THE PRODUCT OF OPERATORS WITH CLOSED RANGE

Richard Bouldin

Abstract. By definition the cosine of the angle between the two subspaces M and N is $\sup\{|\langle f, g\rangle|: f \in M,\ g \in N,\ ||f|| = 1 = ||g||\}$. Assuming that A and B are both Hilbert space operators with closed range, AB has closed range if and only if the angle between BH and $\ker A \cap [\ker A \cap BH]^\perp$ is positive.

In this paper we consider the problem of when the product of two operators with closed range has closed range. Although some sufficient conditions have been determined, without necessary conditions it is impossible to evaluate the strength of the known sufficient conditions. In this paper we find a simple geometric condition which is both necessary and sufficient. Our proof is more geometric than the rather algebraic abstract arguments of index theory that have been used to obtain previous sufficient conditions. We offer some applications of our result.

By "operator" we mean a bounded linear transformation of the complex Hilbert space H into itself. We shall denote the kernel or nullspace of an operator T by $\ker T$ and the orthogonal complement of a subspace M is denoted $M^\perp$. We shall use the elementary fact that a linear transformation T has closed range if and only if there is a positive δ such that $|| Tf || \geq \delta ||f||$ whenever $f \in (\ker T)^\perp$. Essential to our work is the notion of angle between two subspaces, which seems to originate in [3]. The angle between M and N ranges from 0 to $\pi/2$ and its cosine is $\sup\{|\langle f, g\rangle|: f \in M,\ g \in N,\ ||f|| = 1 = ||g||\}$. If either M or N is trivial then the angle is $\pi/2$.

LEMMA. *Let $H_0 = \ker A \cap BH$ and let $H_1 = \ker A \cap (H_0)^\perp$. If $H_2 = (\ker A)^\perp$ then $H = H_0 \oplus H_1 \oplus H_2$. If P and Q are the orthogonal projections onto H_2 and $(H_1)^\perp$, respectively, then $QBH = H_0 \oplus PBH$. Here each direct sum is an orthogonal direct sum.*

PROOF. Let $H_3 = BH \cap (H_0)^\perp$ and note that H_3 is contained in $H_1 \oplus H_2$. In order to show that QH_3 is contained in $H_1 \oplus H_2$ we take $f \in H_3$, $g \in H_0$ and we note that

$$\langle g, Qf \rangle = \langle Qg, f \rangle = \langle g, f \rangle = 0 \ .$$

We also note that $QH_3 \subset QBH$. Since $H_0 \subset BH$ and $H_0 \subset QH$, we have that $H_0 \subset QBH$. Thus $QBH \supset H_0 \oplus QH_3$. Since Q is the identity on QH and $H_0 \subset QH$, it follows that $QH_0 = H_0$. Note that

$$H_0 \oplus QH_3 = QH_0 \oplus QH_3 \supset Q(H_0 \oplus H_3) = QBH$$

and consequently $QBH = H_0 \oplus QH_3$.

It is easy to see that $PBH = P[H_0 \oplus H_3] = PH_3$ and it only remains to show that $QH_3 = PH_3$. This follows form the facts that H_1 is contained in both $\ker P$ and $\ker Q$, that H_2 is contained in both PH and QH, and that H_3 is contained in $H_1 \oplus H_2$.

THEOREM. *Let A and B be operators with closed range and let $H_1 = \ker A \cap (\ker A \cap BH)^\perp$. The angle between H_1 and BH is positive if and only if ABH is closed.*

PROOF. First we show that the angle between H_1 and BH is positive if and only if QBH is closed. Regarding Q as a linear transformation of BH into H, we note that $\ker Q = H_1 \cap BH$ and QBH is closed if and only if Q is bounded away from zero on the unit shell of $BH \cap (H_1 \cap BH)^\perp$. If QBH is not closed then there is a sequence of unit vectors $\{f_n\}$ contained in $BH \cap (H_1 \cap BH)^\perp$ such that $\{\|Qf_n\|\}$ converges to zero, or $\{\|(I - Q)f_n\|\}$ converges to one. Let g_n be $(I - Q)f_n$ divided by its norm and note that $g_n \in H_1$, $f_n \in BH$. Since

$$\begin{aligned}
\langle g_n, f_n \rangle &= \langle (I - Q)f_n/\|(I - Q)f_n\|, f_n \rangle \\
&= \langle (I - Q)^2 f_n, f_n \rangle/\|(I - Q)f_n\| = \langle (I - Q)f_n, (I - Q)f_n \rangle/\|(I - Q)f_n\| \\
&= \|(I - Q)f_n\| \ ,
\end{aligned}$$

we see that $\{|\langle g_n, f_n \rangle|\}$ converges to one and consequently the angle between H_1 and BH is zero.

On the other hand assume there are two sequences of unit vectors $\{g_n\} \subset H_1$ and $\{Bf_n\}$, with $\{f_n\} \subset (\ker B)^\perp$, such that $\{|\langle g_n, Bf_n \rangle|\}$ converges to one. Since

$$\begin{aligned}
|\langle g_n, Bf_n \rangle| &= |\langle (I - Q)g_n, Bf_n \rangle| = |\langle g_n, (I - Q)Bf_n \rangle| \\
&\leqq \|(I - Q)Bf_n\| \leqq 1 \ ,
\end{aligned}$$

we see that $\{\|I - Q)Bf_n\|\}$ converges to one and so $\{\|QBf_n\|\}$ converges to zero. Since each Bf_n is a unit vector the sequence $\{\|f_n\|\}$ must be bounded away from zero, and thus we may assume that $\{f_n\}$ is a sequence of unit vectors and $\{QBf_n\}$ converges to zero. If we show that $\ker B = \ker QB$ then $\{f_n\} \subset (\ker QB)^\perp$ and it follows that QBH is not closed.

Clearly $\ker B \subset \ker QB$; if $QBf = 0$ while $Bf \neq 0$ then it would follow that $\ker Q \cap BH \neq \{0\}$. However, $\ker Q \cap BH = H_1 \cap BH = \{0\}$ and thus $\ker B = \ker QB$.

The preceding lemma makes it clear that QBH is closed if and only if PBH is closed. Thus it suffices to prove that PBH is closed if and only if ABH is closed. Since A is bounded away from zero on the unit shell of PH, if PBH is closed then $A(PBH) = ABH$ is closed. To prove the converse we note that the preceding lemma implies that $(PH)^\perp = \ker A$ and this implies $\ker PB = \ker AB$. If PBH is not closed then there is a sequence of unit vectors $\{f_n\} \subset (\ker PB)^\perp$ such that $\{PBf_n\}$ converges to zero. By continuity $\{APBf_n\}$ converges to zero, and since $APB = AB$, this proves that AB is not bounded away from zero on the unit shell of $(\ker AB)^\perp$. Hence ABH is not closed.

An immediate corollary of the above theorem is the following.

COROLLARY. *Let A and B be operators with closed range such that $\ker A \cap BH = \{0\}$. The operator AB has closed range if and only if the angle between $\ker A$ and BH is positive.*

The following useful corollary has been proved by Goldberg in [6] using the index theory developed in [5].

COROLLARY. *Let A and B be operators with closed range. If $\ker A$ is finite dimensional then AB has closed range.*

PROOF. Since H_1 is finite dimesional, its unit shell is compact and it follows easily that the angle between H_1 and BH can be zero only if they intersect. By the construction of H_1 they do not intersect.

The theorem allows us to give a quick proof of Banach's closed range theorem in the Hilbert space case.

COROLLARY. *The range of T is closed if and only if the range of T^* is closed.*

PROOF. If UR is the polar factorization of T then $\ker U = (RH)^\perp$ and thus if TH is not closed it must be that RH is not closed. Since $R = (T^*T)^{1/2}$ it follows that

$$T^*H = T^*(TH) = R(RH)$$

is not closed. Similarly if T^*H is not closed then $(T^*)^*H = TH$ is not closed.

If T is any operator on H then it induces a one-to-one linear transformation from $(\ker T)^\perp$ onto TH. Define T^+ to be that inverse on TH and to be zero on $(TH)^\perp$. T^+ is called the pseudo-inverse of T and it is

a consequence of the closed graph theorem that T^+ is bounded if and only if TH is closed. See [1] and [2] for relevant discussions. An open problem has been to determine necessary and sufficient conditions for $(AB)^+ = B^+A^+$ to hold. The following corollary is relevant to that problem.

COROLLARY. *If A^+ and B^+ are bounded then $(AB)^+$ is bounded if and only if the angle between* ker $A \cap (\ker A \cap BH)^\perp$ *and BH is positive.*

PROOF. This is immediate from the theorem.

Although it seems unreasonable that the product of two operators with closed range would necessarily have closed range, it seems much more likely that the square of an operator with closed range should have closed range. The following corollary terminates such optimism.

COROLLARY. *There exists an operator T with closed range such that T^2 does not have closed range.*

PROOF. We shall construct T to be a partial isometry; thus it is apparent that the range of T is closed. We shall construct the initial space M and the final space N such that $M^\perp$ has trivial intersection with N and the angle between $M^\perp$ and N is zero. Let $\{e_j, f_j, h_j : j = 1, 2, \cdots\}$ be an orthonormal basis for H. Let $\{a_n\}$ be a sequence of nonnegative numbers converging to zero such that $a_n \leq 1$ for each n. For each n let $b_n = (1 - a_n^2)^{1/2}$ and set $u_n = a_n e_n + b_n h_n$, $v_n = a_n f_n + b_n h_n$. Define $M^\perp$ to the closed span of $\{u_j\}$ and define N to be the closed span of $\{v_j\}$. It is straightforward to see that $M^\perp \cap N$ cannot contain a nonzero vector. The angle between $M^\perp$ and N is clearly zero since $\langle u_j, v_j \rangle = b_n^2 = 1 - a_n^2$. The theorem now implies that the range of T^2 is not closed.

REFERENCES

[1] F. J. BEUTLER, The operator theory of the pseudo-inverse, I, J. Math. Anal. Appl., 10 (1965), 451–470.

[2] F. J. BEUTLER, The operator theory of the pseudo-inverse, II, J. Math. Anal. Appl. 10 (1965), 471–493.

[3] M. J. DIXMIER, Etude sur les varietes et les operaterus de Julia, avec quelques applications, Bull. Soc. Math. France 77 (1949), 11–101.

[4] R. G. DOUGLAS, On majorization, factorization, and range inclusion of operators on Hilbert space, Proc. A.M.S. 17 (1966), 413–415.

[5] I. C. GOHBERG AND M. G. KREIN, The basic propositions on defect numbers, root numbers, and indices of linear operators, A.M.S. Translations 13 (1960), 185–264.

[6] S. GOLDBERG, Unbounded Operators, Theory and Applications, McGraw-Hill Book Co., New York, 1966.

[7] T. N. E. GREVILLE, The pseudo-inverse of a rectangular or singular matrix and its application to the solution of systems of linear equations, Soc. Ind. Appl. Math., 1, (1959)

38–43.

[8] T. N. E. GREVILLE, Some applications of the pseudo-inverse of a matrix, Soc. Ind. Appl. Math., 2 (1960), 15–22.

UNIVERSITY OF GEORGIA