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DERIVED C^*-ALGEBRAS OF PRIMITIVE C^*-ALGEBRAS

Dedicated to Professor Masanori Fukamiya on his 60th birthday

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1. Introduction. It has been known that every derivation of a W^*-algebra is inner (as a corollary, every derivation of a C^*-algebra is weakly inner), and every derivation of a simple C^*-algebra with identity is inner (cf. [6]). Moreover it has been shown that for a simple C^*-algebra $\mathfrak{A}$ with or without identity, there exists a unique primitive C^*-algebra $\mathfrak{D}(\mathfrak{A})$ with identity (called the derived C^*-algebra of $\mathfrak{A}$) such that (1) $\mathfrak{A}$ is an ideal of $\mathfrak{D}(\mathfrak{A})$; (2) for every derivation δ of $\mathfrak{A}$, there is a unique (modulo scalar multiples of identity) element d in $\mathfrak{D}(\mathfrak{A})$ such that $\delta(a) = [d, a]$ $(a \in \mathfrak{A})$; (3) every derivation of $\mathfrak{D}(\mathfrak{A})$ is inner ([7]).

These results make the study of derivations in general C^*-algebras, more or less, possible to reduce to the study of derivations in simple C^*-algebras if the C^*-algebras have only maximal ideals as primitive ideals.

However there are many C^*-algebras which do not have any maximal ideal ([3]). For the study of derivations in these C^*-algebras, it is desirable to analyse derivations in primitive C^*-algebras.

In the present paper, we shall generalize the notion of derived C^*-algebras to primitive C^*-algebras to make possible to reduce the study of derivations in general C^*-algebras to the study of derivations in primitive C^*-algebas.

We shall explain briefly the main result in this paper. Let $\mathfrak{A}$ be a primitive C^*-algebras (more generally, a factorial C^*-algebra) and let $D(\mathfrak{A})$ be the Lie algebra of all derivations on $\mathfrak{A}$. For an arbitrary faithful factorial *-representation $\{\pi, \mathfrak{X}\}$ of $\mathfrak{A}$ on a Hilbert space $\mathfrak{X}$, it is known that a unique (modulo scalar multiples of identity) element d_δ in the weak closure $\overline{\pi(\mathfrak{A})}$ of $\pi(\mathfrak{A})$ such that $\pi(\delta(a)) = [d_\delta, \pi(a)]$ $(a \in \mathfrak{A})$. Now we shall identify $\mathfrak{A}$ with $\pi(\mathfrak{A})$, and let $\mathfrak{D}_\pi(\mathfrak{A})$ be the C^*-subalgebra of $B(\mathfrak{X})$ generated by $\{d_\delta \mid \delta \in D(\mathfrak{A})\}$ and $1_\mathfrak{X}$. Then it is easily imagined that the C^*-algebra $\mathfrak{D}_\pi(\mathfrak{A})$ is closely related to the structure of the Lie algebra $D(\mathfrak{A})$ and so we may apply the C^*-algebra theory to the study of $D(\mathfrak{A})$. However

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$\mathfrak{D}_\pi(\mathfrak{A})$ apparently depends on the choice of the representation $\{\pi, \mathfrak{X}\}$. This is troublesome, since a factorial C^*-algebra has generally uncountably many inequivalent faithful factorial *-representations.

The main result of this paper is that the C^*-algebra $\mathfrak{D}_\pi(\mathfrak{A})$ does not depend on the special choice of the $\{\pi, \mathfrak{X}\}$—namely, for an arbitrary faithful factorial *-representation, we get always the same C^*-algebra $\mathfrak{D}(\mathfrak{A})$; therefore we can associate a unique C^*-algebra $\mathfrak{D}(\mathfrak{A})$ (called the derived C^*-algebra of $\mathfrak{A}$) which is closely related to the structures of $\mathfrak{A}$ and $D(\mathfrak{A})$ to each factorial C^*-algebra $\mathfrak{A}$.

2. **Derived C^*-algebras.** We shall begin with a definition.

DEFINITION 1. A C^*-algebra $\mathfrak{A}$ is called factorial if $\mathfrak{A}$ has a faithful factorial *-representation $\{\pi, \mathfrak{X}\}$ on a Hilbert space $\mathfrak{X}$—i.e. π is faithful and $\overline{\pi(\mathfrak{A})}$ is a factor, where $\overline{\pi(\mathfrak{A})}$ is the weak closure of $\pi(\mathfrak{A})$.

REMARK 1. Every primitive C^*-algebra is clearly factorial, and it is known that every separable factorial C^*-algebra is primitive (cf. [2]).

Let $\mathfrak{A}$ be a factorial C^*-algebra and let $\{\pi, \mathfrak{X}\}$ be a faithful factorial *-representation of $\mathfrak{A}$, and let $D(\mathfrak{A})$ be the Lie algebra of all derivations on $\mathfrak{A}$. For each $\delta \in D(\mathfrak{A})$, there exists an element d_δ in $\overline{\pi(\mathfrak{A})}$ such that $\pi(\delta(a)) = [d_\delta, \pi(a)]$ $(a \in \mathfrak{A})$. Suppose that d'_δ is another element in $\overline{\pi(\mathfrak{A})}$ such that $\pi(\delta(a)) = [d'_\delta, \pi(a)]$ $(a \in \mathfrak{A})$; then $[d_\delta - d'_\delta, \pi(\mathfrak{A})] = 0$; hence $d_\delta - d'_\delta = \lambda 1_\mathfrak{X}$, where λ is a complex number and $1_\mathfrak{X}$ is the identity operator on $\mathfrak{X}$. Now let $\mathfrak{D}_\pi(\mathfrak{A})$ be the C^*-subalgebra of $B(\mathfrak{X})$ generated by $\{d_\delta \mid \delta \in D(\mathfrak{A})\}$ and $1_\mathfrak{X}$, then by the above consideration, $\mathfrak{D}_\pi(\mathfrak{A})$ does not depend on the choices of d_δ's. However it apparently depends on the representation $\{\pi, \mathfrak{X}\}$.

In the following considerations, we shall show that the C^*-algebra $\mathfrak{D}_\pi(\mathfrak{A})$ does not depend on the representation $\{\pi, \mathfrak{X}\}$ either.

Now let δ be a skew-symmetric derivation on $\mathfrak{A}$—i.e. $\delta^* = -\delta$, where $\delta^*(a) = \delta(a^*)^*$ $(a \in \mathfrak{A})$. Then there exists a positive element d_δ in the weak closure $\overline{\pi(\mathfrak{A})}$ such that $\|d_\delta\| = \|\delta\|$, where $\|\delta\|$ is the norm of the derivation δ ([5]). (More generally it is known that for each general derivation δ on $\mathfrak{A}$, there exists an element d_δ in $\overline{\pi(\mathfrak{A})}$ such that $\pi(\delta(a)) = [d_\delta, \pi(a)]$ for $a \in \mathfrak{A}$ and $(1/2)\|\delta\| = \|d_\delta\|$ ([4], [5], [8], [10])).

It is clear that such a positive element is unique, since $\|d_\delta + \lambda 1_\mathfrak{X}\| \gneqq \|d_\delta\|$ if λ is not zero and $d_\delta + \lambda 1_\mathfrak{X} \geqq 0$.

More strongly we have

LEMMA 1 (cf. [1]). *If h is a positive element in $\overline{\pi(\mathfrak{A})}$ such that $\pi(\delta(a)) = [h, \pi(a)]$ $(a \in \mathfrak{A})$ and $\|h\| \leqq \|\delta\|$, then $\|h\| = \|\delta\|$.*

PROOF. Suppose that $||h|| < ||\delta||$ and put $k = h - (||h||/2)1_{\mathfrak{X}}$; then $||\pi(\delta(a))|| = ||[k, \pi(a)]|| \leqq ||k\pi(a)|| + ||\pi(a)k||$. Since π is faithful, $||\delta|| \leqq ||k|| + ||k|| = 2||h - (||h||/2)1_{\mathfrak{X}}|| < 2||h||/2 = ||h|| < ||\delta||$, a contradiction. This completes the proof.

Now we shall define

DEFINITION 2. A C^*-algebra $\mathfrak{B}$ with identity is called a quasi-derived C^*-algebra of a factorial C^*-algebra $\mathfrak{A}$ if it satisfies the following conditions: (1) $\mathfrak{A}$ is a C^*-subalgebra of $\mathfrak{B}$; (2) for each skew-symmetric derivation δ of $\mathfrak{A}$, there exists a positive element d_δ in $\mathfrak{B}$ such that $\delta(a) = [d_\delta, a]$ $(a \in \mathfrak{A})$ and $||\delta|| \geqq ||d_\delta||$; (3) $\mathfrak{B}$ is generated by $\mathfrak{A}$, $\{d_\delta |$ skew-symmetric $\delta \in D(\mathfrak{A})\}$ and the identity 1.

DEFINITION 3. A quasi-derived C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ is called a derived C^*-algebra of $\mathfrak{A}$, if there is no non-zero closed ideal J of $\mathfrak{B}$ such that $\mathfrak{A} \cap J = (0)$.

Then we shall show

THEOREM 1. *Let $\mathfrak{B}$ be a quasi-derived C^*-algebra of a factorial C^*-algebra $\mathfrak{A}$ and let $\{\pi, \mathfrak{X}\}$ be a faithful factorial *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{X}$. Then $\{\pi, \mathfrak{X}\}$ can be uniquely extended to a factorial *-representation $\{\tilde{\pi}, \mathfrak{X}\}$ of $\mathfrak{B}$ on the Hilbert space $\mathfrak{X}$ such that $\tilde{\pi}(\mathfrak{B}) \subseteqq \overline{\pi(\mathfrak{A})}$.*

Moreover if $\mathfrak{B}$ is a derived C^*-algebra of $\mathfrak{A}$, then the extended representation is again faithful.

PROOF. Since any factorial *-representation is a sum of cyclic factorial *-representations, it suffices to assume that the $\{\pi, \mathfrak{X}\}$ is cyclic and so $\{\pi, \mathfrak{X}\}$ is equivalent to a *-representation $\{\pi_\varphi, \mathfrak{X}_\varphi\}$ of $\mathfrak{A}$ on a Hilbert space $\mathfrak{X}_\varphi$ constructed via a state φ on $\mathfrak{A}$. Let $\tilde{\varphi}$ be an extended state of $\mathfrak{B}$ such that $\tilde{\varphi} = \varphi$ on $\mathfrak{A}$. Let $\{\pi_{\tilde{\varphi}}, \mathfrak{X}_{\tilde{\varphi}}\}$ be the *-representation of $\mathfrak{B}$ on a Hilbert space $\mathfrak{X}_{\tilde{\varphi}}$ constructed via $\tilde{\varphi}$. Let E' be the orthogonal projection of $\mathfrak{X}_{\tilde{\varphi}}$ onto the closed subspace $[\pi_{\tilde{\varphi}}(\mathfrak{A})1_{\tilde{\varphi}}]$; then the *-representation $a \to \pi_{\tilde{\varphi}}(a)E'(a \in \mathfrak{A})$ of $\mathfrak{A}$ on $E'\mathfrak{X}_{\tilde{\varphi}}$ is equivalent to $\{\pi_\varphi, \mathfrak{X}_\varphi\}$.

Let $c(E')$ be the central support of E' in $\pi_{\tilde{\varphi}}(\mathfrak{A})'$, where $\pi_{\tilde{\varphi}}(\mathfrak{A})'$ is the commutant of $\pi_{\tilde{\varphi}}(\mathfrak{A})$ on $\mathfrak{X}_{\tilde{\varphi}}$.

Since $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})E'}$ is *-isomorphic to $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}c(E')$, $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}c(E')$ is a factor. Moreover the *-representation $a \to \pi_{\tilde{\varphi}}(a)c(E')$ $(a \in \mathfrak{A})$ is faithful, since $a \to \pi_{\tilde{\varphi}}(a)E'$ $(a \in \mathfrak{A})$ is faithful. For each skew-symmetric $\delta \in D(\mathfrak{A})$, there exists a self-adjoint element k in $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$ such that $\pi_{\tilde{\varphi}}(\delta(a)) = [k, \pi_{\tilde{\varphi}}(a)]$ $(a \in \mathfrak{A})$. Hence $[k, \pi_{\tilde{\varphi}}(a)] = [\pi_{\tilde{\varphi}}(d_\delta), \pi_{\tilde{\varphi}}(a)]$ for $a \in \mathfrak{A}$ and so $k - \pi_{\tilde{\varphi}}(d_\delta) \in \pi_{\tilde{\varphi}}(\mathfrak{A})'$. Since $\pi_{\tilde{\varphi}}(\mathfrak{B})$ is generated by $\pi_{\tilde{\varphi}}(\mathfrak{A})$, $\pi_{\tilde{\varphi}}\{d_\delta |$ skew-symmetric $\delta \in D(\mathfrak{A})\}$ and $1_{\mathfrak{X}_{\tilde{\varphi}}}$, the central element $c(E')$ of $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$ belongs to the center of $\overline{\pi_{\tilde{\varphi}}(\mathfrak{B})}$.

On the other hand, $1_{\tilde{\varphi}} \in E'\mathfrak{X}_{\tilde{\varphi}}$ and so $\pi_{\tilde{\varphi}}(\mathfrak{B})1_{\tilde{\varphi}} \subset c(E')\mathfrak{X}_{\tilde{\varphi}}$; hence $c(E') = 1_{\mathfrak{X}_{\tilde{\varphi}}}$. Therefore the mapping $a \to \pi_{\tilde{\varphi}}(a)$ $(a \in \mathfrak{A})$ on $\mathfrak{X}_\varphi$ is a faithful factorial *-representation of $\mathfrak{A}$, and so we can choose a positive element h in $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$ such that $\pi_{\tilde{\varphi}}(\delta(a)) = [h, \pi_{\tilde{\varphi}}(a)]$ $(a \in \mathfrak{A})$ and $\|h\| = \|\delta\|$.

Let C be the commutative C^*-subalgebra of $B(\mathfrak{X}_{\tilde{\varphi}})$ generated by $\pi_{\tilde{\varphi}}(d_\delta) - h$ and $1_{\mathfrak{X}_{\tilde{\varphi}}}$, and let R be the C^*-subalgebra of $B(\mathfrak{X}_{\tilde{\varphi}})$ generated by $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$ and C. Since $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$ is a factor, R can be canonically identified with the tensor product $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})} \otimes C$ ([9]).

Now suppose that $\pi_{\tilde{\varphi}}(d_\delta) - h$ has a positive spectrum μ_1 and take a character χ_1 on C such that $\chi_1(\pi_{\tilde{\varphi}}(d_\delta) - h) = \mu_1$. Also take a pure state φ_1 on $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$ such that $\varphi_1(h) = \|h\|$. Then we have,

$$\begin{aligned}\|\delta\| &\geqq \|\pi_{\tilde{\varphi}}(d_\delta)\| \geqq |\varphi_1 \otimes \chi_1(h + \pi_{\tilde{\varphi}}(d_\delta) - h)| \\ &= \varphi_1(h) + \chi_1(\pi_{\tilde{\varphi}}(d_\delta) - h) = \|h\| + \mu_1 > \|h\| = \|\delta\|\,,\end{aligned}$$

a contradiction.

Next suppose that $\pi_{\tilde{\varphi}}(d_\delta) - h$ has a negative spectrum μ_2 and take a character χ_2 on C such that $\chi_2(\pi_{\tilde{\varphi}}(d_\delta) - h) = \mu_2$. Now we shall show that h is not invertible in $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$. In fact, if h is invertible in $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$, then there exists a positive number λ such that $h \geqq \lambda 1_{\mathfrak{X}_{\tilde{\varphi}}}$; hence $h - \lambda 1_{\mathfrak{X}_{\tilde{\varphi}}} \geqq 0$ and $\|h - \lambda 1_{\mathfrak{X}_{\tilde{\varphi}}}\| < \|h\| = \|\delta\|$. Since $[h, \pi_{\tilde{\varphi}}(a)] = [h - \lambda 1_{\mathfrak{X}_{\tilde{\varphi}}}, \pi_{\tilde{\varphi}}(a)] = \pi_{\tilde{\varphi}}(\delta(a))$ for $a \in \mathfrak{A}$, and since $\pi_{\tilde{\varphi}}$ is faithful on $\mathfrak{A}$, by Lemma 1 $\|h - \lambda 1_{\mathfrak{X}_{\tilde{\varphi}}}\| = \|\delta\|$, a contradiction. Hence h is not invertible in $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$. Take a pure state φ_2 on $\overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$ such that $\varphi_2(h) = 0$. Then,

$$\varphi_2 \otimes \chi_2(\pi_{\tilde{\varphi}}(d_\delta)) = \varphi_2(h) + \chi_2(\pi_{\tilde{\varphi}}(d_\delta) - h) = \mu_2 < 0\,.$$

On the other hand, $\pi_{\tilde{\varphi}}(d_\delta) \geqq 0$ and $\varphi_2 \otimes \chi_2$ is a state on R, hence $\varphi_2 \otimes \chi_2(\pi_{\tilde{\varphi}}(d_\delta)) \geqq 0$, a contradiction. Therefore, $\pi_{\tilde{\varphi}}(d_\delta) - h = 0$, and so $\overline{\pi_{\tilde{\varphi}}(\mathfrak{B})} = \overline{\pi_{\tilde{\varphi}}(\mathfrak{A})}$. Hence $[\pi_{\tilde{\varphi}}(\mathfrak{B})1_{\tilde{\varphi}}] = [\pi_{\tilde{\varphi}}(\mathfrak{A})1_{\tilde{\varphi}}] = E'\mathfrak{X}_{\tilde{\varphi}} = \mathfrak{X}_{\tilde{\varphi}}$. This implies that the *-representation $\{\pi_{\tilde{\varphi}}, \mathfrak{X}_{\tilde{\varphi}}\}$ of $\mathfrak{B}$ can be considered a *-representation $\{\tilde{\pi}_\varphi, \mathfrak{X}_\varphi\}$ of $\mathfrak{B}$ on the Hilbert space $\mathfrak{X}_\varphi$ such that $\tilde{\pi}_\varphi = \pi_\varphi$ on $\mathfrak{A}$ and $\tilde{\pi}_\varphi(\mathfrak{B}) \subseteqq \overline{\pi_\varphi(\mathfrak{A})}$.

Next we shall show the unicity of the extension. Let $\{\pi', \mathfrak{X}_\varphi\}$ be another *-representation of $\mathfrak{B}$ on the Hilbert space $\mathfrak{X}_\varphi$ such that $\pi' = \pi_\varphi$ on $\mathfrak{A}$ and $\pi'(\mathfrak{B}) \subseteqq \overline{\pi_\varphi(\mathfrak{A})}$. Then $\|\pi'(d_\delta)\| \leqq \|\delta\|$ and $\pi'(d_\delta) \geqq 0$. Moreover $[\pi'(d_\delta), \pi_\varphi(a)] = \pi'(\delta(a)) = \pi_\varphi(\delta(a))$ $(a \in \mathfrak{A})$. Hence by the unicity of such an element, $\pi'(d_\delta) = \tilde{\pi}_\varphi(d_\delta)$ for all skew-symmetric $\delta \in D(\mathfrak{A})$. Therefore we have $\pi' = \tilde{\pi}_\varphi$ on $\mathfrak{B}$—i.e. $\{\tilde{\pi}_\varphi, \mathfrak{X}_\varphi\}$ is unique.

Next suppose that $\mathfrak{B}$ is a derived C^*-algebra of $\mathfrak{A}$. Put $J = \{b \mid \tilde{\pi}_\varphi(b) = 0, b \in \mathfrak{B}\}$; then J is a closed ideal of $\mathfrak{B}$. Since $\tilde{\pi}_\varphi = \pi_\varphi$ on $\mathfrak{A}$, $J \cap \mathfrak{A} = (0)$; hence $J = (0)$. This completes the proof.

REMARK 2. Let I be a closed ideal of a quasi-derived C^*-algebra $\mathfrak{B}$

of $\mathfrak{A}$ such that $\mathfrak{A} \cap I = (0)$; then the quotient C^*-algebra $\mathfrak{B}/I$ can be considered again a quasi-derived C^*-algebra of $\mathfrak{A}$, since $\mathfrak{A}$ can be identified with $\mathfrak{A} + I/I$. By the unicity of the extension in Theorem 1, $\tilde{\pi}(I) = 0$. Now put $I_0 = \{x \mid \tilde{\pi}(x) = 0, x \in \mathfrak{B}\}$; then $I_0 \cap \mathfrak{A} = (0)$. Therefore I_0 is the greatest closed ideal of $\mathfrak{B}$ in all closed ideals I with $I \cap \mathfrak{A} = (0)$. Clearly $\mathfrak{B}/I_0$ is a derived C^*-algebra of the C^*-algebra $\mathfrak{A}$.

Hence we have the following result: Let $\mathfrak{B}$ be a quasi-derived C^*-algebra of a factorial C^*-algebra $\mathfrak{A}$; then there exists the greatest closed ideal I_0 of $\mathfrak{B}$ in all closed ideals I with $I \cap \mathfrak{A} = (0)$, and the quotient C^*-algebra $\mathfrak{B}/I_0$ is a derived C^*-algebra of $\mathfrak{A}$.

THEOREM 2. *Let $\mathfrak{A}$ be a factorial C^*-algebra and let φ be a factorial state on $\mathfrak{A}$ such that the factorial *-representation $\{\pi_\varphi, \mathfrak{X}_\varphi\}$ of $\mathfrak{A}$ is faithful, and let $\mathfrak{B}$ be a quasi-derived C^*-algebra of $\mathfrak{A}$. Then φ has a unique state extension $\tilde{\varphi}$ to $\mathfrak{B}$.*

Moreover if $\mathfrak{B}$ is a derived C^*-algebra of $\mathfrak{A}$, then the extended state $\tilde{\varphi}$ satisfies again the condition that the representation $\{\pi_{\tilde{\varphi}}, \mathfrak{X}_{\tilde{\varphi}}\}$ of $\mathfrak{B}$ is faithful.

PROOF. By the considerations in the proof of Theorem 1, we showed that for an arbitrary state $\tilde{\varphi}$ of $\mathfrak{B}$ with $\tilde{\varphi} = \varphi$ on $\mathfrak{A}$, $\{\pi_{\tilde{\varphi}}, \mathfrak{X}_{\tilde{\varphi}}\} = \{\tilde{\pi}_\varphi, \mathfrak{X}_\varphi\}$ with $\tilde{\pi}_\varphi = \pi_\varphi$ on $\mathfrak{A}$ and $\tilde{\pi}_\varphi(\mathfrak{B}) \subseteqq \overline{\mathfrak{X}(\mathfrak{A})}$, and moreover $\{\tilde{\pi}_\varphi, \mathfrak{X}_\varphi\}$ is unique. Hence $\tilde{\varphi}$ must be unique. Moreover if $\mathfrak{B}$ is a derived C^*-algebra of $\mathfrak{A}$, then $\{\tilde{\pi}_\varphi, \mathfrak{X}_\varphi\}$ is faithful. This completes the proof.

REMARK 3. From Theorem 2, we can conclude the results of the author concerning simple C^*-algebras ([6], [7]). In fact, suppose that $\mathfrak{A}$ is a simple C^*-algebra and let I be the least closed ideal of a quasi-derived C^*-algebra $\mathfrak{B}$ of $\mathfrak{A}$ containing $\mathfrak{A}$. Let S be the set of all bounded self-adjoint linear functionals f on $\mathfrak{B}$ such that $f(\mathfrak{A}) = 0$ and $\|f\| \leqq 1$. If $\mathfrak{A} \subsetneqq I$, there is an extreme point g in S such that $g(I) \neq (0)$. Let $g = g_1 - g_2$ be the orthogonal decomposition of g with $g_1, g_2 \geqq 0$, $\|g\| = \|g_1\| + \|g_2\|$, and let $\xi = g_1 + g_2$. Let $\{\pi_\xi, \mathfrak{X}_\xi\}$ be the *-representation of $\mathfrak{B}$ constructed by ξ. Then the extremity of g implies that $\overline{\pi_\xi(\mathfrak{A})}$ is a factor if $\pi_\xi(\mathfrak{A}) \neq (0)$ (cf. [6], [7]). Since $\mathfrak{A}$ is simple, $\{\pi_\xi, \mathfrak{X}_\xi\}$ is faithful, factorial on $\mathfrak{A}$; hence by Theorem 2, $g_1 \equiv g_2$ on $\mathfrak{B}$ and so $g \equiv 0$ on $\mathfrak{B}$, a contradiction. Hence $\pi_\xi(\mathfrak{A}) = 0$ and so $\pi_\xi(I) = 0$. This implies $g(I) = 0$, a contradiction. Hence $\mathfrak{A} = I$. If $\mathfrak{A}$ has an identity, then $\mathfrak{A} = \mathfrak{B}$, and if $\mathfrak{A}$ has no identity, then $\mathfrak{A}$ is an ideal of $\mathfrak{B}$. Moreover if $\mathfrak{B}$ is a derived C^*-algebra of the simple C^*-algebra $\mathfrak{A}$ in the sense of this paper, then $\mathfrak{B}$ is primitive and so it coincides with the derived C^*-algebra in [7].

Now we shall show a general method to construct quasi-derived C^*-

algebras and derived C^*-algebras of $\mathfrak{A}$. Let $\mathfrak{A}$ be a factorial C^*-algebra, and let $\{\pi_1, \mathfrak{X}_1\}$ be an arbitrary faithful (not necessarily factorial) *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{X}_1$. We shall identify $\mathfrak{A}$ with $\pi_1(\mathfrak{A})$. Then for each skew-symmetric $\delta \in D(\mathfrak{A})$, there exists a positive element e_δ in $\overline{\pi_1(\mathfrak{A})}$ such that $\pi_1(\delta(a)) = [e_\delta, \pi_1(a)]$ $(a \in \mathfrak{A})$ and $\|e_\delta\| \leqq \|\delta\|$ ([5]).

Let $\mathfrak{B}$ be a C^*-subalgebra of $B(\mathfrak{X}_1)$ generated by $\pi_1(\mathfrak{A})$, $\{e_\delta \mid \text{skew-symmetric } \delta \in D(\mathfrak{A})\}$ and $1_{\mathfrak{X}_1}$. Then clearly $\mathfrak{B}$ is a quasi-derived C^*-algebra of $\mathfrak{A}$. Moreover, by Remark 2, there exists the greatest closed ideal I_0 of $\mathfrak{B}$ in all closed ideals I such that $I \cap \mathfrak{A} = (0)$.

Put $\mathfrak{D}_1(\mathfrak{A}) = \mathfrak{B}/I_0$. Since $\mathfrak{A} \cap I_0 = (0)$, we can identify $\mathfrak{A}$ with the image in $\mathfrak{D}_1(\mathfrak{A})$ under the canonical mapping. Then we can easily see that $\mathfrak{D}_1(\mathfrak{A})$ is a derived C^*-algebra of $\mathfrak{A}$.

Now we shall show

THEOREM 3 (The unicity of the derived C^*-algebra). *Let $\mathfrak{A}$ be a factorial C^*-algebra; then there exists a unique derived C^*-algebras $\mathfrak{D}(\mathfrak{A})$ of $\mathfrak{A}$ in the following sense: Let $\mathfrak{D}_1(\mathfrak{A})$, $\mathfrak{D}_2(\mathfrak{A})$ be two derived C^*-algebras of $\mathfrak{A}$; then there exists a *-isomorphism Φ of $\mathfrak{D}_1(\mathfrak{A})$ onto $\mathfrak{D}_2(\mathfrak{A})$ such that* (1) *$\Phi(a) = a$ for $a \in \mathfrak{A}$;* (2) *$\Phi(d_{\delta,1}) = d_{\delta,2}$ for $\delta \in D(\mathfrak{A})$ with $d_{\delta,i} \geqq 0, \|d_{\delta,i}\| = \|\delta\|$ $(i = 1, 2)$, where $\delta(a) = [d_{\delta,1}, a]$ $(a \in \mathfrak{A})$ in $\mathfrak{D}_1(\mathfrak{A})$ and $\delta(a) = [d_{\delta,2}, a]$ $(a \in \mathfrak{A})$ in $\mathfrak{D}_2(\mathfrak{A})$.*

*Moreover, let $\{\pi, \mathfrak{X}\}$ be a faithful factorial *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{X}$, and let $\mathfrak{B}$ be the C^*-subalgebra of $B(\mathfrak{X})$ generated by $\{d_\delta \mid \delta \in D(\mathfrak{A})\}$ and $1_{\mathfrak{X}}$; then $\mathfrak{B}$ is the derived C^*-algebra $\mathfrak{D}(\mathfrak{A})$ of $\mathfrak{A}$, when $\mathfrak{A}$ is identified with the image $\pi(\mathfrak{A})$.*

PROOF. We have shown already that there exists a derived C^*-algebra $\mathfrak{D}_1(\mathfrak{A})$ of $\mathfrak{A}$. Now let $\{\pi, \mathfrak{X}\}$ be a faithful factorial *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{X}$; then by Theorem 1, it can be uniquely extended to a faithful factorial *-representation $\{\tilde{\pi}, \mathfrak{X}\}$ of $\mathfrak{D}_1(\mathfrak{A})$. The image $\tilde{\pi}(\mathfrak{D}_1(\mathfrak{A}))$ is clearly the C^*-algebra generated by $\{d_\delta \mid \delta \in D(\mathfrak{A})\}$ and $1_{\mathfrak{X}}$. Moreover by Lemma 1, it is easily seen that $\tilde{\pi}(d_{\delta,1}) = d_\delta$ if $d_{\delta,1} \geqq 0$, $d_\delta \geqq 0$ and $\|d_{\delta,1}\| = \|d_\delta\| = \|\delta\|$ for all skew-symmetric $\delta \in D(\mathfrak{A})$.

This completes the proof.

Now let $\mathfrak{A}$ be a factorial C^*-algebra and let $\{\pi, \mathfrak{A}\}$ be a faithful factorial *-representation of $\mathfrak{A}$. We shall identify $\mathfrak{A}$ with the image $\pi(\mathfrak{A})$. If $\mathfrak{A}$ is a simple C^*-algebra with identity, then $\mathfrak{D}(\mathfrak{A}) = \mathfrak{A}$. If $\mathfrak{A}$ is a simple C^*-algebra without identity, then $\mathfrak{D}(\mathfrak{D}(\mathfrak{A})) = \mathfrak{D}(\mathfrak{A})$.

We shall denote $\mathfrak{D}(\mathfrak{D}(\mathfrak{A})) = \mathfrak{D}^{(2)}(\mathfrak{A})$, $\mathfrak{D}(\mathfrak{D}(\mathfrak{D}(\mathfrak{A}))) = \mathfrak{D}^{(3)}(\mathfrak{A})$, and so on. Then the following problem would be interesting.

Problem 1. Does there exist a primitive C^*-algebra $\mathfrak{A}$ such that

$\mathfrak{D}(\mathfrak{A}) \subsetneqq \mathfrak{D}^{(2)}(\mathfrak{A})$?

Problem 2. Does there exist a primitive C^*-algebra $\mathfrak{A}$ such that $\mathfrak{D}^{(n)}(\mathfrak{A}) \subsetneqq \mathfrak{D}^{(n+1)}(\mathfrak{A})$ for all positive integers n?

REMARK 4. By using the operation $\mathfrak{D}$, we can obtain an increasing family of C^*-subalgebras $\{\mathfrak{A}_\rho\}$ $(0 \leqq \rho \leqq \alpha)$ of $\overline{\pi(\mathfrak{A})}$, indexed by the ordinals ρ between 0 and a certain ordinal α as follows: (1) $\mathfrak{A}_0 = \mathfrak{A}$; (2) $\mathfrak{A}_{\rho+1} = \mathfrak{D}(\mathfrak{A}_\rho)$ if ρ is not a limit ordinal; (3) $\mathfrak{A}_\rho =$ the uniform closure of $\bigcup_{\rho'<\rho} \mathfrak{A}_{\rho'}$ if ρ is a limit ordinal; (4) $\mathfrak{D}(\mathfrak{A}_\alpha) = \mathfrak{A}_\alpha$. This is clear, since $\mathfrak{D}(\overline{\pi(\mathfrak{A})}) = \overline{\pi(\mathfrak{A})}$. However $\mathfrak{A}_\alpha$ does not generally coincide with $\overline{\pi(\mathfrak{A})}$. For example, let M be a non type I-factor on a separable Hilbert space $\mathfrak{X}$ and let $\mathfrak{A}$ be a uniformly separable C^*-subalgebra of M which is weakly dense in M. Let $\mathfrak{F} = \{\varepsilon_\beta\}_{\beta \in II}$ be a family of C^*-subalgebras of M such that $\mathfrak{A} \subset \varepsilon_\beta$ for $\beta \in II$, and for each faithful factorial $*$-representation $\{\pi_0, \mathfrak{X}_0\}$ of $\mathfrak{A}$, there exists a unique $*$-representation $\{\tilde{\pi}_0, \mathfrak{X}_0\}$ of ε_β such that $\tilde{\pi}_0 = \pi_0$ on $\mathfrak{A}$ and $\tilde{\pi}_0(\varepsilon_\beta) \subset \overline{\pi_0(\mathfrak{A})}$. We shall define an order in $\mathfrak{F}$ by inclusion, and let $\mathfrak{F}_1 = \{\varepsilon_{\beta_1}\}_{\beta_1 \in II_1}$ be a linearly ordered subset of $\mathfrak{F}$ and let ε be the C^*-subalgebra of M generated by $\bigcup_{\beta_1 \in II_1} \varepsilon_{\beta_1}$; then it is clear that ε belongs to $\mathfrak{F}$. Hence by Zorn's lemma, there exists a maximal element in $\mathfrak{F}$. Clearly $\mathfrak{A}_\rho \in \mathfrak{F}$ for all ρ with $0 \leqq \rho \leqq \alpha$.

Now let ε_0 be a maximal element in $\mathfrak{F}$ such that $\mathfrak{A}_\alpha \subseteqq \varepsilon_0$. Since $\mathfrak{A}$ is separable, it is primitive. Now let $\{\pi_1, \mathfrak{X}_1\}$ be a faithful irreducible $*$-representation of $\mathfrak{A}$; then $\mathfrak{X}_1$ is separable. If $\varepsilon_0 = M$, then M have an irreducible $*$-representation on a separable Hilbert space. Since any $*$-representation of M on a separable Hilbert space is σ-continuous ([11]), $\pi_1(M)$ is weakly closed; hence $\pi_1(M) = B(\mathfrak{X}_1)$. This contradicts that M is a non type I-factor. Hence $\mathfrak{A}_\alpha \subsetneqq M$.

The following problem would be intresting.

Problem 3. Let G be a countable, discrete group such that every conjugate class is infinite except for the conjugate class of the identity, and let $U(G)$ be the W^*-algebra generated by the left regular representation; then $U(G)$ is a II_1-factor. Let $\mathfrak{A}$ be the C^*-subalgebra of $U(G)$ generated by the left regular representation. Then what is $\mathfrak{A}_\alpha$?; what is ε_0? Let τ be the unique trace on $U(G)$ and let $\check{\tau}$ be the restriction of τ to $\mathfrak{A}$. Then by Theorem 2, $\check{\tau}$ must be uniquely extended to $\mathfrak{A}_\alpha$. Can we conclude $\mathfrak{A}_\alpha = \mathfrak{A}$?

Next we shall investigate a certain class of derivations. Let $\mathfrak{D}_0(\mathfrak{A})$ be the subset of all elements d in $\mathfrak{D}(\mathfrak{A})$ such that $[d, \mathfrak{A}] \subset \mathfrak{A}$—i.e. the d will define a derivation δ on $\mathfrak{A}$; then $\mathfrak{D}_0(\mathfrak{A})$ is a self-adjoint closed linear subspace of $\mathfrak{D}(\mathfrak{A})$ and moreover it is a Lie subalgebra of $\mathfrak{D}(\mathfrak{A})$ with the

Lie product $[x, y] = xy - yx$ $(x, y \in \mathfrak{D}(\mathfrak{A}))$.

Let J_0 be the least closed ideal of $\mathfrak{A}$ such that $[\mathfrak{A}, \mathfrak{A}] \subset J_0$. Then the quotient C^*-algebra $\mathfrak{A}/J_0$ is commutative and so $\delta | \mathfrak{A}/J_0 = 0$ for all $\delta \in D(\mathfrak{A})$, where $\delta | \mathfrak{A}/J_0$ is the derivation on $\mathfrak{A}/J_0$ induced by δ. Now let d be an element of $\mathfrak{D}_0(\mathfrak{A})$; then it is easily seen that there exists the least closed ideal $J(d)$ of $\mathfrak{A}$ such that $[d, \mathfrak{A}] \subset J(d)$. Clearly $J(d) \subset J_0$ and $\delta | \mathfrak{A}/J(d) = 0$, where δ is the derivation on $\mathfrak{A}$ defined by d.

Now we shall show

THEOREM 4. *Let d be a self-adjoint element of $\mathfrak{D}(\mathfrak{A})$. Suppose that d and d^2 belong to $\mathfrak{D}_0(\mathfrak{A})$; then $d \cdot J(d) \subset J(d)$ and $J(d) \cdot d \subset J(d)$.*

PROOF. Let $\mathfrak{G}$ be the C^*-subalgebra of $\mathfrak{D}(\mathfrak{A})$ generated by $J(d)$, d and 1, and let $\mathfrak{K}$ be the C^*-subalgebra of $\mathfrak{G}$ generated by $J(d)$ and 1. Let $\mathfrak{J}$ be the least closed ideal of $\mathfrak{G}$ containing $J(d)$. Let S be the set of all self-adjoint linear functionals f on $\mathfrak{G}$ such that $f(\mathfrak{K}) = 0$. If $J(d) \subsetneqq \mathfrak{J}$, then there is an extreme point g in S such that $g(\mathfrak{J}) \neq (0)$. Let $g = g_1 - g_2$ be the orthogonal decomposition of g and put $\xi = g_1 + g_2$. Let $\{\pi_\xi, \mathfrak{X}_\xi\}$ be the *-representation of $\mathfrak{G}$; then $\overline{\pi_\xi(\mathfrak{K})}$ is a factor (cf. [6], [7]). If $\pi_\xi(J(d)) \neq (0)$, then $\overline{\pi_\xi(J(d))} = \overline{\pi_\xi(\mathfrak{K})}$. Take self-adjoint elements h, k such that $h \in \overline{\pi_\xi(J(d))}$, $k \in \pi_\xi(J(d))'$ and $\pi_\xi(d) = h + k$. Then $\pi_\xi(d^2) = \pi_\xi(d)^2 = h^2 + 2hk + k^2$. Since $d^2 \in \mathfrak{D}_0(\mathfrak{A})$, $[hk, \pi_\xi(J(d))] \subset [h^2, \pi_\xi(J(d))] + [k^2, \pi_\xi(J(d))] + [\pi_\xi(d^2), \pi_\xi(J(d))] \subset \overline{\pi_\xi(J(d))}$. Hence $[hk, \overline{\pi_\xi(J(d))}] \subset \overline{\pi_\xi(J(d))}$.

On the other hand,

$$[hk, x] = [h, x]k \quad \text{for} \quad x \in \overline{\pi_\xi(J(d))} .$$

If $k = \lambda 1_{\mathfrak{X}_\xi}$ for some complex number λ, then $\pi_\xi(d) \in \overline{\pi_\xi(J(d))}$. Hence $\pi_\xi(\mathfrak{G}) \subset \overline{\pi_\xi(J(d))}$. This implies that $g(\mathfrak{G}) = 0$, a contradiction. Let C be the C^*-subalgebra of $B(\mathfrak{X}_\xi)$ generated by k and $1_{\mathfrak{X}_\xi}$; then $\dim(C) \geqq 2$. Since $\overline{\pi_\xi(J(d))}$ is a factor, the C^*-subalgebra R of $B(\mathfrak{X}_\xi)$ generated by $\overline{\pi_\xi(J(d))}$ and C is canonically identified with $\overline{\pi_\xi(J(d))} \otimes C$. Since k and $1_{\mathfrak{X}_\xi}$ are linearly independent, $[h, x]k \in \overline{\pi_\xi(J(d))}$ implies $[h, x] = 0$ for $x \in \overline{\pi_\xi(J(d))}$ and so $h = \lambda 1_{\mathfrak{X}_\xi}$ for some complex number λ. Hence $[\pi_\xi(d), \pi_\xi(J(d))] = 0$. Since $J(d)$ is a closed ideal of $\mathfrak{A}$, the *-representation $\{\pi_\xi, \mathfrak{X}_\xi\}$ of $J(d)$ can be uniquely extended to a *-repesentation $\{\tilde{\pi}_\xi, \mathfrak{X}_\xi\}$ of $\mathfrak{A}$ such that $\tilde{\pi}_\xi(\mathfrak{A}) \subset \overline{\pi_\xi(J(d))}$ (cf. [7]). Let J_1 be the kernel of $\tilde{\pi}_\xi$; then $[\pi_\xi(d), \tilde{\pi}_\xi(\mathfrak{A})] = \tilde{\pi}_\xi([d, \mathfrak{A}]) = 0$ and so $[d, \mathfrak{A}] \subset J_1$. Hence $J_1 \supset J(d)$ and so $\pi_\xi(J(d)) = 0$, a contradiction. Therefore $\pi_\xi(J(d)) = 0$; hence $\pi_\xi(\mathfrak{J}) = 0$ and so $g(\mathfrak{J}) = 0$, a contradiction. This completes the proof.

COROLLARY 1. *Let $\mathfrak{A}$ be a general C^*-algebra with identity on a Hilbert space $\mathfrak{X}$, and let δ be a skew-symmetric derivation on $\mathfrak{A}$. Suppose* *that there exists a self-adjoint element h in $B(\mathfrak{X})$ such that $\delta(a) = [h, a]$ $(a \in \mathfrak{A})$ and $[h^2, \mathfrak{A}] \subset \mathfrak{A}$. Then if $J(h) = \mathfrak{A}$, the δ is an inner derivation.*

This is clear, since we do not use the fact that $\mathfrak{A}$ is factorial in the proof of Theorem 4.

THEOREM 5. *Let $\mathfrak{L}$ be a C^*-subalgebra of $\mathfrak{D}(\mathfrak{A})$ such that $\mathfrak{A} \subset \mathfrak{L} \subset \mathfrak{D}_0(\mathfrak{A})$; then $\mathfrak{L}J_0 \subset J_0$ and $J_0\mathfrak{L} \subset J_0$.*

PROOF. Let $\mathfrak{G}$ be the C^*-subalgebra of $\mathfrak{D}(\mathfrak{A})$ generated by J_0, $\mathfrak{L}$ and 1, and let $\mathfrak{K}$ be the C^*-subalgebra of $\mathfrak{G}$ generated by J_0 and 1. Let S be the set of all self-adjoint linear functionals f on $\mathfrak{G}$ such that $f(\mathfrak{K}) = 0$. Let $\mathfrak{J}$ be the least closed ideal of $\mathfrak{G}$ containing J_0. If $J_0 \subsetneqq \mathfrak{J}$, then there is an extreme point g in S such that $g(\mathfrak{J}) \neq (0)$. Let $g = g_1 - g_2$ be the orthogonal decomposition of g and put $\xi = g_1 + g_2$. Let $\{\pi_\xi, \mathfrak{X}_\xi\}$ be the *-representation of $\mathfrak{G}$; then $\overline{\pi_\xi(\mathfrak{K})}$ is a factor. If $\pi_\xi(J_0) \neq (0)$, then $\overline{\pi_\xi(J_0)} = \overline{\pi_\xi(\mathfrak{K})}$. For each $d \in \mathfrak{L}$, there exist two elements h_d and k_d such that $h_d \in \overline{\pi_\xi(J_0)}$, $k_d \in \pi_\xi(J_0)'$ and $\pi_\xi(d) = h_d + k_d$. Since $\mathfrak{L}$ is a C^*-algebra containing $\mathfrak{A}$, $\pi_\xi(ad) = \pi_\xi(a)h_d + \pi_\xi(a)k_d$ for $a \in \mathfrak{A}$. Since $[\pi_\xi(ad), \pi_\xi(J_0)] \subset \pi_\xi(J_0)$, $[\pi_\xi(a)k_d, \overline{\pi_\xi(J_0)}] \subset \overline{\pi_\xi(J_0)}$.

On the other hand,

$$[\pi_\xi(a)k_d, x] = [\pi_\xi(a), x]k_d \quad \text{for} \quad x \in \overline{\pi_\xi(J_0)} .$$

Suppose that d is self-adjoint, and let C be the C^*-subalgebra of $B(\mathfrak{X}_\xi)$ generated by k_d and $1_{\mathfrak{X}_\xi}$. Let R be the C^*-subalgebra of $B(\mathfrak{X}_\xi)$ generated by $\overline{\pi_\xi(J_0)}$ and C; then $R = \overline{\pi_\xi(J_0)} \otimes C$. If $k_d = \lambda 1_{\mathfrak{X}_\xi}$ for some complex number λ, then $\pi_\xi(d) \in \overline{\pi_\xi(J_0)}$. If $\pi_\xi(d) \in \overline{\pi_\xi(J_0)}$ for all self-adjoint $d \in \mathfrak{L}$, then $\pi_\xi(\mathfrak{G}) \subset \overline{\pi_\xi(J_0)}$ and so $g(\mathfrak{G}) = 0$, a contradiction. Hence there exists a self-adjoint element d in $\mathfrak{L}$ such that $k_d \neq 1_{\mathfrak{X}_\xi}$ for all complex number λ. Then k_d and $1_{\mathfrak{X}_\xi}$ are linearly independent, so that $[\pi_\xi(a), x]k_d \in \overline{\pi_\xi(J_0)}$ for $x \in \overline{\pi_\xi(J_0)}$ implies $[\pi_\xi(a), x] = 0$ for all $a \in \mathfrak{A}$ and $x \in \overline{\pi_\xi(J_0)}$. Therefore $[\pi_\xi(\mathfrak{A}), \pi_\xi(J_0)] = 0$. Since $\overline{\pi_\xi(J_0)}$ contains $1_{\mathfrak{X}_\xi}$ and since $\pi_\xi(\mathfrak{A})\pi_\xi(J_0) = \pi_\xi(\mathfrak{A}J_0) \subset \pi_\xi(J_0)$, $\overline{\pi_\xi(\mathfrak{A})} = \overline{\pi_\xi(J_0)}$. Hence $[\pi_\xi(\mathfrak{A}), \pi_\xi(\mathfrak{A})] = 0$ and so the kernel J_1 of π_ξ in $\mathfrak{A}$ contains $[\mathfrak{A}, \mathfrak{A}]$; hence $J_0 \subset J_1$. Therefore $\pi_\xi(J_0) = 0$ and so $g(\mathfrak{J}) = 0$, a contradiction. Hence $J_0 = \mathfrak{J}$. This completes the proof.

COROLLARY 2. *Suppose that $\mathfrak{A}$ is a factorial C^*-algebra with identity such that the smallest closed ideal of $\mathfrak{A}$ containing $[\mathfrak{A}, \mathfrak{A}]$ is $\mathfrak{A}$. Then if $\mathfrak{D}_0(\mathfrak{A}) = \mathfrak{D}(\mathfrak{A})$, then $\mathfrak{D}(\mathfrak{A}) = \mathfrak{A}$—namely, every derivation of $\mathfrak{A}$ is inner.*

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