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ON MOD $\mathfrak{C}$ EXCISION THEOREMS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Excision theorems on fibration and cofibration have been proved by P. J. Hilton [4]. The same notion is studied by T. Ganea [3] and Y. Nomura [8].

Let $\mathfrak{C}$ be a class of finite abelian groups. The object of this paper is to show mod $\mathfrak{C}$ excision theorems on fibration and cofibration in the generalized homotopy theory (Theorem 1 and Theorem 2). And we obtain as a special case the general mod $\mathfrak{C}$ suspension theorem shown by B. S. Brown [1].

1. Preliminaries. Throughout this paper, all spaces considered are assumed to have the homotopy type of CW-complexes with base-points denoted by $*$; all maps and homotopies are assumed to preserve base-points.

PX is the space of paths in X emanating from $*$, and ΩX is the loop space. If $f: X \to Y$ is any map, C_f is the space obtained by attaching to Y the reduced cone over X by means of f. X is embedded in CX by $x \to (x, 1)$, and ΣX is the reduced suspension.

By applying the mapping track functor, any map $f: X \to Y$ is converted into a homotopy equivalent fibre map $p: E \to Y$, yielding the homotopy commutative diagram

$$\begin{array}{ccccc} E_f & \xrightarrow{\zeta_f} & X & \xrightarrow{f} & Y \\ \| & & \downarrow{\scriptstyle h} & & \| \\ E_f & \xrightarrow{i} & E & \xrightarrow{p} & Y, \end{array}$$

where $E = \{(x, \lambda) \in X \times Y^I \mid f(x) = \lambda(1)\}$, $p(x, \lambda) = \lambda(0)$, $E_f = \{(x, \lambda) \in X \times PY \mid f(x) = \lambda(1)\}$, $i =$ the inclusion map, $\zeta_f(x, \lambda) = x$, $h(x) = (x, \lambda_x)$ and $\lambda_x(t) = f(x)$ for $t \in I$. Then the sequence $E_f \xrightarrow{\zeta_f} X \xrightarrow{f} Y$ is called the extended fibration.

Dually, by applying the mapping cylinder functor, any map f is converted into a homotopy equivalent cofibre map $q: X \to M_f$, yielding the homotopy commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{q} & M_f & \xrightarrow{j} & C_f \\ \| & & \downarrow k & & \| \\ X & \xrightarrow{f} & Y & \xrightarrow{\eta_f} & C_f , \end{array}$$

where $M_f =$ the mapping cylinder of f, $q(x) = (x, 0)$, $\eta_f(y) = y$, $k(x, t) = f(x)$ for $(x, t) \in X \times I$ and $k(y) = y$ for $y \in Y$. Then the sequence $X \xrightarrow{f} Y \xrightarrow{\eta_f} C_f$ is called the extended cofibration.

Throughout this paper, we assume that all groups considered are finitely generated, $\mathfrak{C}$ denotes a Serre's class of finite abelian groups and that $\bar{\mathfrak{C}}$ is defined as in [1; p. 684]. Let G be a (finitely generated) abelian group. Then $G_{\mathfrak{C}}$ means the largest subgroup of G which is in $\mathfrak{C}$. A sequence $A \xrightarrow{f} B \xrightarrow{g} D$ of abelian groups and homomorphisms is said to be (mod $\mathfrak{C}$) exact if and only if $gf(A) \in \mathfrak{C}$ and $g^{-1}(D_{\mathfrak{C}})/f(A) \in \mathfrak{C}$. A homomorphism $f: A \to B$ is said to be (mod $\mathfrak{C}$) monomorphic if and only if $0 \to A \to B$ is (mod $\mathfrak{C}$) exact and to be (mod $\mathfrak{C}$) epimorphic if and only if $A \to B \to 0$ is (mod $\mathfrak{C}$) exact.

LEMMA 1.1. *Let $A \xrightarrow{f} B \xrightarrow{g} D$ be a sequence of abelian groups and homomorphisms such that $gf(A) \in \mathfrak{C}$. Then the condition $g^{-1}(D_{\mathfrak{C}})/f(A) \in \mathfrak{C}$ is equivalent to the condition $g^{-1}(gf(A))/f(A) \in \mathfrak{C}$.*

PROOF. It is obvious that the condition $g^{-1}(D_{\mathfrak{C}})/f(A) \in \mathfrak{C}$ means the condition $g^{-1}(gf(A))/f(A) \in \mathfrak{C}$. Since

$$\frac{g^{-1}(D_{\mathfrak{C}})/f(A)}{g^{-}(gf(A))/f(A)} \cong \frac{g^{-1}(D_{\mathfrak{C}})}{g^{-1}(gf(A))} \cong \frac{D_{\mathfrak{C}} \cap g(B)}{gf(A)} \in \mathfrak{C} ,$$

it follows immediately that the condition $g^{-1}(gf(A))/f(A) \in \mathfrak{C}$ means the condition $g^{-1}(D_{\mathfrak{C}})/f(A) \in \mathfrak{C}$.

COROLLARY 1.2. *A homomorphism $f: A \to B$ is* (mod $\mathfrak{C}$) *monomorphic if and only if Kernel f is in $\mathfrak{C}$ and is* (mod $\mathfrak{C}$) *epimorphic if and only if Cokernel f is in $\mathfrak{C}$.*

For the rest, we shall use notations due to Hilton [5].

2. The mod $\mathfrak{C}$ excision theorem on fibration. Let $F \xrightarrow{i} X \xrightarrow{f} Y$ be a fibration. We may consider, by [3; Proposition 1.6], the homotopy commutative diagram

$$\begin{array}{ccccccc} F & \xrightarrow{i} & X & \xrightarrow{\eta_i} & C_i & \xrightarrow{\sigma} & \Sigma F \\ & & \| & & \downarrow r & & \downarrow s \\ & & X & \xrightarrow{f} & Y & \xrightarrow{\eta_f} & C_f \end{array}$$

in which $r(x) = f(x)$, $r(y, t) = *$ for $x \in X$, $(y, t) \in CF$, $s(y, t) = (i(y), 1 - t)$ for $(y, t) \in \Sigma F$ and σ is the identification map.

PROPOSITION 2.1 [cf. 3; Proposition 2.1]. *Let $F \xrightarrow{i} X \xrightarrow{f} Y$ be a fibration in which X and Y are 1-connected and F is strongly simple* (*see* [10; p. 510]). *If $\pi_q(Y) \in \mathfrak{C}$ for $q < m$ and $\pi_q(F) \in \mathfrak{C}$ for $q < n$, then the induced homomorphisms*

$$r_\#: H_q(C_i) \longrightarrow H_q(Y) \quad \textit{and} \quad s_\#: H_q(\Sigma F) \longrightarrow H_q(C_f)$$

are (mod $\mathfrak{C}$) *monomorphic for $q < m + n$ and are* (mod $\mathfrak{C}$) *epimorphic for $q \leqq m + n$.*

PROOF. According to [10; 9.6. 18], loop space ΩY is strongly simple. Since $\pi_q(F) \in \mathfrak{C}$ for $q < n$ and $\pi_{q+1}(Y) \cong \pi_q(\Omega Y) \in \mathfrak{C}$ for $q < m - 1$, by using [10; 9.6. Theorem 20], we have $H_q(F) \in \mathfrak{C}$ for $q < n$ and $H_q(\Omega Y) \in \mathfrak{C}$ for $q < m - 1$. Let $F * \Omega Y$ denote the join of F and ΩY. Since

$$H_{t+1}(F * \Omega Y) \cong \sum_{p+q=t} H_p(F) \otimes H_q(\Omega Y) \oplus \sum_{p+q=t-1} \operatorname{Tor}(H_p(F), H_q(\Omega Y))$$

and all groups considered are finitely generated, it follows that $H_{t+1}(F * \Omega Y) \in \mathfrak{C}$ for $t < m + n - 1$. Hence we have $\pi_{t+1}(F * \Omega Y) \in \mathfrak{C}$ for $t < m + n - 1$. Now we consider the "fibration" $F * \Omega Y \xrightarrow{j} C_i \xrightarrow{r} Y$ (see [3; p. 298]). Then, by using the exact sequence

$$\longrightarrow \pi_{q+1}(C_i) \xrightarrow{r_*} \pi_{q+1}(Y) \longrightarrow \pi_q(F * \Omega Y) \xrightarrow{j_*} \pi_q(C_i) \longrightarrow ,$$

it follows that C_i is 1-connected and $r_*: \pi_q(C_i) \to \pi_q(Y)$ is (mod $\mathfrak{C}$) monomorphic for $q < m + n$ and is (mod $\mathfrak{C}$) epimorphic for $q \leqq m + n$. Hence, by the (mod $\mathfrak{C}$) Whitehead theorem [1; Theorem 4], $r_\#: H_q(C_i) \to H_q(Y)$ is (mod $\mathfrak{C}$) monomorphic for $q < m + n$ and is (mod $\mathfrak{C}$) epimorphic for $q \leqq m + n$.

Next, we shall prove that $s_\#$ has the same property. According to [3; Proposition 1.6], there exists a homotopy equivalence $\zeta: C_r \to C_s$ in the homotopy commutative diagram

$$\begin{array}{ccccc} C_i & \xrightarrow{r} & Y & \longrightarrow & C_r \\ \downarrow & & \downarrow \eta_f & & \downarrow \zeta \\ \Sigma F & \xrightarrow{s} & C_f & \longrightarrow & C_s . \end{array}$$

Then we have $H_q(r) \cong H_q(C_r) \cong H_q(C_s) \cong H_q(s)$. Since $H_q(r) \in \mathfrak{C}$ for $q \leqq m+n$, so is $H_q(s)$. That is, $s_\sharp: H_q(\Sigma F) \to H_q(C_f)$ is (mod $\mathfrak{C}$) monomorphic for $q < m+n$ and is (mod $\mathfrak{C}$) epimorphic for $q \leqq m+n$.

For a given abelian group G, let $K'(G, n)$ be a polyhedron with abelian fundamental group such that $H_i(K'(G, n)) = 0$ for $i \neq n$ and $H_n(K'(G, n)) = G$. Then $\Sigma K'(G, n-1)$ is a $K'(G, n)$. We define the nth homotopy group of B with coefficient in G by

$$\pi_n(G; B) = \pi(\Sigma K'(G, n-1), B) \qquad (n \geqq 2)\,.$$

LEMMA 2.2. *Let G be a (finitely generated) abelian group. If $\pi_q(B) \in \mathfrak{C}$ for $q = n,\ n+1$ $(n \geqq 2)$, then $\pi_n(G; B) \in \mathfrak{C}$ $(n \geqq 3)$ and $\pi_2(G; B) \in \bar{\mathfrak{C}}$.*

PROOF. Let $G = F/R$ be a representation of G as quotient of a finitely generated and free abelian group F by the subgroup R. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}\,(G, \pi_q(B)) \longrightarrow \mathrm{Hom}\,(F, \pi_q(B)) \longrightarrow \mathrm{Hom}\,(R, \pi_q(B))\,.$$

Since $\pi_q(B) \in \mathfrak{C}$ for $q = n,\ n+1$ $(n \geqq 2)$, $\mathrm{Hom}\,(F, \pi_q(B))$ and $\mathrm{Hom}\,(R, \pi_q(B))$ are in $\mathfrak{C}$. Hence $\mathrm{Hom}\,(G, \pi_n(B))$ and $\mathrm{Ext}\,(G, \pi_{n+1}(B))$ are also in $\mathfrak{C}$. By the universal coefficient theorem for homotopy groups [5], the sequence

$$0 \longrightarrow \mathrm{Ext}\,(G, \pi_{n+1}(B)) \longrightarrow \pi_n(G; B) \longrightarrow \mathrm{Hom}\,(G, \pi_n(B)) \longrightarrow 0$$

is exact. Therefore, it follows immediately that $\pi_n(G; B) \in \mathfrak{C}$ $(n \geqq 3)$ and $\pi_2(G; B) \in \bar{\mathfrak{C}}$.

LEMMA 2.3 [1; Lemma 8]. *If $G \in \mathfrak{C}$, then $\pi_n(G; B) \in \mathfrak{C}$ for $n \geqq 3$ and $\pi_2(G; B) \in \bar{\mathfrak{C}}$.*

LEMMA 2.4. *Let $f: X \to Y$ be a map with 1-connected spaces X and Y, and let B be a space such that $\pi_q(B) = 0$ for all sufficiently large q. Suppose that $f_\sharp: H_q(X) \to H_q(Y)$ is (mod $\mathfrak{C}$) monomorphic for $q < N$ and is (mod $\mathfrak{C}$) epimorphic for $q \leqq N$. Then $f^*: \pi(Y, \Omega^r B) \to \pi(X, \Omega^r B)$ $(r \geqq 2)$ is (mod $\mathfrak{C}$) monomorphic if $\pi_q(B) \in \mathfrak{C}$ for $q > N + r$ and is (mod $\mathfrak{C}$) epimorphic if $\pi_q(B) \in \mathfrak{C}$ for $q \geqq N + r$ (when $r = 1$, Kernel $f^* \in \bar{\mathfrak{C}}$ if $\pi_q(B) \in \mathfrak{C}$ for $q > N + 1$).*

PROOF. We can consider f to be a cofibration with the cofibre $C_f = F$. Then F is 1-connected and $H_q(F) \in \mathfrak{C}$ for $q \leqq N$. Hence we consider the Eckmann-Hilton decomposition of F (see [5]):

$$\begin{array}{l} F \\ \uparrow \\ \vdots \\ \uparrow \\ F_s \longrightarrow K'(H_s(F), s) \\ \uparrow \\ F_{s-1} \\ \uparrow \\ \vdots \\ \uparrow \\ F_2 = K'(H_2(F), 2)\,. \end{array}$$

Since $H_q(F) \in \mathfrak{C}$ for $q \leqq N$, it follows from Lemma 2.3 that $\pi(K'(H_s(F), s), \Omega^{r-1}B) \cong \pi_{s+r-1}(H_s(F); B) \in \mathfrak{C}$ for $2 \leqq s \leqq N$ and $r \geqq 2$. By Lemma 2.2, we have $\pi(K'(H_s(F), s), \Omega^{r-1}B) \in \mathfrak{C}$ for $s \geqq N+1$ if $\pi_q(B) \in \mathfrak{C}$ for $q \geqq N+r$. In the diagram above, we shall prove by induction on s that $\pi(F, \Omega^{r-1}B) \in \bar{\mathfrak{C}}$. It holds certainly that $\pi(F_2, \Omega^{r-1}B) = \pi(K'(H_2(F), 2), \Omega^{r-1}B) \in \mathfrak{C} \subset \bar{\mathfrak{C}}$. Assume that $\pi(F_{s-1}, \Omega^{r-1}B) \in \bar{\mathfrak{C}}$. In the exact sequence

$$\pi(K'(H_s(F), s), \Omega^{r-1}B) \longrightarrow \pi(F_s, \Omega^{r-1}B) \longrightarrow \pi(F_{s-1}, \Omega^{r-1}B)\,,$$

the two extreme groups are in $\bar{\mathfrak{C}}$. Thus $\pi(F_s, \Omega^{r-1}B) \in \bar{\mathfrak{C}}$. Since $\pi_q(B) = 0$ for all sufficiently large q, only a finite number of non-trivial extentions are required for building up to $\pi(F, \Omega^{r-1}B)$. Hence $\pi(F, \Omega^{r-1}B) \in \bar{\mathfrak{C}}$ if $\pi_q(B) \in \mathfrak{C}$ for $q \geqq N+r$.

We shall consider the commutative diagram

$$\begin{array}{ccc} \pi(\Sigma^2 Y, \Omega^{r-2}B) & \xrightarrow{(\Sigma^2 f)^*} & \pi(\Sigma^2 X, \Omega^{r-2}B) \\ \uparrow\cong & & \uparrow\cong \\ \pi(Y, \Omega^r B) & \xrightarrow{f^*} & \pi(X, \Omega^r B) \end{array}$$

and the exact sequence

$$\pi(\Sigma^2 F, \Omega^{r-2}B) \longrightarrow \pi(\Sigma^2 Y, \Omega^{r-2}B) \xrightarrow{(\Sigma^2 f)^*} \pi(\Sigma^2 X, \Omega^{r-2}B) \longrightarrow \pi(\Sigma F, \Omega^{r-2}B)\,.$$

Then Cokernel $(\Sigma^2 f)^*$ is in $\bar{\mathfrak{C}}$ and is an abelian group, hence it is in $\mathfrak{C}$. By the diagram above, Cokernel f^* is in $\mathfrak{C}$, that is, f^* is (mod $\mathfrak{C}$) epimorphic. If $\pi_q(B) \in \mathfrak{C}$ for $q > N+r$, by the same way, we have that f^* is (mod $\mathfrak{C}$) monomorphic ($r \geqq 2$) and Kernel f^* is in $\bar{\mathfrak{C}}$ ($r = 1$).

THEOREM 1. *Let* $F \xrightarrow{i} X \xrightarrow{f} Y$ *be a fibration in which* X *and* Y *are 1-connected and* F *is strongly simple, and let* B *be a space such that*

$\pi_q(B) = 0$ for all sufficiently large q. Suppose that $\pi_q(Y) \in \mathfrak{C}$ for $q < m$ and $\pi_q(F) \in \mathfrak{C}$ for $q < n$. Then the excision homomorphisms

$$\varepsilon_1\colon \pi_1(f, \Omega^r B) \longrightarrow \pi_1(F, \Omega^r B) \text{ and } \varepsilon_2\colon \pi(Y, \Omega^r B) \longrightarrow \pi_1(i, \Omega^r B) \quad (r \geqq 2)$$

are (mod $\mathfrak{C}$) *monomorphic if $\pi_q(B) \in \mathfrak{C}$ for $q > r + m + n$ and are* (mod $\mathfrak{C}$) *epimorphic if $\pi_q(B) \in \mathfrak{C}$ for $q \geqq r + m + n$ (when $r = 1$, Kernel $\varepsilon_i \in \bar{\mathfrak{C}}$, $i = 1, 2$, if $\pi_q(B) \in \mathfrak{C}$ for $q > m + n + 1$).*

PROOF. We may consider two commutative squares

$$\begin{array}{ccc} \pi_1(f, \Omega^r B) & \xrightarrow{\varepsilon_1} & \pi_1(F, \Omega^r B) \\ \cong\uparrow \varepsilon_f & & \cong\uparrow \kappa \\ \pi(C_f, \Omega^r B) & \xrightarrow{(-s)^*} & \pi(\Sigma F, \Omega^r B) \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi(Y, \Omega^r B) & \xrightarrow{\varepsilon_2} & \pi_1(i, \Omega^r B) \\ \| & & \cong\uparrow \varepsilon_i \\ \pi(Y, \Omega^r B) & \xrightarrow{r^*} & \pi(C_i, \Omega^r B) \end{array}$$

in which ε_f and ε_i are excision isomorphisms induced by extended cofibrations and κ is the natural equivalence in [4]. By Proposition 2.1, $s_\sharp\colon H_q(\Sigma F) \rightarrow H_q(C_f)$ and $r_\sharp\colon H_q(C_i) \rightarrow H_q(Y)$ are (mod $\mathfrak{C}$) monomorphic for $q < m + n$ and are (mod $\mathfrak{C}$) epimorphic for $q \leqq m + n$. Hence, by using Lemma 2.4, we obtain the desired results.

COROLLARY 2.5 (The general (mod $\mathfrak{C}$) loop theorem). *Let Y be 1-connected and $\pi_q(Y) \in \mathfrak{C}$ for $q < m$, and let B be a space such that $\pi_q(B) = 0$ for all sufficiently large q. Then the loop homomorphism $\Omega\colon \pi(Y, \Omega^r B) \rightarrow \pi(\Omega Y, \Omega^{r+1} B)$ is* (mod $\mathfrak{C}$) *monomorphic if $\pi_q(B) \in \mathfrak{C}$ for $q > r + 2m - 1$ and is* (mod $\mathfrak{C}$) *epimorphic if $\pi_q(B) \in \mathfrak{C}$ for $q \geqq r + 2m - 1$ (when $r = 1$, Kernel $\Omega \in \bar{\mathfrak{C}}$ if $\pi_q(B) \in \mathfrak{C}$ for $q > 2m$).*

PROOF. Consider the standard fibration $\Omega Y \xrightarrow{i} PY \xrightarrow{f} Y$. Since $\pi_q(\Omega Y) \cong \pi_{q+1}(Y) \in \mathfrak{C}$ for q $< m - 1$, $\varepsilon_2\colon \pi(Y, \Omega^r B) \rightarrow \pi_1(i, \Omega^r B)$ is (mod $\mathfrak{C}$) monomorphic if $\pi_q(B) \in \mathfrak{C}$ for $q > r + 2m - 1$ and is (mod $\mathfrak{C}$) epimorphic if $\pi_q(B) \in \mathfrak{C}$ for $q \geqq r + 2m - 1$. By using the commutative diagram

$$\begin{array}{ccc} \pi(Y, \Omega^r B) & \xrightarrow{-\Omega} & \pi(\Omega Y, \Omega^{r+1} B) \\ \| & & \cong\downarrow J \\ \pi(Y, \Omega^r B) & \xrightarrow{\varepsilon_2} & \pi_1(i, \Omega^r B)\ , \end{array}$$

we have that ε_2 and Ω are equivalent, which proves the Corollary 2.5.

3. The mod $\mathfrak{C}$ excision theorem on cofibration. We shall consider the dual cases stated in Section 2. Let $K(G, n)$ be Eilenberg-MacLane space whose ith homotopy group vanishes for $i \neq n$ and whose nth homotopy group is G.

LEMMA 3.1 [1; Lemma 11]. *If $G \in \mathfrak{C}$ and A is any space, then $\pi(A, K(G, n)) \in \mathfrak{C}$ $(n \geqq 1)$.*

PROPOSITION 3.2. *Let $f: X \to Y$ be a map with 1-connected spaces X and Y, and let A be a space such that $H_q(A) = 0$ for all sufficiently large q. Suppose that $f_*: \pi_q(X) \to \pi_q(Y)$ is (mod $\mathfrak{C}$) monomorphic for $q < N$ and is (mod $\mathfrak{C}$) epimorphic for $q \leqq N$. Then $f_*: \pi(\Sigma^r A, X) \to \pi(\Sigma^r A, Y)$ $(r \geqq 2)$ is (mod $\mathfrak{C}$) monomorphic if $H^q(A) \in \mathfrak{C}$ for $q \geqq N - r$ and is (mod $\mathfrak{C}$) epimorphic if $H^q(A) \in \mathfrak{C}$ for $q > N - r$ (when $r = 1$, Kernel $f_* \in \bar{\mathfrak{C}}$ if $H^q(A) \in \mathfrak{C}$ for $q \geqq N - 1$).*

PROOF. We take f to be a fibration with fibre $F = E_f$. Then F is 0-connected and $\pi_q(F) \in \mathfrak{C}$ for $q \leqq N - 1$. Hence we may consider the Postonikov system for F:

$$\begin{array}{l} F \\ \downarrow \\ \vdots \\ \downarrow \\ F_s \longleftarrow K(\pi_s(F), s) \\ \downarrow \\ F_{s-1} \\ \downarrow \\ \vdots \\ \downarrow \\ F_1 = K(\pi_1(F), 1)\ . \end{array}$$

Since $\pi_q(F) \in \mathfrak{C}$ for $q \leqq N - 1$, it follows from Lemma 3.1 that $\pi(\Sigma^{r-1}A, K(\pi_s(F), s)) \in \mathfrak{C}$ for $1 \leqq s \leqq N - 1$ and $r \geqq 2$. The other hand, we have $\pi(\Sigma^{r-1}A, K(\pi_s(F), s) \cong H^s(\Sigma^{r-1}A; \pi_s(F)) \cong H^{s-r+1}(A; \pi_s(F)) \cong H^{s-r+1}(A) \otimes \pi_s(F) \oplus \mathrm{Tor}\,(H^{s-r+2}(A), \pi_s(F))$ because all groups considered are finitely generated. Hence it follows that $\pi(\Sigma^{r-1}A, K(\pi_s(F), s)) \in \mathfrak{C}$ for $s \geqq N$ if $H^q(A) \in \mathfrak{C}$ for $q > N - r$. Assume that $\pi(\Sigma^{r-1}A, F_{s-1}) \in \bar{\mathfrak{C}}$ $(s - 1 \geqq 1)$. In the exact sequence $\pi(\Sigma^{r-1}A, K(\pi_s(F), s)) \to \pi(\Sigma^{r-1}A, F_s) \to \pi(\Sigma^{r-1}A, F_{s-1})$, two extreme groups are in $\bar{\mathfrak{C}}$. Thus $\pi(\Sigma^{r-1}A, F_s) \in \bar{\mathfrak{C}}$. Since $H_q(A) = 0$ for all sufficiently large q, it follows from the universal coefficient theorem for cohomology groups that $\pi(\Sigma^{r-1}A, F_s) \cong \pi(\Sigma^{r-1}A, F_{s+1})$ for all sufficiently large s. Hence we have $\pi(\Sigma^{r-1}A, F) \in \bar{\mathfrak{C}}$. By the exact sequence

$$\pi(\Sigma^{r-2}A, \Omega^2 F) \longrightarrow \pi(\Sigma^{r-2}A, \Omega^2 X) \xrightarrow{(\Omega^2 f)_*} \pi(\Sigma^{r-2}A, \Omega^2 Y) \longrightarrow \pi(\Sigma^{r-2}A, \Omega F)\ ,$$

we have Cokernel $(\Omega^2 f)_* \in \mathfrak{C}$, and hence Cokernel $f_* \in \mathfrak{C}$. This implies that f_* is (mod $\mathfrak{C}$) epimorphic. If $H^q(A) \in \mathfrak{C}$ for $q \geqq N - r$, by the same way, we obtain that f_* is (mod $\mathfrak{C}$) monomorphic ($r \geqq 2$) and Kernel f_* is in $\bar{\mathfrak{C}}$ ($r = 1$).

Let X be a space with two distinguished subspaces A and B such that $C = A \cap B \ni *$. Consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{j_1} & A \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\ B & \xrightarrow{j_2} & X, \end{array} \tag{3.3}$$

where each map is inclusion.

PROPOSITION 3.4 [cf. 7; Theorem 1.1]. *In* (3.3), *suppose that*

(1) *X, A, B, and C are* 1-*connected,*

(2) *$\pi_q(i_2) \in \mathfrak{C}$ for $q < m$,*

(3) *$\pi_q(j_2) \in \mathfrak{C}$ for $q < n$, and*

(4) *$(j_1, j_2)_\#: H_q(i_1) \to H_q(i_2)$ is* (mod $\mathfrak{C}$) *monomorphic for $q < m + n - 2$ and is* (mod $\mathfrak{C}$) *epimorphic for $q \leqq m + n - 2$.*

Then $(j_1, j_2)_*: \pi_q(i_1) \to \pi_q(i_2)$ is (mod $\mathfrak{C}$) *monomorphic for $q < m + n - 2$ and is* (mod $\mathfrak{C}$) *epimorphic for $q \leqq m + n - 2$.*

COROLLARY 3.5. *In the following commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle i} \\ CX & \xrightarrow{j} & C_f \end{array}$$

where each map is the inclusion, suppose that X and Y are 1-*connected and that $\pi_q(X) \in \mathfrak{C}$ for $q < m$ and $\pi_q(f) \in \mathfrak{C}$ for $q < n$. Then $(f, j)_*$: $\pi_q(\iota) \to \pi_q(i)$ is* (mod $\mathfrak{C}$) *monomorphic for $q < m + n - 1$ and is* (mod $\mathfrak{C}$) *epimorphic for $q \leqq m + n - 1$.*

PROOF. Consider the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\iota} & CX & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle 1} \\ Y & \xrightarrow{i} & C_f & \longrightarrow & \Sigma X. \end{array}$$

Since ι and i are cofibrations, it follows that $H_q(j) \cong H_q(f)$ and $H_q(i) \cong H_q(\iota)$ for all q. Hence, by the assumption and the generalized relative Hurewicz isomorphism theorem [10], we have that $\pi_q(j) \in \mathfrak{C}$ for $q < n$

and $\pi_q(i) \in \mathfrak{C}$ for $q < m + 1$. Furthermore, one can easily verify that C_f is 1-connected. Thus, according to Proposition 3.4, the required results hold.

COROLLARY 3.6. *Let $X \xrightarrow{f} Y \xrightarrow{p} F$ be an inclusion cofibration with cofibre F and let X and Y be 1-connected. If $\pi_q(X) \in \mathfrak{C}$ for $q < m$ and $\pi_q(f) \in \mathfrak{C}$ for $q < n$, then excision homomorphisms $\varepsilon_1' = (*, p)_*: \pi_q(f) \to \pi_q(F)$ and $\varepsilon_2': \pi_{q-1}(X) \to \pi_q(p)$ are* (mod $\mathfrak{C}$) *monomorphic for $q < m + n - 1$ and are* (mod $\mathfrak{C}$) *epimorphic for $q \leqq m + n - 1$.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{p} & F \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle i} & & \| \\ CX & \xrightarrow{j} & C_f & \xrightarrow{h} & F . \end{array}$$

Let Φ be the map $(f, j): \iota \to i$. Then the results of Corollary 3.5 implies $\pi_q(\Phi) \in \mathfrak{C}$ for $q \leqq m + n - 1$, and we have $\pi_q(\Phi) \cong \pi_q(\Phi^T)$ (see [2; Proposition 7.4]). Thus $\pi_q(\Phi^T) \in \mathfrak{C}$ for $q \leqq m + n - 1$, that is, $(\iota, i)_*: \pi_q(f) \to \pi_q(j)$ is (mod $\mathfrak{C}$) monomorphic for $q < m + n - 1$ and is (mod $\mathfrak{C}$) epimorphic for $q \leqq m + n - 1$. Moreover, since $(*, h): (CX, C_f) \to (*, F)$ is a homotopy equivalence, we have $(*, h)_*: \pi_q(j) \to \pi_q(F)$ for all q. Hence, by the commutative diagram

$$\begin{array}{ccc} \pi_q(f) & \xrightarrow{(\iota, i)_*} & \pi_q(j) \\ \| & & \downarrow{\scriptstyle (*, h)_*} \\ \pi_q(f) & \xrightarrow{\varepsilon_1'} & \pi_q(F) , \end{array}$$

ε_1' has the required property. Furthermore, it follows from the (mod $\mathfrak{C}$) five lemma (see [1]) that ε_2' has also the same property.

Let $X \xrightarrow{f} Y \xrightarrow{p} F$ be a cofibration. We shall consider the homotopy commutative diagram (see [3; Lemma 3.1])

$$\begin{array}{ccccccc} E_f & \xrightarrow{\zeta_f} & X & \xrightarrow{f} & Y & & \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle 1} & & \\ \Omega F & \xrightarrow{\mu} & E_p & \xrightarrow{\zeta_p} & Y & \xrightarrow{p} & F \end{array}$$

in which $d(x, \eta)(t) = p \cdot \eta(1 - t)$ for $(x, \eta) \in E_f$, $e(x) = (f(x), *)$ for $x \in X$ and $\mu(\nu) = (*, \nu)$ for $\nu \in \Omega F$.

LEMMA 3.7. *Under the assumptions of Corollary* 3.6, *the induced homomorphisms $e_*: \pi_q(X) \to \pi_q(E_p)$ and $d_*: \pi_q(E_f) \to \pi_q(\Omega F)$ are* (mod $\mathfrak{C}$) *monomorphic for $q < m + n - 2$ and are* (mod $\mathfrak{C}$) *epimorphic for $q \leqq m + n - 2$.*

PROOF. We can see easily that two squares shown under are both commutative:

$$(3.8)\qquad \begin{array}{ccc} \pi_q(X) & \xrightarrow{e_*} & \pi_q(E_p) \\ \| & & \cong\big\downarrow \varepsilon_p' \\ \pi_q(X) & \xrightarrow{\varepsilon_2'} & \pi_{q+1}(p) \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_q(E_f) & \xrightarrow{(-d)_*} & \pi_q(\Omega F) \\ \cong\big\downarrow \varepsilon_f' & & \cong\big\downarrow \kappa \\ \pi_{q+1}(f) & \xrightarrow{\varepsilon_1'} & \pi_{q+1}(F) \end{array}$$

in which ε_p' and ε_f' are excision isomorphisms induced by extended fibration and κ is the natural equivalence in [4]. Then the required results are equivalent to that ε_1' and ε_2' have the same properties, which follows from Corollary 3.6.

THEOREM 2. *Let $X \xrightarrow{f} Y \xrightarrow{p} F$ be a cofibration in which X and Y are 1-connected and $\pi_2(f) = 0$. Let A be a space such that $H_q(A) = 0$ for all sufficiently large q. Suppose that $\pi_q(X) \in \mathfrak{C}$ for $q < m$ and $\pi_q(F) \in \mathfrak{C}$ for $q < n$. Then the excision homomorphisms $\varepsilon_1': \pi_1(\Sigma^r A, f) \to \pi_1(\Sigma^r A, F)$ and $\varepsilon_2': \pi(\Sigma^r A, X) \to \pi_1(\Sigma^r A, p)$ $(r \geqq 2)$ are* (mod $\mathfrak{C}$) *monomorphic if $H^q(A) \in \mathfrak{C}$ for $q \geqq m + n - r - 2$ and are* (mod $\mathfrak{C}$) *epimorphic if $H^q(A) \in \mathfrak{C}$ for $q > m + n - r - 2$ (when $r = 1$, Kernel $\varepsilon_i' \in \bar{\mathfrak{C}}$ for $i = 1, 2$ if $H^q(A) \in \mathfrak{C}$ for $q \geqq m + n - 3$).*

PROOF. The cofibration may replace by an inclusion cofibration, hence we assume that f is an inclusion map with $F = Y/X$. Then we can see as in (3.8) that ε_1' and ε_2' are equivalent to $d_*: \pi(\Sigma^r A, E_f) \to \pi(\Sigma^r A, \Omega F)$ and $e_*: \pi(\Sigma^r A, X) \to \pi(\Sigma^r A, E_p)$, respectively. Hence, by using Proposition 3.2 and Lemma 3.7, we obtain the desired results.

COROLLARY 3.9. (The general (mod $\mathfrak{C}$) suspension theorem). *Let X be 1-connected and $\pi_q(X) \in \mathfrak{C}$ for $q < m$, and let A be a space such that $H_q(A) = 0$ for all sufficiently large q. Then the suspension homomorphism $\Sigma: \pi(\Sigma^r A, X) \to \pi(\Sigma^{r+1} A, \Sigma X)$ $(r \geqq 2)$ is* (mod $\mathfrak{C}$) *monomorphic if $H^q(A) \in \mathfrak{C}$ for $q \geqq 2m - r - 1$ and is* (mod $\mathfrak{C}$) *epimorphic if $H^q(A) \in \mathfrak{C}$ for $q > 2m - r - 1$ (when $r = 1$, Kernel $\Sigma \in \bar{\mathfrak{C}}$ if $H^q(A) \in \mathfrak{C}$ for $q \geqq 2m - 2$).*

PROOF. Consider the standard cofibration $X \xrightarrow{f} CX \xrightarrow{p} \Sigma X$ and the commutative diagram

$$\begin{array}{ccc} \pi(\Sigma^r A, X) & \xrightarrow{-\Sigma} & \pi(\Sigma^{r+1} A, \Sigma X) \\ \| & & \cong\big\downarrow J \\ \pi(\Sigma^r A, X) & \xrightarrow{\varepsilon_2'} & \pi_1(\Sigma^r A, p)\,. \end{array}$$

Since $\pi_q(\Sigma X) \in \mathfrak{C}$ for $q < m + 1$, by the Theorem 2, we obtain the desired results.

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