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A GENERATION THEOREM FOR SEMIGROUPS OF GROWTH ORDER α

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Introduction. This paper is concerned with the generation of (operator) semigroups of growth order α.

Extending the notion of a semigroup of class (C_0), Da Prato [1] introduces the notion of a semigroup of growth order n, n is a nonnegative integer. Roughly speaking, a semigroup $\{T(t); t > 0\}$ of bounded linear operators on a Banach space is of growth order n if $\|t^n T(t)\|$ is bounded as t tends to zero; in particular, $\{T(t)\}$ is of growth order 0 if and only if it belongs to class (C_0). In [1], Da Prato gave a characterization for the Laplace transform of $t^n T(t)$ through the notion of a closable linear operator of type n and its resolvent of order n. Namely, if A_0 is the infinitesimal generator of a semigroup $\{T(t)\}$ of growth order n, then A_0 is of type n and its resolvent $S(\lambda, A_0)$ of order n is equal to the Laplace transform of $t^n T(t)$ and satisfies a certain stability condition. Viceversa if B is of type n and its resolvent $S(\lambda, B)$ of order n satisfies the stability condition mentioned above, then there exists a unique semigroup of growth order n such that $S(\lambda, B) = S(\lambda, A_0)$, where A_0 is the infinitesimal generator of the constructed semigroup. This result was generalized by Zafievskii [10] to the case of fractional α (cf. also Sobolevskii [8]). So, if it can be shown that $B = A_0$, then their result is proved to be a characterization for the infinitesimal generator of a semigroup of growth order α. But, this is not expected in general as noted in [2].

The purpose of this paper is to give a characterization for the *closure* of the infinitesimal generator of a semigroup of growth order α. We first clarify some properties of the closure of the infinitesimal generator and then modify the construction of the semigroup stated in [1]. In this way, we obtain a criterion for a closed linear operator in a Banach space to be the closure of the infinitesimal generator of a semigroup of growth order α.

The main result of this paper is stated in §1 and the proof of it is given in §3 and §4. §2 is devoted to the preliminaries.

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1. Statement of the result. Let X be a complex Banach space. We denote by $\mathscr{B}(X)$ the set of all bounded linear operators on X to X. A one-parameter family $\{T(t); t > 0\}$ in $\mathscr{B}(X)$ is called a *semigroup* on X if $T(t + s) = T(t)T(s)$ for $t, s > 0$ and if $T(t)$ is strongly continuous for $t > 0$. We denote by A_0 the *infinitesimal generator* of $\{T(t)\}$, i.e.,

$$A_0 u = \lim_{h \to 0+} h^{-1}[T(h)u - u]$$

whenever the limit exists. If A_0 is closable, then the closure of A_0 is called the *complete infinitesimal generator* of $\{T(t)\}$.

DEFINITION 1.1. Let $\alpha > 0$. Then a semigroup $\{T(t)\}$ on X is said to be of *growth order* α if it satisfies the following three conditions:

(i) If $T(t)u = 0$ for all $t > 0$ then $u = 0$.

(ii) $\| t^\alpha T(t) \|$ is bounded as t tends to zero.

(iii) $X_0 = \bigcup_{t>0} T(t)[X]$ is dense in X.

A semigroup of growth order α has the complete infinitesimal generator (see [1], Theorem 1.1; cf. also Lemma 3.1 below). Examples of semigroups of growth order α will be found in Krein [4].

Let A be a closed linear operator (with domain $D(A)$ and range $R(A)$) in X. Then a linear manifold D contained in $D(A)$ is called a *core* of A if the closure of the restriction of A to D is again A (see Kato [3], III-§ 5.3).

Now our result is given by

THEOREM 1.2. *Let n be the integral part of $\alpha > 0$. Then a closed linear operator A in X is the complete infinitesimal generator of a semigroup of growth order α if and only if the following four conditions are satisfied:*

(I) *There is a real number ω such that for each $\xi > \omega$, $R(\xi - A)$ contains $D(A^{n+1})$ and $(\xi - A)^{-1}$ exists.*

(II) *There is a constant $M > 0$ such that for $v \in D(A^{n+1})$,*

$$\| (\xi - A)^{-m} v \| \leqq \frac{M}{(m-1)!} \frac{\Gamma(m - \alpha)}{(\xi - \omega)^{m-\alpha}} \| v \|, \ \xi > \omega, \ m = k(n + 1) \, ,$$

where $k = 1, 2, \cdots$.

(III) *$D(A^{n+2})$ is a core of A and $D(A)$ is dense in X.*

(IV) *For some $b > \omega$, $(b - A)^{n+1}$ is closable.*

The following corollary is announced by Zabreiko-Zafievskii [9].

COROLLARY 1.3. *Let $0 < \alpha < 1$. Then a closed linear operator A in X is the complete infinitesimal generator of a semigroup of growth order α if and only if*

(I′) *there is a real number* ω *such that for each* $\xi > \omega$, $(\xi - A)^{-1}$ *exists and belongs to* $\mathscr{B}(X)$,

(II′) *there is a constant* $M > 0$ *such that for* $\xi > \omega$ *and* $m \geqq 1$,

$$\|(\xi - A)^{-m}\| \leqq \frac{M}{(m-1)!}\,\frac{\Gamma(m-\alpha)}{(\xi-\omega)^{m-\alpha}}\,,$$

(III′) $D(A)$ *is dense in* X.

2. Preliminaries. Let $\{T(t); t > 0\}$ be a semigroup on X. We denote by ω_0 the *type* of $\{T(t)\}$: $\omega_0 = \lim_{t\to\infty} t^{-1}\log\|T(t)\|$ (it is well known that ω_0 is finite or $-\infty$), and by Σ the *continuity set* of $\{T(t)\}$:

$$\Sigma = \{u \in X;\ \|T(t)u - u\| \to 0 \text{ as } t \to 0+\}\,.$$

Then $X_0 = \bigcup_{t>0} T(t)[X]$ is dense in Σ and we have

LEMMA 2.1 (see [7], § 2). *Let* A_0 *be the infinitesimal generator of a semigroup on* X. *Then for each* λ *with* $\mathrm{Re}\,\lambda > \omega_0$, $R(\lambda - A_0) \supset \Sigma$ *and* $(\lambda - A_0)^{-1}$ *exists.*

Let Ω be the restriction of A_0 to

$$D(\Omega) = \{u \in D(A_0);\ A_0 u \in \Sigma\}\,.$$

Then we have

LEMMA 2.2. *For each* λ *with* $\mathrm{Re}\,\lambda > \omega_0$, $R(\lambda - \Omega) = \Sigma$ *and* $(\lambda - \Omega)^{-1}$ *exists; furthermore,*

$$(\lambda - \Omega)^{-1}v = \int_0^\infty e^{-\lambda t}T(t)v\,dt\,, \quad v \in \Sigma\,. \tag{2.1}$$

PROOF. Let $\mathrm{Re}\,\lambda > \omega_0$. To see that $R(\lambda - \Omega) = \Sigma$, it suffices to show that $R(\lambda - \Omega) \supset \Sigma$. Let $v \in \Sigma$ and set

$$J(\lambda)v = \int_0^\infty e^{-\lambda t}T(t)v\,dt\,. \tag{2.2}$$

Then it follows that $J(\lambda)v \in D(A_0)$ and $A_0J(\lambda)v = \lambda J(\lambda)v - v$ (see [7], § 2). But since $J(\lambda)v \in \Sigma$, we see that $J(\lambda)v \in D(\Omega)$ and

$$v = (\lambda - \Omega)J(\lambda)v\,, \qquad v \in \Sigma\,. \tag{2.3}$$

This shows that $\Sigma \subset R(\lambda - \Omega)$. Since $\lambda - \Omega$ is invertible (see Lemma 2.1), (2.1) follows from (2.3) and (2.2). q.e.d.

Now let us introduce the notion of a semigroup of class $(\mathfrak{S}_k)$.

DEFINITION 2.3 (see [6], § 3). Let A_0 be the infinitesimal generator of a semigroup $\{T(t)\}$ on X. Then $\{T(t)\}$ is said to be of class $(\mathfrak{S}_k)$, where k is a nonnegative integer, if it satisfies the following three conditions:

(α_1) A_0 has the closure A: $A = \bar{A}_0$, and there is $\omega > \omega_0$ such that for each $\xi > \omega$, $(\xi - A)^{-1}$ exists.

(α_2) $D(A^k) \subset \Sigma$.

(α_3) X_0 is dense in X.

Since $\Omega \subset A_0$, Ω has the closure $\bar{\Omega}$ if A_0 is closable. In this connection, we have

LEMMA 2.4 (see [6], §3). *Suppose that $\{T(t)\}$ satisfies condition* (α_1). *Then $\bar{\Omega} = \bar{A}_0$.*

As mentioned in §1, $A = \bar{A}_0$ is called the complete infinitesimal generator of $\{T(t)\}$.

LEMMA 2.5. *Let A be the complete infinitesimal generator of a semigroup of class* ($\mathfrak{S}_k$). *Then A has the following properties*:

(d$_1$) *There is a real number ω such that for each $\xi > \omega$, $R(\xi - A)$ contains $D(A^k)$ and $(\xi - A)^{-1}$ exists.*

(d$_2$) *$D(A^{k+1})$ is a core of A and $D(A)$ is dense in X.*

PROOF. Since $\Sigma \subset R(\xi - A)$ for each $\xi > \omega_0$ (see Lemma 2.1), (d$_1$) follows from Definition 2.3. (d$_2$) is proved in [6]. q.e.d.

3. Complete infinitesimal generators. In this section we shall prove the "only if" part of Theorem 1.2. The following lemma shows that condition (i) is stronger than condition (α_1).

LEMMA 3.1 (see [7], §3). *Let A_0 be the infinitesimal generator of a semigroup $\{T(t)\}$ on X. Suppose that $\{T(t)\}$ satisfies condition* (i). *Then A_0 has the closure A: $A = \bar{A}_0$, and*

$$(3.1)\qquad (d/dt)T(t)u = A_0T(t)u = T(t)Au\ , \quad u \in D(A)\ , \quad t > 0\ .$$

Furthermore, for each λ with $\mathrm{Re}\,\lambda > \omega_0$, *$(\lambda - A)^{-1}$ exists and*

$$(3.2)\qquad (\lambda - A)^{-(m+1)}v = (1/m!)\int_0^\infty t^m e^{-\lambda t}T(t)v\,dt\ , \quad v \in \Sigma\ , \quad m \geqq 0\ .$$

LEMMA 3.2. *Suppose that $\{T(t)\}$ satisfies condition* (i), *and let $A = \bar{A}_0$. Let m be a positive integer and λ be a complex number with* $\mathrm{Re}\,\lambda > \omega_0$. *Then for $u \in D(A^m)$,*

$$(3.3)\qquad (m-1)!\ T(t)u = \int_0^\infty s^{m-1}e^{-\lambda s}T(t+s)(\lambda - A)^m u\,ds\ , \qquad t > 0\ .$$

Furthermore, $(\lambda - A)^m$ is closable.

PROOF. To see that (3.3) holds, let $u \in D(A)$ and $\mathrm{Re}\,\lambda > \omega_0$. Then we have by (3.1) that for $t > 0$,

$$(d/ds)e^{-\lambda s}T(t+s)u = -e^{-\lambda s}T(t+s)(\lambda - A)u\ , \quad s \geqq 0\ .$$

Integrating this equality, we obtain (3.3) with $m = 1$. Thus, it suffices to show that

$$\int_0^\infty s^m e^{-\lambda s}T(t+s)(\lambda - A)uds = m\int_0^\infty s^{m-1}e^{-\lambda s}T(t+s)uds\ .$$

But, to see this, it suffices to note that $(d/ds)[s^m e^{-\lambda s}T(t+s)u] = ms^{m-1}e^{-\lambda s}T(t+s)u - s^m e^{-\lambda s}T(t+s)(\lambda - A)u$.

Next we prove that $(\lambda - A)^m$ is closable. Let $\{u_p\}$ be a sequence in $D(A^m)$ such that $u_p \to 0$ and $(\lambda - A)^m u_p \to v$. Setting $u = u_p$ in (3.3) and going to the limit $p \to \infty$, we obtain

$$\int_0^\infty s^{m-1}e^{-\lambda s}T(t+s)vds = 0\ , \quad t > 0\ .$$

This implies that $T(t)v = 0$ for $t > 0$. Therefore, $v = 0$ by condition (i). Thus, $(\lambda - A)^m$ is closable. q.e.d.

LEMMA 3.3. *Suppose that* $\{T(t)\}$ *satisfies conditions* (i) *and* (ii). *Let* $A = \bar{A}_0$, *and let* n *be the integral part of* $\alpha > 0$. *Then*

$$D(A^{n+1}) \subset \Sigma\ . \tag{3.4}$$

PROOF. Since $T(t) \in \mathscr{B}(X)$, it follows from (3.3) and condition (ii) that

$$T(t)\Big[n!\,u - \int_0^\infty s^n e^{-\lambda s}T(s)(\lambda - A)^{n+1}uds\Big] = 0\ , \quad t > 0\ .$$

Therefore, by condition (i) we obtain

$$n!\,u = \int_0^\infty s^n e^{-\lambda s}T(s)(\lambda - A)^{n+1}uds\ , \quad u \in D(A^{n+1})\ , \quad \operatorname{Re}\lambda > \omega_0\ .$$

Let $\omega > \omega_0$. Then there is a constant $M_1 > 0$ such that $\| T(t) \| \leqq M_1 e^{\omega t}$ for $t \geqq 1$. Therefore, by condition (ii) we can find a constant $M > 0$ such that

$$\| t^\alpha T(t) \| \leqq Me^{\omega t}\ , \quad t > 0\ . \tag{3.5}$$

Let $b > \omega$ and $0 < t \leqq 1$. Then

$$n!\,T(t)u = \int_0^\infty s^n e^{-bs}T(t+s)(b - A)^{n+1}uds\ .$$

Since $s^n e^{-bs}\| T(t+s) \| \leqq$ const. $s^{n-\alpha}e^{-(b-\omega)s} \in L(0, \infty)$, we see by the principle of dominated convergence that $T(t)u \to u$ for $u \in D(A^{n+1})$ as $t \to 0+$. This shows that (3.4) holds. q.e.d.

The next lemma completes the proof of the "only if" part of Theorem 1.2.

LEMMA 3.4. *Let $\{T(t)\}$ be a semigroup of growth order α, and let $A = \bar{A}_0$. Then A satisfies conditions* (I)–(IV) *of Theorem* 1.2.

PROOF. It follows from Lemmas 3.1 and 3.3 that A is the complete infinitesimal generator of a semigroup of class $(\mathfrak{S}_{n+1})$, where n is the integral part of α. So, we see from Lemma 2.5 that A satisfies conditions (I) and (III). Also, it follows from Lemma 3.2 that A satisfies condition (IV). Therefore, it remains to show that A satisfies condition (II) for some $\omega > \omega_0$, where ω_0 is the type of $\{T(t)\}$. We have by (3.2) and (3.4) that for $v \in D(A^{n+1})$ and $\xi > \omega_0$,

$$(\xi - A)^{-m}v = [1/(m-1)!]\int_0^\infty t^{m-1}e^{-\xi t}T(t)v\,dt\,, \qquad m \geqq 1\,.$$

Hence, in virtue of (3.5), condition (II) can be easily verified. q.e.d.

The "only if" part of Corollary 1.3 follows from Lemma 3.4.

4. Construction of the semigroups. In this section we shall prove the "if" part of Theorem 1.2. Obviously, it suffices to consider the case of $\omega = 0$.

LEMMA 4.1. *Let A be a closed linear operator in X satisfying conditions* (I)–(III) *with $\omega = 0$. Then for each $\xi > 0$ there exists $S(\xi, A) \in \mathscr{B}(X)$ such that*

(a) *$AS(\xi, A)u = S(\xi, A)Au$ for $u \in D(A)$,*

(b) *$S(\xi, A)(\xi - A)^{n+1}v = v$ for $v \in D(A^{n+1})$,*

(c) *$S(\xi, A)$ is invertible if and only if $(\xi - A)^{n+1}$ is closable.*

PROOF. Let $(\xi - A)^{-(n+1)} \mid D(A^{n+1})$ be the restriction of $(\xi - A)^{-(n+1)}$ to $D(A^{n+1})$. Since $D(A^{n+1})$ is dense in X (see condition (III)), it follows from condition (II) that $(\xi - A)^{-(n+1)} \mid D(A^{n+1})$ admits a closure $S(\xi, A)$ in $\mathscr{B}(X)$:

$$S(\xi, A) = \overline{(\xi - A)^{-(n+1)} \mid D(A^{n+1})}\,.$$

Next we shall show that $S(\xi, A)$ has the properties (a)–(c).

(a) Let $u \in D(A)$. Then, since $D(A^{n+2})$ is a core of A, there exists a sequence $\{u_p\}$ in $D(A^{n+2})$ such that $u_p \to u$ and $Au_p \to Au$. We have by the definition of $S(\xi, A)$ that

$$AS(\xi, A)u_p = A(\xi - A)^{-(n+1)}u_p = (\xi - A)^{-(n+1)}Au_p = S(\xi, A)Au_p\,.$$

Going to the limit $p \to \infty$, the desired equality follows from the closedness of A.

(b) Let $v \in D(A^{n+1})$. Then we have $v = (\xi - A)^{n+1}(\xi - A)^{-(n+1)}v =$

$(\xi - A)^{n+1}S(\xi, A)v$. On the other hand, by (a), $S(\xi, A)(\xi - A)^{n+1}v = (\xi - A)^{n+1}S(\xi, A)v$. Thus, we obtain the desired equality.

(c) First suppose that $(\xi - A)^{n+1}$ has the closure $\overline{(\xi - A)^{n+1}}$. Since $D(A^{n+1})$ is dense in X, for each $w \in X$ there exists a sequence $\{w_p\}$ in $D(A^{n+1})$ such that $w_p \to w$. Since $(\xi - A)^{n+1}S(\xi, A)w_p = w_p$, it follows that $\overline{(\xi - A)^{n+1}}S(\xi, A)w = w$. So, if $S(\xi, A)w = 0$ then $w = 0$. Thus, $S(\xi, A)$ is invertible.

Conversely, suppose that $S(\xi, A)$ is invertible. Let $\{v_p\}$ be a sequence in $D(A^{n+1})$ such that $v_p \to 0$ and $(\xi - A)^{n+1}v_p \to v$. Then we have by (b) that $S(\xi, A)(\xi - A)^{n+1}v_p = v_p$. Going to the limit $p \to \infty$, we obtain $S(\xi, A)v = 0$ and therefore $v = 0$. Consequently, $(\xi - A)^{n+1}$ is closable.

q.e.d.

REMARK 4.2. If $(\xi - A)^{n+1}$ has the closure $\overline{(\xi - A)^{n+1}}$, then

$$S(\xi, A) = [\overline{(\xi - A)^{n+1}}]^{-1}, \qquad \xi > 0 .$$

Also, for each integer $k \geqq 2$, $S^k(\xi, A) = [S(\xi, A)]^k$ is the closure of $(\xi - A)^{-k(n+1)} \mid D(A^{n+1})$. So, it follows from condition (II) that

$$\| S^k(\xi, A) \| \leqq \frac{M}{(m-1)!} \frac{\Gamma(m - \alpha)}{\xi^{m-\alpha}}, \quad \xi > 0, \quad m = k(n+1), \tag{4.1}$$

where $k = 1, 2, \cdots$. Since

$$\lim_{m\to\infty} \frac{(m-1)!\, m^{-\alpha}}{\Gamma(m-\alpha)} = \lim_{m\to\infty} \frac{m!\, m^{-\alpha}}{\Gamma(m+1-\alpha)} = 1 ,$$

we see that there exists a constant $M' > 0$ such that

$$\xi^{k(n+1)} \| S^k(\xi, A) \| \leqq M' \left[\frac{\xi}{k(n+1)} \right]^{\alpha}, \quad \xi > 0, \quad k = 1, 2, \cdots . \tag{4.2}$$

LEMMA 4.3. *Let A be as in Lemma* 4.1. *Then for each integer $m \geqq 2$, $D(A^m)$ is a core of A.*

PROOF. Let k be a positive integer. Since $D(A^{n+2})$ is a core of A, it suffices to show that if $D(A^{k+n+1})$ is a core of A then so is $D(A^{k+2(n+1)})$. To see this, suppose that $D(A^{k+n+1})$ is a core of A. For each $u \in D(A^{k+n+1})$ we shall construct a sequence $\{u_p\}$ in $D(A^{k+2(n+1)})$ such that $u_p \to u$ and $Au_p \to Au$.

Let $b > 0$. Then each $u \in D(A^{k+n+1})$ can be written as $u = S(b, A)v$ for some $v \in D(A^k)$. In fact, set $v = (b - A)^{n+1}u \in D(A^k)$. Then it follows from Lemma 4.1(b) that $S(b, A)v = S(b, A)(b - A)^{n+1}u = u$. Since $D(A^{k+n+1})$ is a core of A, for each $v \in D(A^k)$ there exists a sequence $\{v_p\}$ in $D(A^{k+n+1})$ such that $v_p \to v$ and $Av_p \to Av$. Setting $u_p = (b - A)^{-(n+1)}v_p \in D(A^{k+2(n+1)})$, we see that $u_p = S(b, A)v_p \to S(b, A)v = u \in D(A^{k+n+1})$ and

$$Au_p = S(b, A)Av_p \to S(b, A)Av = Au ,$$

where we have used Lemma 4.1(a). q.e.d.

Since $D(A)$ is dense in X, it follows from Lemma 4.3 that for each integer $m \geqq 2$, $D(A^m)$ is dense in X.

LEMMA 4.4. *Let $S(\xi, A)$ be as in Lemma 4.1. Set*

$$A_i = (n+1)^{-1}[i^{n+2}S(i, A) - i] \in \mathscr{B}(X) , \qquad i = 1, 2, \cdots ,$$

and $T_i(t) = e^{tA_i}$. Then we have

$$\| T_i(t) \| \leqq 1 + Nt^{-\alpha} , \qquad t > 0 , \tag{4.3}$$

where $N = 2M'(n+2)!$, and

$$(\xi - A_i)^{-(n+1)} = (1/n!)\int_0^\infty e^{-\xi t} t^n T_i(t)\, dt , \qquad \xi > 0 . \tag{4.4}$$

Hence for each $\xi > 0$, $\| (\xi - A_i)^{-(n+1)} \|$ is bounded as i tends to infinity. Furthermore,

$$AT_i(t)u = T_i(t)Au \text{ for } u \in D(A) \text{ and } t > 0 . \tag{4.5}$$

PROOF. It suffices to prove (4.3). Let $t > 0$. Then we have

$$T_i(t) = \exp\left(\frac{-it}{n+1}\right) \sum_{k=0}^{\infty} \frac{t^k i^{k(n+2)}}{k!\,(n+1)^k} S^k(i, A) .$$

So, it follows from (4.2) that

$$\begin{aligned} \| T_i(t) \| &\leqq \exp\left(\frac{-it}{n+1}\right)\left(1 + \sum_{k=1}^{\infty}\left(\frac{it}{n+1}\right)^k \frac{M'}{k!}\left[\frac{i}{k(n+1)}\right]^{\alpha}\right) \\ &= \exp\left(\frac{-it}{n+1}\right)\left(1 + M't^{-\alpha} \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{it}{n+1}\right)^k \left[\frac{it}{k(n+1)}\right]^{\alpha}\right) . \end{aligned}$$

Let k_0 be the integral part of $it/(n+1)$. Then we have

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{it}{n+1}\right)^k \left[\frac{it}{k(n+1)}\right]^{\alpha} \\ &\leqq \sum_{k=1}^{k_0} \frac{1}{k!}\left(\frac{it}{n+1}\right)^k \left[\frac{it}{k(n+1)}\right]^{n+1} + \sum_{k=k_0+1}^{\infty} \frac{1}{k!}\left(\frac{it}{n+1}\right)^k . \end{aligned}$$

Since $(k+n+1)! \leqq (n+2)!\,k!\,k^{n+1}$ for $k \geqq 1$, we obtain $S \leqq (n+2)!\,2\exp[it/(n+1)]$. q.e.d.

In the rest of this section, we assume for simplicity that α is an integer: $\alpha = n$, since the proof for the case of $\alpha \neq n$ is essentially the same.

LEMMA 4.5. *Let $S(i, A)$ be as in Lemma 4.4. Then there exists a constant $K > 0$ such that*

$$\| i^k S(i, A)u \| \leqq K i^{-1} \| u \|_{n+k}, \quad k = 1, 2, \cdots, n, \tag{4.6}$$

where $\| u \|_m = \| u \| + \| Au \| + \cdots + \| A^m u \|$ for $u \in D(A^m)$.

PROOF. We shall prove (4.6) by induction. It follows from Lemma 4.1(b) that for $u \in D(A^{n+1})$,

$$\begin{aligned} i^{n+1}S(i, A)u - u &= S(i, A)[i^{n+1} - (i - A)^{n+1}]u \\ &= \sum_{p=1}^{n+1} {}_{n+1}C_p(-1)^{p+1} i^{n+1-p} S(i, A)A^p u . \end{aligned} \tag{4.7}$$

Dividing the both sides of (4.7) by i^n, we obtain

$$iS(i, A)u = i^{-n}u + \sum_{p=1}^{n+1} {}_{n+1}C_p(-1)^{p+1} i^{1-p} S(i, A)A^p u .$$

So, it follows from (4.1) that

$$\| iS(i, A)u \| \leqq i^{-n} \| u \| + (M/n!) \sum_{p=1}^{n+1} {}_{n+1}C_p i^{-p} \| A^p u \| .$$

Hence we obtain (4.6) with $k = 1$.

Now let $2 \leqq m \leqq n - 1$ and suppose that (4.6) holds for each $k \leqq m$. Dividing (4.7) by i^{n-m}, we have

$$i^{m+1}S(i, A)u = i^{-(n-m)}u + \sum_{p=1}^{n+1} {}_{n+1}C_p(-1)^{p+1} i^{m+1-p} S(i, A)A^p u .$$

Consequently, we see that for $u \in D(A^{n+m+1})$,

$$\begin{aligned} \| i^{m+1}S(i, A)u \| \leqq i^{-(n-m)} \| u \| &+ \text{const.} \sum_{p=1}^{m} {}_{n+1}C_p i^{-1} \| A^p u \|_{n+m+1-p} \\ &+ (M/n!) \sum_{p=m+1}^{n+1} {}_{n+1}C_p i^{m-p} \| A^p u \| . \end{aligned}$$

Thus, we obtain (4.6) with $k = m + 1$. q.e.d.

The next lemma shows that the sequence $\{A_i\}$ approximates A in a certain sense.

LEMMA 4.6. *Let A_i be as in Lemma 4.4. Then there exists a constant $L > 0$ such that for $u \in D(A^{2(n+1)})$,*

$$\| (A_i - A)u \| \leqq L i^{-1} \| u \|_{2(n+1)}, \quad i = 1, 2, \cdots; \tag{4.8}$$

hence we obtain

$$\| (A_i - A_j)u \| \leqq L(1/i + 1/j) \| u \|_{2(n+1)}, \quad i, j = 1, 2, \cdots, \tag{4.9}$$

$$\| A_i u \| \leqq (L + 1) \| u \|_{2(n+1)}, \quad i = 1, 2, \cdots . \tag{4.10}$$

PROOF. We see from (4.7) that for $u \in D(A^{n+1})$,

$$i^{n+2}S(i, A)u - iu = (n+1)i^{n+1}S(i, A)Au + \sum_{k=0}^{n-1} {}_{n+1}C_k(-1)^{n-k}i^{k+1}S(i, A)A^{n+1-k}u$$
$$= (n+1)\left[Au + \sum_{k=0}^{n} {}_{n+1}C_k(-1)^{n-k}i^k S(i, A)A^{n+2-k}u\right]$$
$$+ \sum_{k=1}^{n} {}_{n+1}C_{k-1}(-1)^{n-k+1}i^k S(i, A)A^{n+2-k}u .$$

Since $A_i = (n+1)^{-1}[i^{n+2}S(i, A) - i]$, it follows from (4.6) that

$$\|(A_i - A)u\| \leqq \text{const.} \sum_{k=0}^{n} {}_{n+1}C_k i^{-1}\|A^{n+2-k}u\|_{n+k}$$
$$+ \text{const.} \sum_{k=1}^{n} {}_{n+1}C_{k-1} i^{-1}\|A^{n+2-k}u\|_{n+k}$$
$$\leqq L i^{-1}\|u\|_{2(n+1)}. \qquad \text{q.e.d.}$$

LEMMA 4.7. *Let $T_i(t)$ be as in Lemma 4.4 and let $m = 2(n+1)^2$. Then for each $\varepsilon > 0$ there exists a constant $M_\varepsilon > 0$ such that for $u \in D(A^m)$ and $\varepsilon \leqq t \leqq 1/\varepsilon$,*

$$\|T_i(t)u - T_j(t)u\| \leqq M_\varepsilon(1/i + 1/j)\|u\|_m , \quad i, j = 1, 2, \cdots . \tag{4.11}$$

PROOF. Let $m = 2(n+1)^2$. Then it follows from (4.9) that there exists a constant $L > 0$ such that for $u \in D(A^m)$,

$$\|(A_i - A_j)^k u\| \leqq L(1/i + 1/j)\|u\|_m , \quad k = 1, 2, \cdots, n+1 . \tag{4.12}$$

Now we have the identity

$$T_i(t) - T_j(t) = T_i(t/2)T_j(t/2)\sum_{k=1}^{n}[1 + (-1)^{k-1}](t/2)^k(A_i - A_j)^k/k!$$
$$+ (t^{n+1}/n!)\int_0^{1/2} s^n[(-1)^n T_i(ts)T_j(t-ts) + T_j(ts)T_i(t-ts)]ds(A_i - A_j)^{n+1}$$

(see [1], Lemmas 4.3 and 4.4). Consequently, (4.11) follows from (4.3) and (4.12). q.e.d.

LEMMA 4.8. *For each $t > 0$ there exists $T(t) \in \mathscr{B}(X)$ given by*

$$T(t) = \text{strong} \lim_{i \to \infty} T_i(t) \tag{4.13}$$

such that

$$\|T(t)\| \leqq 1 + Nt^{-n} , \qquad t > 0 , \tag{4.14}$$

$$S(\xi, A) = (1/n!)\int_0^\infty e^{-\xi t}t^n T(t)dt , \qquad \xi > 0 , \tag{4.15}$$

$$AT(t)u = T(t)Au\,, \quad u \in D(A)\,, \quad t > 0\,. \tag{4.16}$$

PROOF. Let $m = 2(n+1)^2$. Then $D(A^m)$ is dense in X as noted after Lemma 4.3. Therefore, it follows from Lemma 4.7 and (4.3) that the limit (4.13) exists and (4.14) holds.

Now we prove (4.15). In view of (4.4), it suffices to show that

$$S(\xi, A) = \text{strong} \lim_{i\to\infty} (\xi - A_i)^{-(n+1)}\,, \qquad \xi > 0\,. \tag{4.17}$$

First we note that for $v \in D(A^{n+1})$,

$$\begin{aligned}(\xi - A_i)^{-(n+1)}v &- S(\xi, A)v \\ &= (\xi - A_i)^{-(n+1)}[(\xi - A)^{n+1} - (\xi - A_i)^{n+1}]S(\xi, A)v\,.\end{aligned}$$

Let $u \in D(A^m)$, $m = 2(n+1)^2$. Then it follows from (4.10) and (4.8) that $\|(\xi - A_i)^{n+1}u - (\xi - A)^{n+1}u\| \to 0$ as $i \to \infty$. Since $D(A^m)$ is dense in X, we obtain (4.17). Finally, (4.16) follows from (4.5) since A is closed.

q.e.d.

LEMMA 4.9. *Let A be a closed linear operator in X satisfying condition* (I)–(IV) *with $\omega = 0$. Then the family $\{T(t); t > 0\}$ of operators given by* (4.13) *forms a semigroup of growth order n.*

PROOF. First we note that for each $\varepsilon > 0$ the convergence (4.13) is uniform with respect to t on the interval $[\varepsilon, 1/\varepsilon]$. Since $T_i(t)$ is continuous in t, $T(t)$ is strongly continuous for $t > 0$. Also, the semigroup property of $\{T(t)\}$ follows from that of $\{T_i(t)\}$. Thus, $\{T(t)\}$ forms a semigroup on X.

Next we prove that $\{T(t)\}$ is of growth order n. To this end, let Σ be the continuity set of $\{T(t)\}$. Then we see from (4.15) and (4.14) that $S(\xi, A)u \in \Sigma$ for $u \in X$ and $\xi > 0$ (cf. Lemma 3.3). Now let $T(t)u = 0$ for $t > 0$ and let b be the real number in condition (IV). Then $T(t)S(b, A)u = S(b, A)T(t)u = 0$ for $t > 0$. But since $S(b, A)u \in \Sigma$, we have $S(b, A)u = 0$. Noting that $S(b, A)$ is invertible (see Lemma 4.1(c)), we obtain $u = 0$. Namely, $\{T(t)\}$ satisfies condition (i). (4.14) shows that condition (ii) is satisfied. Thus, it remains to show that X_0 is dense in X. Since X_0 is dense in Σ, it suffices to show that Σ is dense in X. Noting that $S(b, A) = [\overline{(b - A)^{n+1}}]^{-1}$ (see Remark 4.2), we have that

$$D(A^{n+1}) \subset \Sigma\,. \tag{4.18}$$

Consequently, Σ is dense in X since so is $D(A^{n+1})$. q.e.d.

LEMMA 4.10. *Let A and $\{T(t)\}$ be as in Lemma* 4.9, *and let Ω be as in* §2. *Then for $\xi > 0$,*

$$(\xi - A)^{-1}v = (\xi - \Omega)^{-1}v\,, \qquad v \in \Sigma\,. \tag{4.19}$$

PROOF. Since $(d/dt)T_i(t) = T_i(t)A_i$, we see from Lemmas 4.8 and 4.6 that for $u \in D(A^{2(n+1)})$,

$$T(t)u - T(\varepsilon)u = \int_\varepsilon^t T(s)Auds\ , \qquad t \geqq \varepsilon > 0\ . \tag{4.20}$$

Noting that $D(A^{2(n+1)})$ is a core of A, we see that (4.20) holds for all $u \in D(A)$. Consequently, we have $(d/dt)T(t)u = T(t)Au$ for $u \in D(A)$ and $t > 0$. Let $w \in D(A^{n+1})$. Then it follows from (4.16) and condition (I) that $T(s)w \in R(\xi - A)$ for $\xi > 0$. So we obtain

$$(d/dt)[e^{-\xi t}T(t)(\xi - A)^{-1}T(s)w] = -e^{-\xi t}T(t+s)w\ .$$

Since $(\xi - A)^{-1}T(s)w \in \Sigma$ (see (4.18)), it follows that

$$(\xi - A)^{-1}T(s)w = \int_0^\infty e^{-\xi t}T(t+s)wdt\ . \tag{4.21}$$

Now we have by (4.14) that

$$\|(\xi - A)^{-1}T(s)w\| \leqq (1 + Ns^{-n})\xi^{-1}\|w\|\ , \qquad w \in D(A^{n+1})\ .$$

Since $D(A^{n+1})$ is dense in X and since $(\xi - A)^{-1}T(s)$ is closed, we see that (4.21) holds for all $w \in X$. Let $v \in \Sigma$. Then we have $(\xi - A)^{-1}T(s)v = T(s)\int_0^\infty e^{-\xi t}T(t)vdt = T(s)(\xi - \Omega)^{-1}v$ (see (2.1); note that $\omega_0 \leqq 0$). Therefore, $(\xi - A)^{-1}T(s)v \to (\xi - \Omega)^{-1}v$ as $s \to 0+$. Since $T(s)v \to v$ as $s \to 0+$, (4.19) follows from the closedness of $(\xi - A)^{-1}$. q.e.d.

The following lemma completes the proof of the "if" part of Theorem 1.2.

LEMMA 4.11. *Let A and $\{T(t)\}$ be as in Lemma 4.9. Then A is equal to the complete infinitesimal generator of $\{T(t)\}$.*

PROOF. We see from (4.20) and (4.18) that

$$T(t)u - u = \int_0^t T(s)Auds\ , \quad u \in D(A^{n+2})\ , \quad t > 0\ .$$

Hence it follows that $A\,|\,D(A^{n+2}) \subset A_0$.

Now let Ω be as in Lemma 4.10. Then, since $R(\xi - \Omega) = \Sigma$ for $\xi > 0$ (see Lemma 2.2), (4.19) implies that $\Omega \subset A$. Therefore, Ω has the closure $\bar{\Omega}$ such that $\bar{\Omega} \subset A$. But since $\bar{\Omega} = \bar{A}_0$ (see Lemma 2.4), it follows that $A_0 \subset A$. Thus we have proved that

$$A\,|\,D(A^{n+2}) \subset A_0 \subset A\ .$$

Since $D(A^{n+2})$ is a core of A, this shows that $\bar{A}_0 = A$. q.e.d.

LEMMA 4.12. *Let A be a closed linear operator in X satisfying con-*

ditions (I′)-(III′) *of Corollary* 1.3. *Then A satisfies conditions* (I)-(IV) *with* $n = 0$.

PROOF. It suffices to show that $D(A^2)$ is a core of A. Namely, it suffices to show that for each $u \in D(A)$ there exists a sequence $\{u_p\}$ in $D(A^2)$ such that $u_p \to u$ and $Au_p \to Au$. To see this, let $u \in D(A)$ and $b > \omega$. Set $v = (b - A)u$. Since $D(A)$ is dense in X, there exists a sequence $\{v_p\}$ in $D(A)$ such that $v_p \to v$. Setting $u_p = (b - A)^{-1}v_p$, $\{u_p\}$ has the required property. q.e.d.

The "if" part of Corollary 1.3 follows from Lemma 4.12 and Theorem 1.2. Also, examining the proof of the generation theorem for semigroups of class (1, A), we can obtain another proof of Corollary 1.3 (see Miyadera [5]).

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