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ON BOUNDED FUNCTIONS IN THE ABSTRACT HARDY SPACE THEORY*

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1. This paper is a continuation of our former work [6]. The purpose of this note is to study the essential ranges of bounded functions in abstract Hardy spaces in the sense of H. König. Let (X, Σ, m) be a probability measure space and H a weak* closed subalgebra of the sup-norm algebra L^∞ of the bounded m-measurable functions, satisfying $1 \in H$ and $\int uvdm = \int udm \int vdm$ for any $u, v \in H$. The main result we want to show is the following: For every non-constant $u \in H$ there exists a unique Carathéodory domain A such that $m\{x; u(x) \in \bar{A}\} = 1$ and $m\{x; |u(x) - b| < \varepsilon\} > 0$ for any $\varepsilon > 0$ and any $b \in \partial A$. We shall show it in the following form: The polynomial convex hull K of the value carrier of a non-constant $u \in H$ coincides with the polynomial convex hull of the closure $\bar{A}$ of a component A of the interior of K and it holds further $m\{x; u(x) \in \bar{A}\} = 1$ (Theorem A in Section 3). "Carathéodory domain" and "value carrier" are defined in Section 3. In Section 2 we shall give several lemmas. The main lemma is Lemma 2. The key tools we shall use frequently are some well-known theorems on polynomial approximation, such as Mergelyan's theorem etc. All properties we shall show follow essentially from the multiplicativity of the integration on H.

A prototype of our space H is the classical $H^\infty(T)$: Let $T = \{|z| = 1\}$ and consider the normalized Lebesgue measure L on T. Let $H^\infty(U)$ be the set of all bounded holomorphic functions in the open unit disc $U = \{|z| < 1\}$. As is well-known, every $f \in H^\infty(U)$ defines a radial limit function $f(e^{i\theta})$: $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e.. We denote the set of all such limiting functions by $H^\infty(T)$. Then it is well-known that $H^\infty(T)$ is weak* closed and satisfies all conditions for our space H. The author would like to acknowledge several helpful conversations with Professor Heinz König.

2. We shall start with some definitions:

Definition 1. Let K be a compact set in the complex plane $\boldsymbol{C}$. The algebra $C(K)$ consists of the continuous functions on K, endowed with

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the supremum norm. The algebra $P(K)$ consists of the functions in $C(K)$ which can be approximated uniformly on K by polynomials in z. The algebra $R(K)$ consists of the functions in $C(K)$ which can be approximated uniformly on K by rational functions with poles off K. For a set $A \subset \boldsymbol{C}$ we denote by $\|f\|_A$ the supremum norm of an $f \in C(A)$.

We shall give a lemma on integrals $\int u dm$ of $u \in H$.

LEMMA 1. *Let $K \subset \boldsymbol{C}$ be a compact set and $\hat{K}$ its polynomial convex hull. Then for any $u \in H$ with $m\{x; u(x) \in K\} = 1$ we have $\int u dm \in \hat{K}$.*

If u is in particular not constant, we have $\int u dm \in \hat{K}^0$: the interior of $\hat{K}$.

PROOF. Since the integration is multiplicative on H, we have $\int P(u) dm = P\left(\int u dm\right)$ for any polynomial $P(z)$. Hence we get

$$\left|P\left(\int u dm\right)\right| \leqq \int |P(u)| dm \leqq \sup_{z \in K} |P(z)| .$$

Therefore we have $\int u dm \in \hat{K}$. Suppose next that u is not constant and $\int u dm \in \partial \hat{K}$. Set $a = \int u dm$. Since $\hat{K}$ is polynomially convex, $\hat{K}^c$ is connected. Hence by Gonchar's criterion for peak points for $R(K)$ every boundary point of $\hat{K}$ is a peak point for $R(\hat{K})$. Hence there exists a function $f(z) \in R(\hat{K})$ such that $f(a) = 1$ and $|f(z)| < 1$ for $z \in \hat{K} \backslash \{a\}$. By Mergelyan's theorem we have $R(\hat{K}) = P(\hat{K})$. Therefore there is a sequence of polynomials $P_n(z)$ converging to $f(z)$ uniformly on $\hat{K}$. Since $P_n(u) \in H$ and they converge to $f(u)$ in the sup-topology, we have $f(u) \in H$ and

$$1 = f(a) = f\left(\int u dm\right) = \lim_{n \to \infty} P_n\left(\int u dm\right) = \lim_{n \to \infty} \int P_n(u) dm = \int f(u) dm .$$

Hence we have $f(u) = 1$ a.e., that is, $u = a$ a.e., which is a contradiction. This completes the proof.

As a consequence we have a sufficient condition for a $u \in H$ to be constant.

COROLLARY 1. *Let $K \subset \boldsymbol{C}$ be a compact set such that K has no interior point and K^c is connected. Then every $u \in H$ with $m\{x; u(x) \in K\} = 1$ is constant.*

Now using Lemma 1 we can show the following fundamental lemma.

LEMMA 2. *Let K be a compact set in $\boldsymbol{C}$ with connected complement.*

Let $u \in H$ be non-constant and $m\{x; u(x) \in K\} = 1$. Then there is a unique component A of the interior K^0 of K with $\int udm \in A$ and for this component it holds $m\{x; u(x) \in \bar{A}\} = 1$. This component is naturally a simply connected domain.

In order to prove this lemma we need two lemmas.

LEMMA 3. *Let u, K be the same as in Lemma 2. Then the number $\int udm$ belongs to a unique component A of K^0 and it holds $m\{x; u(x) \in \partial K \cup \bar{A}\} = 1$.*

LEMMA 4. *Let K be a compact set in $\mathbf{C}$ with connected complement and A be a component of K^0. Then there exist polynomials $P_n(z)$ with $\|P_n\|_K \leqq 1$ such that $P_n(z) \to 0$ for all $z \in \partial K \backslash \bar{A}$ and $P_n(z) \to 1$ for all $z \in A$.*

PROOF OF LEMMA 3. Since u is not constant, by Lemma 1 the interior K^0 of K is not empty and it holds $\int udm \in K^0$. Hence there exists a unique component A of K^0 with $\int udm \in A$. Let $f(z) = 1$ on A and $= 0$ on $K^0 \backslash A$, so that $f(z)$ is bounded and holomorphic on K^0. Since K^c is connected, by a version of Farrell-Rubel-Shields theorem (Gamelin [1], p. 154) there is a sequence of polynomials $P_n(z)$ with $\|P_n\|_K \leqq \|f\|_{K^0} = 1$ such that $P_n(z) \to f(z)$ for all $z \in K^0$. We consider the set $\{P_n(u)\}_{n=1}^{\infty}$. Since $\|P_n(u)\|_\infty \leqq 1$ and H is weak* closed, there exist a $v \in H$ with $\|v\|_\infty \leqq 1$ and a subsequence $\{P_{n_j}(u)\}$ of $\{P_n(u)\}$ such that $P_{n_j}(u) \to v$ in the weak* topology. Since $\int P_n(u)dm = P_n\left(\int udm\right)$ and $\int udm \in A$, we get $\int P_n(u)dm \to 1$ by the choice of P_n. Hence we have $\int vdm = 1$. Since $\|v\|_\infty \leqq 1$, by Lemma 1 we have $v = 1$. As $\|P_n(u)\|_\infty \leqq 1$, we have $\mathrm{Re}\,(1 - P_n(u)) \geqq 0$ and using Kolmogorov's inequality we have for any $0 < p < 1$

$$\cos p\pi/2 \int \left| P_n(u) - i\,\mathrm{Im}\left(\int P_n(u)dm\right)\right|^p dm \leqq \left(\int \mathrm{Re}\,(1 - P_n(u))dm\right)^p .$$

Since $P_{n_j}(u) \to 1$ in the weak* topology, we obtain $\int (1 - P_{n_j}(u))dm \to 0$ and so $\int \mathrm{Re}\,(1 - P_{n_j}(u))dm \to 0$ and $\mathrm{Im}\left(\int P_{n_j}(u)dm\right) \to 0$ as $j \to \infty$. Hence there exists a subsequence of $\{P_{n_j}\}$, which we write as $\{Q_n\}$, such that $Q_n(u) \to 1$ a.e. on X. Since $Q_n(z) \to 0$ for $z \in K^0 \backslash A$, we get $m\{x; u(x) \in K^0 \backslash A\} = 0$ and hence $m\{x; u(x) \in \partial K \cup \bar{A}\} = 1$. This completes the proof of Lemma 3.

PROOF OF LEMMA 4. Since K^c is connected, we have $A(K) = P(K) = R(K)$ by Mergelyan's theorem and hence $R(K)$ is dirichlet on ∂K. Hence every homomorphism $\phi: R(K) \to C$ has a unique representing measure on ∂K. For any $a \in A$ we denote by m_a the unique representing measure for the evaluation homomorphism at a. As is known, m_a and m_b are mutually boundedly absolutely continuous for any $a, b \in A$, i.e., there is a constant $c = c(a, b)$ such that $c^{-1}m_a \leqq m_b \leqq cm_a$. Now let us fix a point $a_0 \in A$ and let $E = \partial K \backslash \bar{A}$. Then E is an F_σ set, i.e., a union of an increasing sequence of closed sets in C. Further we have $m_{a_0}(E) = 0$, since m_{a_0} is supported on ∂A. Hence by Forelli's lemma (Gamelin [1], p. 43) there are $f_n \in R(K)$ such that $\|f_n\|_K \leqq 1$, $f_n(z) \to 0$ for all $z \in E$ and $f_n \to 1$ m_{a_0}-a.e. on ∂A. Since m_a is absolutely continuous with respect to m_{a_0} for any $a \in A$, we have $f_n \to 1$ m_a-a.e. on ∂A and so $f_n(a) = \int f_n dm_a \to 1$ for all $a \in A$. As $R(K) = P(K)$, it is easily seen that there are polynomials $P_n(z)$ with $\|P_n\|_K \leqq 1$ such that $P_n(z) \to 0$ for all $z \in \partial K \backslash \bar{A}$ and $P_n(z) \to 1$ for all $z \in A$. That completes the proof of Lemma 4.

PROOF OF LEMMA 2. Using Lemma 3 and Lemma 4 we apply the argument in the proof of Lemma 3 and obtain the desired conclusion.

As immediate consequences of Lemma 2 we have the following corollaries, whose proofs we omit.

COROLLARY 2. *Let A, B be two compact sets in C such that $(A \cup B)^c$ is connected and $A \cap B$ consists of only one point or is empty. Then for every $u \in H$ with $m\{x; u(x) \in A \cup B\} = 1$ it holds either $m\{x; u(x) \in A\} = 1$ or $m\{x; u(x) \in B\} = 1$.*

COROLLARY 3 ([6] Theorem 4). *Let A, B be two disjoint compact sets in C such that $(A \cup B)^c$ is connected. Let J be a Jordan arc joining a boundary point of A with a boundary point of B such that the set $J \cap (A \cup B)$ consists of the end points of J. Then for every $u \in H$ with $m\{x; u(x) \in A \cup B \cup J\} = 1$ it holds $m\{x; u(x) \in A\} = 1$ or $m\{x; u(x) \in B\} = 1$ or u is constant.*

Now a bounded domain in C is said to be a Jordan domain if its boundary is a Jordan curve.

COROLLARY 4. *Let D_1, D_2 be Jordan domains with $D_1 \cap D_2 \neq \emptyset$. For any non-constant $u \in H$ with $m\{x; u(x) \in \bar{D}_j\} = 1$ $(j = 1, 2)$ there exists a Jordan domain $D \subset D_1 \cap D_2$ with $m\{x; u(x) \in \bar{D}\} = 1$.*

PROOF. The set $K = \overline{D_1 \cap D_2} = \bar{D}_1 \cap \bar{D}_2$ is compact and the interior of K is $D_1 \cap D_2$. Further K^c is clearly connected. By a theorem of Kerékjártó every component of $D_1 \cap D_2$ is also a Jordan domain. Hence by Lemma 2 we have the desired conclusion.

3. We shall next define "value carrier" and state our main result once more and prove it.

DEFINITION 2. The value carrier $\omega(h)$ of a measurable function h on X is defined to be the set of all complex numbers $a \in C$ such that $m\{x; |h(x) - a| < \varepsilon\} > 0$ for all $\varepsilon > 0$. Thus $\omega(h)$ is closed and not empty.

DEFINITION 3. Let G be a bounded simply connected domain, and let G_∞ be the component of $(\bar{G})^c$ containing the point at infinity. Then G is said to be a Carathéodory domain if G and G_∞ have the same boundary.

THEOREM A. *Let $u \in H$ be not constant. Then the polynomial convex hull $\widehat{\omega(u)}$ of $\omega(u)$ coincides with the polynomial convex hull of the closure $\bar{A}$ of a component A of $(\widehat{\omega(u)})^0$ containing $\int udm$ and it holds $m\{x; u(x) \in \bar{A}\} =$ 1. In particular A is a bounded simply connected domain and it holds $\partial\widehat{\omega(u)} = \partial A$, and hence A is a Carathéodory domain.*

PROOF. Let $K = \widehat{\omega(u)}$. Then one can see easily that $m\{x; u(x) \in K\} =$ 1 and (*) for any $\varepsilon > 0$ and any $a \in \partial K$ it holds $m\{x; |u(x) - a| < \varepsilon\} > 0$. K^c is connected, since K is polynomially convex. Now let A be the component of K^0 with $\int udm \in A$. Then by Lemma 2 we have $m\{x; u(x) \in \bar{A}\} = 1$. Hence the property (*) of K implies $\partial K \subset \bar{A}$. Since K is polynomially convex, we have $K = \widehat{\partial K} \subset \hat{\bar{A}}$ and so $K = \hat{\bar{A}}$. The last assertion is then clear. We have thus proved the theorem.

REMARK. For every $h \in L^\infty$ the set $\widehat{\omega(h)}$ is the unique compact set K such that (i) K^c is connected, (ii) $m\{x; h(x) \in K\} = 1$, and (iii) $m\{x;$ $|h(x) - a| < \varepsilon\} > 0$ for any $\varepsilon > 0$ and any $a \in \partial K$. In fact, let K_1, K_2 be two compact sets in C satisfying (i), (ii) and (iii). Then by (ii) for K_1 and (iii) for K_2 we have $\partial K_2 \subset K_1$. Hence by (i) for K_1, K_2 we have $K_2 = \widehat{\partial K_2} \subset K_1$. Similarly we have $K_1 \subset K_2$ and hence $K_1 = K_2$.

Using this remark we see that Theorem A is equivalent to the following Theorem B.

THEOREM B. *For every non-constant $u \in H$ there exists a unique compact set K satisfying the following conditions*: (i) *K^c is connected.* (ii) $m\{x; u(x) \in K\} = 1$. (iii) $m\{x; |u(x) - a| < \varepsilon\} > 0$ *for any $\varepsilon > 0$ and any $a \in \partial K$. Further there exists a unique component A of the interior of K containing $\int udm$. This component is simply connected and we have $K =$ $\hat{\bar{A}}$, $\partial K = \partial A$ and $m\{x; u(x) \in \bar{A}\} = 1$. In particular K is connected.*

Now using Corollary 4 to Lemma 2 one can represent the set K in

Theorem B as follows.

COROLLARY 5. *Let u, K be the same as in Theorem B. Then we have*

$$K = \bigcap_{D \in \Omega} \bar{D} ,$$

where Ω *is the set of all Jordan domains* D *with* $m\{x; u(x) \in \bar{D}\} = 1$.

PROOF. Ω is clearly not empty. Set $L = \bigcap_{D \in \Omega} \bar{D}$. Then by Corollary 4, L is not empty and compact. L^c is clearly connected. Since every $\bar{D}$ is closed, L satisfies the condition (ii) in Theorem B. It is also easily seen that L satisfies the condition (iii) in Theorem B. Hence by Theorem B we have $L = K$. This completes the proof.

REMARKS. 1. In Corollary 5 one can not replace $\bar{D}$ by D, which is shown by the following example: Let us consider the classical $H^\infty(T)$. Let $u(e^{i\theta}) = e^{i\theta}$ and $C_\alpha = \{|z + e^{i\alpha}| < 2\}$ $(0 \leqq \alpha < 2\pi)$. Then we have $L\{e^{i\theta}; u(e^{i\theta}) \in C_\alpha\} = 1$ and C_α are Jordan domains. Since $\bigcap_{0 \leqq \alpha < 2\pi} C_\alpha = U = \{|z| < 1\}$, we have $L\{e^{i\theta}; u(e^{i\theta}) \in \bigcap_{D \in \Omega} D\} = 0$.

2. In Theorem B one can not expect in general that K^0 is a Jordan domain or a simply connected domain. In fact, let D be the "cornucopia" (Figure 1), which is a ribbon winding the outside of the circle $T = \{|z| = 1\}$ and accumulating on that circle. Let $f(z)$ be a conformal map from U onto D. Then we have $f(e^{i\theta}) \in H^\infty(T)$ and $L\{e^{i\theta}; f(e^{i\theta}) \in \partial D\} = 1$. K in Theorem B is then $\bar{D} \cup \bar{U}$ and the interior of K is $D \cup U$, which is disconnected. And A in Theorem B is D. This shows that the simply connected domain A in Theorem B is in general not a Jordan domain.

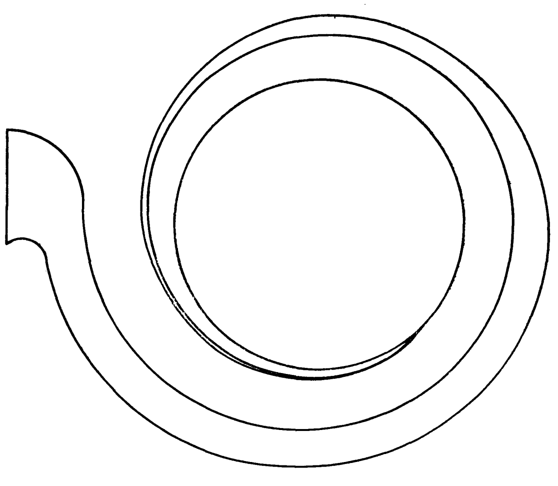

FIGURE 1. The cornucopia.

4. As topics related to the preceding section we shall show the following results.

PROPOSITION 1. *Let D be the cornucopia in Remark 2 in Section 3. If $u \in H$ and $m\{x; u(x) \in \bar{D}\} = 1$, then it holds $m\{x; u(x) \in \bar{D} \backslash T\} = 1$ or $m\{x; u(x) \in T\} = 1$.*

PROOF. Let $w = f(z)$ be a conformal mapping from D onto $|w| < 1$. Then the boundary element $\{|z| = 1\}$ corresponds by Carathéodory's theorem to a point on the unit circle $\{|w| = 1\}$. We may assume that this point is $w = 1$. One sees in this case that $f(z)$ is continuous on $\bar{D}$. Let $g(z) = f(z)$ for $z \in \bar{D}$, $= 1$ for $z \in U = \{|z| < 1\}$. Then $g(z)$ is continuous on $\bar{D} \cup \bar{U}$ and holomorphic on $D \cup U$. Since $(\bar{D} \cup \bar{U})^c$ is clearly connected, by Mergelyan's theorem there exists a sequence of polynomials converging to $g(z)$ uniformly on $\bar{D} \cup \bar{U}$. Hence as before we have $g(u) \in H$ and $|g(u)| \leqq 1$ a.e.. By our generalization of Löwner's lemma ([6]) we have $0 = m\{x; g(u(x)) = 1\} = m\{x; |u(x)| = 1\}$ or $g(u(x)) = 1$ a.e.. The latter implies $m\{x; u(x) \in T\} = 1$. This completes the proof.

PROPOSITION 2. *Let D be the simply connected domain bounded by the arcs*

$$\gamma: \begin{cases} 0 < x \leqq 2/3\pi\,, & y = \sin x^{-1} + x \\ x = 2/3\pi\,, & -1 \leqq y \leqq 2/3\pi - 1 \\ 2/3\pi \geqq x > 0\,, & y = \sin x^{-1} \\ x = 0\,, & -1 \leqq y \leqq 1 \end{cases}$$

where $z = x + iy$. If $u \in H$ and $m\{x; u(x) \in \bar{D}\} = 1$, then it holds $m\{x; u(x) \in \bar{D} \backslash i[-1, 1]\} = 1$ or u is constant.

PROOF. In a similar way to the proof of Proposition 1 we see that $m\{x; u(x) \in \bar{D} \backslash i[-1, 1]\} = 1$ or $m\{x; u(x) \in i[-1, 1]\} = 1$. In the latter case u is constant by Corollary 1.

Combining Proposition 2 with Lemma 2 we have the following result.

PROPOSITION 3. *Let D be as in Proposition 2 and $D' = \{z = x + iy; -x + iy \in D\}$. If $u \in H$ is not constant and $m\{x; u(x) \in \bar{D} \cup \bar{D}'\} = 1$, then it holds $m\{x; u(x) \in \bar{D} \backslash i[-1, 1]\} = 1$ or $m\{x; u(x) \in \bar{D}' \backslash i[-1, 1]\} = 1$.*

As an application we have the following: Let D be as in Proposition 2 and $K = \bar{D}$. Let m be the unique representing measure on ∂K for the homomorphism from $P(K)$ to $C: f \to f(a)$, where a is a point in D. Then we have $m\{i[-1, 1]\} = 0$.

5. Final remark: All our results hold for any $u \in L^\infty$ such that

$\int u^n dm = \left(\int u dm\right)^n$ $(n = 0, 1, 2, \cdots)$. We have only to take the weak* closure of the set of all finite linear combinations of $\{u^n\}_{n=0}^{\infty}$.

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