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THE NUMBER THEORETIC FUNCTIONS

SHÔICHI TAKAHASHI

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k, A finite algebraic number field.

A_k, The set of all integral ideals of k.

$\mathfrak{o}_k$, The unit ideal of A_k.

R_k, The set of all complex-valued functions on A_k.

We shall define a summation and a product in R_k as the following. For any pair $f, g \in R_k$ and $\mathfrak{a} \in A_k$ we set

$$\begin{aligned}(f + g)(\mathfrak{a}) &= f(\mathfrak{a}) + g(\mathfrak{a}) \\ (f \circ q)(\mathfrak{a}) &= \sum_{\mathfrak{b} \mid \mathfrak{a}} f(\mathfrak{b}) g(\mathfrak{a}/\mathfrak{b}) \\ &= \sum_{\mathfrak{a}_1 \mathfrak{a}_2 = \mathfrak{a}} f(\mathfrak{a}_1) g(\mathfrak{a}_2) \, .\end{aligned}$$

THEOREM 1. *R_k is a commutative ring with respect to the summation and product mentioned above.*

PROOF. This is well known for the case of rational ground field. And for the case of k, the same method holds.

Now, we set the function $e_k \in R_k$ as the following.

$$e_k(\mathfrak{a}) = \begin{cases} 1, & \mathfrak{a} = \mathfrak{o}_k \\ 0, & \mathfrak{a} \neq \mathfrak{o}_k \, . \end{cases}$$

Then, for $f \in R_k$, $\mathfrak{a} \in A_k$, we get

$$\begin{aligned}(e_k \circ f)(\mathfrak{a}) &= \sum_{\mathfrak{a}_1 \mathfrak{a}_2 = \mathfrak{a}} e_k(\mathfrak{a}_1) f(\mathfrak{a}_2) \\ &= e_k(\mathfrak{o}_k) f(\mathfrak{a}) \\ &= f(\mathfrak{a}) \, .\end{aligned}$$

Therefore, the function e_k is the unit element in R_k. Now, the prime ideals of A_k are countable. Therefore we can take some numbering.

$$\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n, \cdots \, .$$

We take the dictionary order in A_k as the following. For

$$\mathfrak{a} = \prod_{i=1}^{n} \mathfrak{p}_i^{e(\mathfrak{a}, i)} \, , \qquad \mathfrak{b} = \prod_{i=1}^{n} \mathfrak{p}_i^{e(\mathfrak{b}, i)}$$

we set $\mathfrak{a} < \mathfrak{b}$ when the following holds.

$$\begin{aligned} e(\mathfrak{a}, n) &= e(\mathfrak{b}, n) \\ e(\mathfrak{a}, n-1) &= e(\mathfrak{b}, n-1) \\ &\cdots \\ e(\mathfrak{a}, k+1) &= e(\mathfrak{b}, k+1) \\ e(\mathfrak{a}, k) &< e(\mathfrak{b}, k) \,. \end{aligned}$$

The order in A_k is totally order. When

$$\mathfrak{a} = \mathfrak{p}_1^{a_1}\mathfrak{p}_2^{a_2} \cdots \mathfrak{p}_n^{a_n} \cdots$$
$$a_n \neq 0,\ a_{n+1} = a_{n+2} = \cdots = 0 \,,$$

we call n the length of $\mathfrak{a}$ and write $l(\mathfrak{a})$.

LEMMA 1. *Any sub-set S of A_k has the minimum element in the sense of the above dictionary order.*

PROOF. We set

$$\begin{aligned} n &= \mathrm{Min}\,\{l(\mathfrak{b}) \mid \mathfrak{b} \in S\} \\ b_n^0 &= \mathrm{Min}\,\{b_n \mid \mathfrak{b} \in S,\ l(\mathfrak{b}) = n,\ \mathfrak{b} = \mathfrak{p}_1^{b_1}\mathfrak{p}_2^{b_2} \cdots \mathfrak{p}_n^{b_n}\} \\ b_{n-1}^0 &= \mathrm{Min}\,\{b_{n-1} \mid \mathfrak{b} \in S,\ l(\mathfrak{b}) = n,\ \mathfrak{b} = \mathfrak{p}_1^{b_1}\mathfrak{p}_2^{b_2} \cdots \mathfrak{p}_{n-1}^{b_{n-1}}\mathfrak{p}_n^{b_n^0}\} \\ &\cdots \\ b_1^0 &= \mathrm{Min}\,\{b_1 \mid \mathfrak{b} \in S,\ l(\mathfrak{b}) = n,\ \mathfrak{b} = \mathfrak{p}_1^{b_1}\mathfrak{p}_2^{b_2^0} \cdots \mathfrak{p}_{n-1}^{b_{n-1}^0}\mathfrak{p}_n^{b_n^0}\} \,. \end{aligned}$$

then the element

$$\mathfrak{b}_0 = \mathfrak{p}_1^{b_1^0}\mathfrak{p}_2^{b_2^0} \cdots \mathfrak{p}_n^{b_n^0}$$

is the minimum element of S.

LEMMA 2. $\mathfrak{a}_0 < \mathfrak{a},\ \mathfrak{b}_0 \leqq \mathfrak{b} \Rightarrow \mathfrak{a}_0\mathfrak{b}_0 < \mathfrak{a}\mathfrak{b}$.

THEOREM 2. *The ring R_k is an integral domain.*

PROOF. We take $f, g \in R_k$, $f \neq O_k$, $g \neq O_k$. Then, let $\mathfrak{a}_0$ be the minimum element of $\mathfrak{a}$ such that $f(\mathfrak{a}) \neq 0$ holds. And let $\mathfrak{b}_0$ be the minimum element of $\mathfrak{b}$ such that $f(\mathfrak{b}) \neq 0$ holds. Then

$$\begin{aligned} (f \circ g)(\mathfrak{a}_0\mathfrak{b}_0) &= \sum_{\mathfrak{a}\mathfrak{b}=\mathfrak{a}_0\mathfrak{b}_0} f(\mathfrak{a})g(\mathfrak{b}) \\ &= \sum_{\substack{\mathfrak{a}_0 \leqq \mathfrak{a} \\ \mathfrak{b}_0 \leqq \mathfrak{b} \\ \mathfrak{a}\mathfrak{b}=\mathfrak{a}_0\mathfrak{b}_0}} f(\mathfrak{a})g(\mathfrak{b}) \\ &= f(\mathfrak{a}_0)g(\mathfrak{b}_0) \\ &\neq 0 \end{aligned}$$

holds. Therefore

$$f \circ g \neq O_k \,.$$

We call $f \in R_k$ "multiplicative" when the following equality holds. For $\mathfrak{a}, \mathfrak{b} \in A_k$, $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{o}_k$

$$f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b}) .$$

Moreover, when

$$f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b}), \quad \text{for any} \quad \mathfrak{a}, \mathfrak{b} \in A_k$$

we call f "completely multiplicative". Next, when the both $f, g \in R_k$ are multiplicative, for $\mathfrak{a}, \mathfrak{b} \in A_k$, $(\mathfrak{a}, \mathfrak{b}) = \mathfrak{o}_k$

$$\begin{aligned}(f \circ g)(\mathfrak{a}\mathfrak{b}) &= \sum_{\mathfrak{c} \mid \mathfrak{a}\mathfrak{b}} f(\mathfrak{c})/g(\mathfrak{a}\mathfrak{b}/\mathfrak{c}) \\ &= \sum_{\substack{\mathfrak{a}' \mid \mathfrak{a} \\ \mathfrak{b}' \mid \mathfrak{b}}} f(\mathfrak{a}'\mathfrak{b}')g(\mathfrak{a}\mathfrak{b}/\mathfrak{a}'\mathfrak{b}') \\ &= \sum_{\substack{\mathfrak{a}' \mid \mathfrak{a} \\ \mathfrak{b}' \mid \mathfrak{b}}} f(\mathfrak{a}')f(\mathfrak{b}')g(\mathfrak{a}/\mathfrak{a}')g(\mathfrak{b}/\mathfrak{b}') \\ &= \Big(\sum_{\mathfrak{a}' \mid \mathfrak{a}} f(\mathfrak{a}')g(\mathfrak{a}/\mathfrak{a}')\Big)\Big(\sum_{\mathfrak{b}' \mid \mathfrak{b}} f(\mathfrak{b}')g(\mathfrak{b}/\mathfrak{b}')\Big) \\ &= (f \circ g)(\mathfrak{a}) \cdot (f \circ g)(\mathfrak{b})\end{aligned}$$

holds. Therefore, the function $f \circ g$ is also multiplicative.

Now, we set the function $l_k \in R_k$ as the following

$$l_k(\mathfrak{a}) = 1 , \qquad \forall \mathfrak{a} \in A_k .$$

Obviously the function l_k is completely mutiplicative. For a non-negative rational integer e, let ${}_xH_e$ be a polynomial of one variable x with e-degree as the following

$${}_xH_e = \begin{cases} 1, \ e = 0 \\ \dfrac{1}{e!}(x + e - 1)(x + e - 2) \cdots (x + 1)x, \ e \geqq 1 . \end{cases}$$

Then, for any complex number α we define the multiplicative function $l_k^{(\alpha)}$ as the following:

$$l_k^{(\alpha)}(\mathfrak{p}^e) = {}_\alpha H_e .$$

THEOREM 3. (i) *If we restrict α to the rational integer, $l_k^{(\alpha)}$ has the same mean as the grouptheoretical power in R_k.*

(ii) *For any complex number α, β*

$$l_k^{(\alpha)} \circ l_k^{(\beta)} = l_k^{(\alpha+\beta)}$$

holds.

PROOF. (i) Let f be a rational integer,

(a) $f = 0$

$$l_k^0(\mathfrak{p}^e) = \begin{cases} 1, & e = 0 \\ 0, & e \geqq 1 \end{cases}$$
$$= {}_0H_e$$
$$= l_k^{(0)}(\mathfrak{p}^e) .$$

(b) $f > 0$

$$l_k^f(\mathfrak{p}^e) = \begin{cases} 1, e = 0 \\ \sum_{a_1+a_2+\cdots+a_f=e} l_k(\mathfrak{p}^{a_1}) l_k(\mathfrak{p}^{a_2}) \cdots l_k(\mathfrak{p}^{a_f}) \end{cases}$$
$$= \sum_{a_1+a_2+\cdots+a_f=e} 1$$
$$= {}_fH_e$$
$$= l_k^{(f)}(\mathfrak{p}^e) .$$

(c) $f < 0$

We take the function $\mu_k \in R_k$ as the following

$$\mu_k(\mathfrak{a}) = \begin{cases} 0, & \mathfrak{p}^2 | \mathfrak{a} \\ (-1)^k, & \mathfrak{a} = \mathfrak{p}_{i_1}\mathfrak{p}_{i_2} \cdots \mathfrak{p}_{i_k} . \end{cases}$$

Then

$$\mu_k \circ l_k = e_k$$

holds. For $\mu_k \circ l_k(\mathfrak{o}_k) = \mu_k(\mathfrak{o}_k) l_k(\mathfrak{o}_k) = 1 = e_k(\mathfrak{o}_k)$. When $\mathfrak{a} > \mathfrak{o}_k$, $\mathfrak{a} = \mathfrak{p}_{i_1}^{a_1}\mathfrak{p}_{i_2}^{a_2} \cdots \mathfrak{p}_{i_k}^{a_k}$, and $a_i > 0$ $(i = 1, 2, \cdots, k)$,

$$\mu_k \circ l_k(\mathfrak{a}) = \sum_{\mathfrak{b} | \mathfrak{a}} \mu_k(\mathfrak{b})$$
$$= 1 + \sum_j \mu_k(\mathfrak{p}_{i_j}) + \sum_{j,j'} \mu(\mathfrak{p}_{i_j}\, \mathfrak{p}_{i_{j'}}) + \cdots$$
$$= 1 - k + \binom{k}{2} - \cdots$$
$$= (1-1)^k$$
$$= 0$$
$$= e_k(\mathfrak{a})$$

holds. On the other hand,

$$\mu_k^{-f}(\mathfrak{p}^e) = \begin{cases} 0, & e > -f \\ (-1)^e{}_{-f}C_e, & e \leqq -f . \end{cases}$$

Therefore we get

$$l_k^{(f)}(\mathfrak{p}^e) = \mu_k^{-f}(\mathfrak{p}^e) = {}_fH_e .$$

(ii) We get

$$(l_k^{(\alpha)} \circ l_k^{(\beta)})(\mathfrak{p}^e) = l_k^{(\alpha)}(\mathfrak{o}_k) l_k^{(\beta)}(\mathfrak{p}^e) + l_k^{(\alpha)}(\mathfrak{p}) l_k^{(\beta)}(\mathfrak{p}^{e-1}) + \cdots + l_k^{(\alpha)}(\mathfrak{p}^e) l_k^{(\beta)}(\mathfrak{o}_k)$$
$$= {}_\alpha H_0 \cdot {}_\beta H_e + {}_\alpha H_1 \cdot {}_\beta H_{e-1} + \cdots + {}_\alpha H_e \cdot {}_\beta H_0 \, .$$

On the other hand,

$$l_k^{(\alpha+\beta)}(\mathfrak{p}^e) = {}_{\alpha+\beta} H_e \, .$$

Now, it is sufficient that the following polynomial identity of two variables x, y

$${}_{x+y}H_e = {}_x H_0 \cdot {}_y H_e + {}_x H_1 \cdot {}_y H_{e-1} + \cdots + {}_x H_e \cdot {}_y H_0$$

holds. Also it is sufficient that for the special values

$$(x, y) = (p, q), \quad (p, q = 0, 1, 2, \cdots, e)$$

holds. This is trivial. For $f \in R_k$, we consider the following function of variable s

$$\zeta_k(s, f) = \sum_{\mathfrak{a} \in A_k} \frac{f(\mathfrak{a})}{N(\mathfrak{a})^s} \, .$$

We take $f, g \in R_k$. Then for values of s such that $\zeta_k(s, f)$, $\zeta_k(s, g)$ together absolutely converge

$$\zeta_k(s, f) \zeta_k(s, g) = \zeta_k(s, f \circ g)$$

holds. Especially, the function

$$\zeta_k(s) = \zeta_k(s, l_k) = \sum_{\mathfrak{a} \in A_k} \frac{1}{N(\mathfrak{a})^s}$$

absolutely converges for the values $\operatorname{Re} s > 1$. Next, for a complex value α, we define

$$((\zeta_k)(s))^{(\alpha)} = \sum_{\mathfrak{a} \in A_k} \frac{l_k^{(\alpha)}(\mathfrak{a})}{N(\mathfrak{a})^s}$$

for the complex values s such that the right-hand side absolutely converges.

THEOREM 4. (i) *If we restrict α to the rational integers, $(\zeta_k(s))^{(\alpha)}$ has the same mean as natural power $(\zeta_k(s))^\alpha$.*

(ii) *For complex number α, β*

$$(\zeta_k(s))^{(\alpha)} (\zeta_k(s))^{(\beta)} = (\zeta_k(s))^{(\alpha+\beta)}$$

holds.

PROOF. See the theorem 3.

Let $k \subset K$ be a finite algebraic extention. Then, we shall define a map

$$\bar{N}_{K/k} \colon R_K \longrightarrow R_k$$

as the following. For $F \in R_K$, $\mathfrak{a} \in A_k$, we set

$$\bar{N}_{K/k}F(\mathfrak{a}) = \sum_{\substack{N_{K/k}\mathfrak{A}=\mathfrak{a} \\ \mathfrak{A} \in A_k}} F(\mathfrak{A})$$

provided that the right-hand side represents 0 when there is no $\mathfrak{A}$ such that

$$\mathfrak{A} \in A_k, \quad N_{K/k}\mathfrak{A} = \mathfrak{a} .$$

Now, for $F, G \in R_k$ and $\mathfrak{a} \in A_k$

$$\begin{aligned} \bar{N}_{K/k}(F+G)(\mathfrak{a}) &= \sum_{\substack{\mathfrak{A} \in A_k \\ N_{K/k}\mathfrak{A}=\mathfrak{a}}} (F+G)(\mathfrak{A}) \\ &= \sum_{\substack{\mathfrak{A} \in A_k \\ N_{K/k}\mathfrak{A}=\mathfrak{a}}} F(\mathfrak{A}) + \sum_{\substack{\mathfrak{A} \in A_k \\ N_{K/k}\mathfrak{A}=\mathfrak{a}}} G(\mathfrak{A}) \\ &= \bar{N}_{K/k}F(\mathfrak{a}) + \bar{N}_{K/k}G(\mathfrak{a}) \end{aligned}$$

holds. Therefore we get

$$\bar{N}_{K/k}(F+G) = \bar{N}_{K/k}F + \bar{N}_{K/k}G .$$

Next,

$$\begin{aligned} \bar{N}_{K/k}(F \circ G)(\mathfrak{a}) &= \sum_{\substack{\mathfrak{A} \in A_k \\ N_{K/k}\mathfrak{A}=\mathfrak{a}}} (F \circ G)(\mathfrak{A}) \\ &= \sum_{\substack{\mathfrak{A} \in A_k \\ N_{K/k}\mathfrak{A}=\mathfrak{a}}} \sum_{\mathfrak{A}_1\mathfrak{A}_2=\mathfrak{A}} F(\mathfrak{A}_1)G(\mathfrak{A}_2) \end{aligned}$$

holds. On the other hand,

$$\begin{aligned} ((\bar{N}_{K/k}F)\circ(\bar{N}_{K/k}G))(\mathfrak{a}) &= \sum_{\mathfrak{a}=\mathfrak{a}_1\mathfrak{a}_2} (\bar{N}_{K/k}F(\mathfrak{a}_1))(\bar{N}_{K/k}G(\mathfrak{a}_2)) \\ &= \sum_{\mathfrak{a}=\mathfrak{a}_1\mathfrak{a}_2} \Bigg(\sum_{\substack{\mathfrak{A}_1 \in A_k \\ N_{K/k}\mathfrak{A}_1=\mathfrak{a}_1}} F(\mathfrak{A}_1) \Bigg) \Bigg(\sum_{\substack{\mathfrak{A}_2 \in A_k \\ N_{K/k}\mathfrak{A}_2=\mathfrak{a}_2}} G(\mathfrak{A}_2) \Bigg) \\ &= \sum_{\substack{\mathfrak{A} \in A_k \\ N_{K/k}\mathfrak{A}=\mathfrak{a}}} \sum_{\mathfrak{A}=\mathfrak{A}_1\mathfrak{A}_2} F(\mathfrak{A}_1)G(\mathfrak{A}_2) \end{aligned}$$

holds. Therefore we get

$$\bar{N}_{K/k}(F \circ G) = (\bar{N}_{K/k}F)\circ(\bar{N}_{K/k}G) .$$

From the above the map

$$\bar{N}_{K/k}\colon R_K \longrightarrow R_k$$

is a into ring homomorphism. Still more, if $F \in R_k$ is multiplicative, $\bar{N}_{K/k}F \in R_k$ is also multiplicative.

Let be $\mathfrak{a}_1, \mathfrak{a}_2 \in A_k$, $(\mathfrak{a}_1, \mathfrak{a}_2) = \mathfrak{o}_k$, then

$$\begin{aligned}
\bar{N}_{K/k}F(\mathfrak{a}_1\mathfrak{a}_2) &= \sum_{\substack{\mathfrak{A}\in A_k \\ N_{K/k}\mathfrak{A}=\mathfrak{a}_1\mathfrak{a}_2}} F(\mathfrak{A}) \\
&= \sum_{\substack{\mathfrak{A}_1,\mathfrak{A}_2\in A_k \\ N_{K/k}\mathfrak{A}_1=\mathfrak{a}_1 \\ N_{K/k}\mathfrak{A}_2=\mathfrak{a}_2}} F(\mathfrak{A}_1\mathfrak{A}_2),\ (\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{o}_k \\
&= \sum_{\substack{\mathfrak{A}_1,\mathfrak{A}_2\in A_k \\ N_{K/k}\mathfrak{A}_1=\mathfrak{a}_1 \\ N_{K/k}\mathfrak{A}_2=\mathfrak{a}_2}} F(\mathfrak{A}_1)F(\mathfrak{A}_2) \\
&= \Big(\sum_{\substack{\mathfrak{A}_1\in A_k \\ N_{K/k}\mathfrak{A}_1=\mathfrak{a}_1}} F(\mathfrak{A}_1)\Big)\Big(\sum_{\substack{\mathfrak{A}_2\in A_k \\ N_{K/k}\mathfrak{A}_2=\mathfrak{a}_2}} F(\mathfrak{A}_2)\Big) \\
&= (\bar{N}_{K/k}F(\mathfrak{a}_1))(\bar{N}_{K/k}F(\mathfrak{a}_2))
\end{aligned}$$

holds. Now, let K/k be a non-ramified abelian extention of degree n. We take a prime ideal $\mathfrak{p}$ in A_k, then

$$\mathfrak{p} = \mathfrak{P}_1\mathfrak{P}_2 \cdots \mathfrak{P}_g,\quad \mathfrak{P}_i\text{: a prime ideal in } A_k$$
$$N_{K/k}\mathfrak{P}_i = \mathfrak{p}^f,\ fg = n$$

holds. Next, we set the completely multiplicative functions in R_k $\chi_0, \chi_1, \cdots, \chi_{n-1}$ as the following:

$$\chi_i(\mathfrak{p}) = \zeta_f^i,\ \zeta_f = e^{2\pi i/f},\quad i = 0, 1, \cdots, n-1\,.$$

THEOREM 5. $\bar{N}_{K/k}(l_K) = \chi_0\circ\chi_1\circ\cdots\circ\chi_{n-1}$.

PROOF. The both-sides are multiplicative in R_k. Therefore it is sufficient that we show that for any prime ideal $\mathfrak{p}\in A_k$ the both-sides are equal for $\mathfrak{p}^a$. Now,

$$\bar{N}_{K/k}l_K(\mathfrak{p}^a) = \begin{cases} 0, & f\nmid a \\ {}_gH_{a/f}, & f\mid a \end{cases}$$

holds. Therefore $\bar{N}_{K/k}l_K(\mathfrak{p}^a)$ is equal to the coefficient of x^a in the following formal power series

$$(1 + x^f + x^{2f} + \cdots)^g = \frac{1}{(1-x^f)^g}\,.$$

On the other hand, $\chi_0\circ\chi_1\circ\cdots\circ\chi_{n-1}(\mathfrak{p}^a)$ is equal to the coefficient of x^a in the following formal power series.

$$\begin{aligned}
&(1 + \chi_0(\mathfrak{p})x + \chi_0(\mathfrak{p})^2x^2 + \cdots) \\
&\quad\cdot(1 + \chi_1(\mathfrak{p})x + \chi_1(\mathfrak{p})^2x^2 + \cdots) \\
&\qquad\cdots \\
&\qquad\cdot(1 + \chi_{n-1}(\mathfrak{p})x + x_{n-1}(\mathfrak{p})^2x^2 + \cdots)
\end{aligned}$$

$$= \frac{1}{1-\chi_0(\mathfrak{p})x} \cdot \frac{1}{1-\chi_1(\mathfrak{p})x} \cdot \cdots \cdot \frac{1}{1-\chi_{n-1}(\mathfrak{p})x}$$

$$= \frac{1}{\prod_{i=0}^{n-1}(1-\zeta_f^i x)}$$

Now, the equality

$$(1-x^f)^g = \prod_{i=0}^{n-1}(1-\zeta_i^f x)$$

is trivial.

DEPARTMENT OF MATHEMATICS
YAMAGATA UNIVERSITY
YAMAGATA, JAPAN