[a]$ is the integral part of a. This may be proved as follows. It follows from the assumption that

$$(10)\qquad \sup_{0<\alpha<\eta} \frac{1}{\alpha} \int_0^\alpha (T_t f)(x) dt > 0 \quad \text{on} \quad E .$$

Since the integral $1/\alpha \int_0^\alpha (T_t f)(x) dt$ is a continuous function of the variable $\alpha > 0$ for a.a. x,

$$(11)\qquad \lim_{v\to\infty} \sup_{\substack{0<\alpha<\eta \\ \alpha \in Q_v}} \frac{1}{\alpha} \int_0^\alpha (T_t f)(x) dt = \sup_{0<\alpha<\eta} \frac{1}{\alpha} \int_0^\alpha (T_t f)(x) dt \text{ for a.a. } x ,$$

where Q_v is the set of fractions with the denominator v (v is a positive integer). We can choose a positive number ε' $(0 < \varepsilon' < 1)$ by (10) such that

$$(12)\qquad \text{if } m(A) < \varepsilon', \text{ then } \mu(A) < \varepsilon^p/3$$

and

$$(13)\qquad \mu(E - E(\varepsilon')) < \varepsilon^p/3 ,$$

where

$$(14)\qquad \mu(A) = K^p \int_A (f^-(x))^p dm$$

and

$$E(\varepsilon') = \left\{x: \sup_{0<\alpha<\eta} \frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt > 2\varepsilon'\right\} \cap E .$$

It follows from (11) and (14) by the Egorov theorem that there exists an integer r such that if $v \geqq r$,

$$\sup_{\substack{\alpha\in Q_v \\ 0<\alpha<\eta}} \frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt > \varepsilon' \tag{15}$$

for any x in a set F_1 with $F_1 \subset E(\varepsilon')$ and $\mu(E(\varepsilon') - F_1) < \varepsilon^p/3$. By (4) there exists a positive integer l such that

$$\left\|\frac{1}{[l(j/r)]}\sum_{i=0}^{[l(j/r)]}(T_{1/l}^i f) - \frac{r}{j}\int_0^{j/r}(T_t f)(x)dt\right\| < \varepsilon'^2/[r\eta] \tag{16}$$

$$(j = 1, 2, \cdots, [r\eta]) .$$

And it follows from this that

$$\left|\frac{1}{[l(j/r)]}\sum_{i=0}^{[l(j/r)]}(T_{1/l}^i f)(x) - \frac{r}{j}\int_0^{j/r}(T_t f)(x)dt\right| < \varepsilon' \tag{17}$$

$$(j = 1, 2, \cdots, [r\eta]) ,$$

for any x except on a set F_2 with $m(F_2) < \varepsilon'$. By (12), $\mu(F_2) < \varepsilon^p/3$. Letting $F = F_1 \cap F_2^c$ we have (8) and (9) by (13), (14), (15) and (17).

We denote $T_{1/l}$ by T so that (8) and (9) are written by (18) and (19), respectively.

$$\sup_{0\leqq j\leqq [l\eta]} \sum_{i=0}^{j} (T^i f)(x) > 0 \quad \text{on} \quad F \tag{18}$$

and

$$K^p \int_{E-F} (f^-(x))^p dm < \varepsilon^p . \tag{19}$$

We use the Chacon-Ornstein lemma.

LEMMA (Chacon-Ornstein) [3]. *If* $\sup_{0\leqq j\leqq N} \sum_{i=0}^{j} (T^i f)(x) > 0$ *on* F, *then there exist sequences of non-negative functions* $\{d_k\}$ *and* $\{f_k\}$ $(0 \leqq k \leqq N)$ *such that*

$$T^n f^+ = \sum_{k=0}^{n} T^{n-k} d_k + f_n \quad (0 \leqq n \leqq N) \tag{20}$$

and

$$\sum_{k=0}^{N} d_k \leqq f^- \quad \textit{and} \quad \sum_{k=0}^{N} d_k = f^- \ \textit{on}\ F . \tag{21}$$

REMARK. Though the lemma was proved by them under the assumptions that T is a positive linear operator on $L_1(X)$ with $\|T\| \leqq 1$ and $N = \infty$, (20) and (21) hold good without the assumptions.

Let us apply the lemma with $N = [l\eta]$. Put $n = [l(\delta - \eta)]$ and $S_n f = \sum_{k=0}^{n-1} T^k f$. We have by (5), remembering $T = T_{1/l}$,

$$\|(S_n/n)T^N f^+\| \leqq 1/n \sum_{k=0}^{n-1} \|T^{k+N} f^+\| \leqq (1+\varepsilon)\|f^+\| . \tag{22}$$

We have by (22), (20) and $f_N \geqq 0$.

$$\begin{aligned}(1+\varepsilon)\|f^+\| \geqq \|(S_n/n)T^N f^+\| &\geqq \left\|(S_n/n)\sum_{k=0}^{N} T^{N-k} d_k\right\| \\ &\geqq \left\|(S_n/n)\sum_{k=0}^{N} d_k\right\| - \left\|(S_n/n)\sum_{k=0}^{N}(T^{N-k}d_k - d_k)\right\| .\end{aligned} \tag{23}$$

Since we have

$$\begin{aligned}&\left\|(S_n/n)\sum_{k=0}^{N}(T^{N-k}d_k - d_k)\right\| \\ &\leqq \left\|1/n \sum_{k=0}^{N}\sum_{j=0}^{N-k-1} T^{n+j} d_k\right\| + \left\|1/n \sum_{k=0}^{N}\sum_{j=0}^{N-1} T^j d_k\right\| ,\end{aligned} \tag{24}$$

we have by the positivity of T, $d_k \geqq 0\ (0 \leqq k \leqq N)$, (6) and (21)

$$\left\|\sum_{k=0}^{N}(S_n/n)(T^{N-k}d_k - d_k)\right\| \leqq (2NK/n)\left\|\sum_{k=0}^{N} d_k\right\| \leqq (2NK/n)\|f^-\| . \tag{25}$$

Therefore, we have by (23) and (25)

$$(1+\varepsilon)\|f^+\| + (2NK/n)\|f^-\| \geqq \left\|(S_n/n)\sum_{k=0}^{N} d_k\right\| . \tag{26}$$

Since we have by (5),

$$\|(S_n/n)f^-\chi_E - f^-\chi_E\| \leqq 1/n \sum_{k=0}^{n} \|T^k f^-\chi_E - f^-\chi_E\| \leqq \varepsilon\|f^-\chi_E\|$$

we have by (21) and (6),

$$\begin{aligned}(1-\varepsilon)\|f^-\chi_E\| \leqq \|(S_n/n)f^-\chi_E\| &\leqq \left\|(S_n/n)\sum_{k=0}^{N} d_k\chi_F\right\| + \|(S_n/n)f^-\chi_{E-F}\| \\ &\leqq \left\|(S_n/n)\sum_{k=0}^{N} d_k\right\| + K\|f^-\chi_{E-F}\| .\end{aligned} \tag{27}$$

We have by (26) and (27)

$$(1-\varepsilon)\|f^-\chi_E\| \leqq (1+\varepsilon)\|f^+\| + (2NK/n)\|f^-\| + K\|f^-\chi_{E-F}\| . \tag{28}$$

We have by (7), (19) and (28),

$$(1-\varepsilon)\|f^-\chi_E\| \leqq (1+\varepsilon)\|f^+\| + 2\varepsilon .$$

Arbitrariness of ε implies Lemma 2.

THE PROOF OF THEOREM 1. If the theorem does not hold, then there exist a positive number δ $(0 < \delta < 1)$, a function f and a set E such that

$$\limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt - f(x) > \delta \text{ on } E \text{ and } 0 < m(E) < \infty . \tag{29}$$

Let ε' be an arbitrary positive number with $0 < \varepsilon' < 1/10$. Put $\varepsilon = \varepsilon'\delta$. By Lemma 1 we can choose a function g (put $g = T_sf$ for a sufficiently small s) such that

(30) $|f(x) - g(x)| < \varepsilon$ for any x except on a set with a measure less than $\varepsilon \min(m(E), 1)$,

$$||f - g|| < \varepsilon \tag{31}$$

and

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_tg)(x)dt = g(x) \text{ a.e. .} \tag{32}$$

Then we have by (29), (30) and (32)

$$\begin{aligned}\limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha T_t(f - g)(x)dt \\ = \limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt - f(x) + f(x) - g(x) > \delta/2 \text{ on } F ,\end{aligned} \tag{33}$$

where $F = E \cap \{x: |f - g|(x) < \varepsilon\}$ and therefore by (30),

$$m(E - F) < \varepsilon \min(m(E), 1) . \tag{34}$$

Again by Lemma 1 we can choose a non-negative function h (put $h(x) = T_s(1 - \varepsilon/2)\chi_F(x)$ for a sufficiently small s) such that

(35) $1 - \varepsilon \leqq h(x) \leqq 1$ on G with $G \subset F$ and $m(F - G) < \varepsilon \min(m(E), 1)$,

and

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_th)(x)dt = h(x) \text{ a.e. .} \tag{36}$$

Then we have by (33), (35) and (36)

$$\limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha T_t(f - g - \delta h/2)(x)dt > 0 \text{ on } G . \tag{37}$$

By Lemma 2, $h \geqq 0$ and (31)

$$\int_G ((f - g - \delta h/2)^-(x))^p dm \leqq \int_X ((f - g - \delta h/2)^+(x))^p dm \tag{38}$$
$$< \int ((f - g)^+(x))^p dm < \varepsilon^p .$$

Since we have $(f - g - \delta h/2)^-(x) > \delta/3$ on G by (30), (34) and (35) we have remembering $\varepsilon = \varepsilon'\delta$ $(0 < \varepsilon' < 1/10,\ 0 < \delta < 1)$,

$$m(E) \leqq m(G) + 2\varepsilon < (3\varepsilon')^p + 2\varepsilon' \leqq (3^p + 2)\varepsilon' . \tag{39}$$

Arbitrariness of ε' implies that $m(E) = 0$. This contradicts the assumption (29) and the proof is complete.

4. The proof of Theorem 2. In this paragraph we shall give two different proofs of Theorem 2.

LEMMA 3. *Let (T_t) $(t \geqq 0)$ be a semi-group of linear contractions on $L_1(X)$ satisfying (E). Then there exists a semi-group of positive linear contractions $(\tilde{T}_t)$ $(t \geqq 0)$ on $L_p(X)$ for any $p\,(1 \leqq p < \infty)$ (($\tilde{T}_t$) is called the linear modulus of (T_t).) such that*

(3.1) *for any $t \geqq 0$ and $f \in L_p(X)$,*

$$(\tilde{T}_t|f|)(x) \geqq |T_t f|(x) \text{ for a.a. } x ,$$

(3.2) *if $f \in L_p(X)$, then we have for almost all x,*

$$\int_0^\alpha (\tilde{T}_t|f|)(x)dt \geqq \int_0^\alpha |T_t f|(x)dt \text{ for any } \alpha \geqq 0 ,$$

and

(3.3) $\operatorname{ess.sup}_x |(\tilde{T}_t f)(x)| \leqq \operatorname{ess.sup}_x |f(x)|$

for any $f \in L_1(X) \cap L_\infty(X)$ and $t \geqq 0$.

The proofs of (3.1) and (3.2) $(p = 1)$ are found in [7]. The proof of (3.1) and (3.2) follow easily from this. The proof of (3.3) may be obtained by the fact that the operator norm of the linear modulus of bounded linear operator T is that of T [4] and the proofs of (3.1) and (3.2) $(p = 1)$ in [7].

THE PROOF OF THEOREM 2. (T_t) $(t \geqq 0)$ is extended to a semi-group of linear contractions on $L_p(X)$ for any $p\,(1 \leqq p < \infty)$. The same symbol (T_t) is used for the extended semi-group. Let $f \in L_p(X)$ and ε an arbitrary positive number. By Lemma 1, we can choose a function g (put $g = T_s f$ for a sufficiently small s) in $L_p(X)$ such that

$$||f - g|| < \varepsilon^2 , \tag{40}$$

and

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_t g)(x)dt = g(x) \text{ a.e. .} \tag{41}$$

Then we have

$$\limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt - f(x)\right| \leqq \limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha T_t(f-g)(x)dt\right| + \lim_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha (T_t g)(x)dt - g(x)\right| + |g(x)-f(x)| . \tag{42}$$

Let $(\tilde{T}_t)$ be the linear modulus of (T_t). Then we have by (3.2) of Lemma 3 and (41)

$$\leqq \limsup_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha \tilde{T}_t|f-g|(x)dt + |g(x)-f(x)| . \tag{43}$$

Applying Theorem 1 to $(\tilde{T}_t)$ we have

$$\limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt - f(x)\right| \leqq 2|f(x)-g(x)| .$$

Since we have $|f(x)-g(x)| < \varepsilon$ for any x except on a set with a measure less than ε by (40) and ε is arbitrary we have Theorem 2.

THE SECOND PROOF OF THEOREM 2. We shall give another proof of Theorem 2 by making use of Lemma 1 and the following Lemma 4. Let a be an arbitrary positive number. Put

$$f^{a-}(x) = \min(a, |f(x)|)\frac{f(x)}{|f(x)|} \tag{44}$$

and

$$f^{a+}(x) = (\max(a, |f(x)|) - a)\frac{f(x)}{|f(x)|} ,$$

where $f^{a-}(x)$, $f^{a+}(x) = 0$ whenever $f(x) = 0$.

LEMMA 4. *Let (T_t) $(t \geqq 0)$ be a semi-group of linear contractions satisfying (E). If $f \in L_p(X)$ $(1 \leqq p < \infty)$ and*

$$\sup_{0<\alpha\leqq 1}\left|\frac{1}{\alpha}\int_0^\alpha (T_t f)(x)dt\right| > a \text{ on } E(a) \quad (a > 0) , \tag{45}$$

then we have

$$\int_{E(a)} (a - |f^{a-}(x)|)dm \leqq \int_X |f^{a+}(x)|dm . \tag{46}$$

PROOF. Let ε be an arbitrary positive number and F any subset of

$E(a)$ with a finite measure. By the same argument in the proof of Lemma 2 (from (8) to (19)), there exist a positive integer l and a set G such that

$$\sup_{0<j\leqq l}\left|\frac{1}{j}\sum_{i=0}^{j-1}(T_{1/l}^{i}f)(x)\right| > a \text{ on } G \tag{47}$$

and

$$G\subset F \quad\text{and}\quad m(F-G)<\varepsilon\,. \tag{48}$$

We use the Chacon lemma [2]. Let us denote $T_{1/l}$ by T.

LEMMA (Chacon [2]). *If* $f\in L_p(X)(1\leqq p<\infty)$ *and*

$$\sup_{1\leqq j\leqq l}\left|\frac{1}{j}\sum_{k=0}^{j-1}(T^{k}f)(x)\right| > a \text{ on } G\,,$$

then we have

$$\int_G (a-|f^{a-}(x)|)dm \leqq \int_X |f^{a+}(x)|\,dm\,. \tag{49}$$

We have (49) by (47) and lemma. Since ε is arbitrary we have by (48) and (49),

$$\int_F (a-|f^{a-}(x)|)dm \leqq \int |f^{a+}(x)|\,dm \tag{50}$$

for any subset $F(F\subset E(a))$ with $m(F)<\infty$.

Since the measure space (X, B, m) is σ-finite we have Lemma 4 by (50).

THE PROOF OF THEOREM 2. Let $f\in L_p(X)$ $(1\leqq p<\infty)$ and ε an arbitrary positive number with $0<\varepsilon<1$. We can choose a function g in $L_p(X)$ by Lemma 1 (put $g=T_sf$ for a sufficiently small s) such that

$$\|f-g\|<\varepsilon^3 \tag{51}$$

and

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha (T_tg)(x)dt = g(x) \text{ a.e.}\,. \tag{52}$$

Then we have

$$\limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt - f(x)\right| \leqq \limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha T_t(f-g)(x)dt\right| + \limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha (T_tg)(x)dt - g(x)\right| + |g(x)-f(x)|\,. \tag{53}$$

By (52) and (53)

(54) $$\limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt - f(x)\right|$$
$$\leqq \limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha T_t(f-g)(x)dt\right| + |g(x)-f(x)|\,.$$

Put

(55) $$E = \left\{x: \limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha T_t(f-g)(x)dt\right| > \varepsilon\right\}.$$

Then we have by Lemma 4,

(56) $$\int_E (\varepsilon - |(f-g)^{\varepsilon-}(x)|)dm \leqq \int |(f-g)^{\varepsilon+}(x)|\,dm\,.$$

By the definition of $(f-g)^{\varepsilon+}$ and (51) we have

(57) $$\int |(f-g)^{\varepsilon+}(x)|\,dm \leqq \int_{A(\varepsilon)} |f(x)-g(x)|\,dm$$
$$\leqq m\{x: \varepsilon < |f(x)-g(x)| \leqq 1\} + \int |f(x)-g(x)|^p dm \leqq \varepsilon^{2p} + \varepsilon^{3p}\,,$$

where $A(\varepsilon) = \{x: |f(x)-g(x)| > \varepsilon\}$. There exists a subset F of E by (51) such that

(58) $$|f(x)-g(x)| < \varepsilon/2 \text{ on } F \text{ and } m(E-F) < (2\varepsilon^2)^p\,.$$

Since we have by the definition of $(f-g)^{\varepsilon-}$ and (58)

$$\varepsilon - |(f-g)^{\varepsilon-}(x)| \geqq 0 \text{ and } \varepsilon - |(f-g)^{\varepsilon-}(x)| \geqq \varepsilon/2 \text{ on } F\,,$$

we have by (56), (57) and (58),

(59) $$m(E) \leqq m(F) + (2\varepsilon^2)^p \leqq (2/\varepsilon)(\varepsilon^{2p} + \varepsilon^{3p}) + 2^p\varepsilon^{2p} < (2^p + 4)\varepsilon\,.$$

We have by (51)

(60) $|g(x) - f(x)| < \varepsilon$ for any x except on a set with a measure less than ε.

And therefore we have by (54), (59) and (60)

$$\limsup_{\alpha\to 0}\left|\frac{1}{\alpha}\int_0^\alpha (T_tf)(x)dt - f(x)\right| < 2\varepsilon$$

for any x except on a set with a measure less than $(2^p + 5)\varepsilon$. Arbitrariness of ε implies Theorem 2.

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REMARK. The author was informed after he wrote the note that Professor R. Sato of Josai University proved independently Theorems 1 and 2 by different methods. Sato's proof is based on the maximal ergodic lemma of the author in [6]. He proved it for any semi-group of positive bounded linear operators on $L_p(X)$ $(1 \leqq p < \infty)$ with $m(X) < \infty$ [12].

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