Tôhoku Math. Journ.
25 (1973), 177-184.

SPECIAL CONFORMALLY FLAT SPACES AND CANAL HYPERSURFACES

Dedicated to Professor Shigeo Sasaki on his 60th birthday

BANG-YEN CHEN AND KENTARO YANO

(Received May 25, 1972; Revised December 5, 1972)

About forty years ago, three Russian mathematicians (Kagan [1, 2], Rachevsky [4, 5], Shapiro [7, 8]) introduced and studied the so-called subprojective spaces and they obtained many interesting results. In the present paper, the authors would like to introduce the notion of special conformally flat spaces which generalizes that of subprojective spaces.

In § 1, we shall give some formulas and definitions which we use later.[1] In § 2, we shall prove that every conformally flat hypersurface of a euclidean space (hence, of a conformally flat space) is special, and conversely, every special conformally flat space can be isometrically immersed in a euclidean space as a hypersurface. In the last section, we shall prove that every canal hypersurface of a euclidean space is a special conformally flat space and it is a subprojective space if and only if it is a surface of revolution.

1. Preliminaries. Let M_n be an n-dimensional Riemannian space with metric $ds^2 = g_{ji}du^j du^i$; $h, i, j, \cdots = 1, 2, \cdots, n$, where $\{u^h\}$ is a local coordinate system. We denote by $\left\{ {h \atop ji} \right\}$ the Christoffel symbols formed with g_{ji} and by ∇_j the operator of covariant differentiation with respect to $\left\{ {h \atop ji} \right\}$. We denote by $K_{kji}{}^h$ the Riemann-Christoffel curvature tensor of M_n:

$$K_{kji}{}^h = \partial_k \left\{ {h \atop ji} \right\} - \partial_j \left\{ {h \atop ki} \right\} + \left\{ {h \atop kt} \right\} \left\{ {t \atop ji} \right\} - \left\{ {h \atop jt} \right\} \left\{ {t \atop ki} \right\} , \tag{1.1}$$

where $\partial_k = \partial/\partial u^k$. Then the Ricci tensor and the scalar curvature are given respectively by

$$K_{ji} = K_{tji}{}^t \tag{1.2}$$

and

$$K = g^{ji} K_{ji} , \tag{1.3}$$

[1] Manifolds, mappings, functions, $\cdots$ are assumed to be sufficiently differentiable and we shall restrict ourselves only to connected manifolds of dimension $n \geqq 3$.

where g^{ji} are contravariant components of the fundamental metric tensor.

We define a tensor field L_{ji} of type (0, 2) by

$$L_{ji} = -\frac{K_{ji}}{n-2} + \frac{Kg_{ji}}{2(n-1)(n-2)} . \tag{1.4}$$

The conformal curvature tensor $C_{kji}{}^h$ is then given by

$$C_{kji}{}^h = K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^h g_{ji} - L_j{}^h g_{ki} , \tag{1.5}$$

where δ_k^h are the Kronecker deltas and $L_k{}^h = L_{kt}g^{th}$.

A Riemannian manifold M_n is called a *conformally flat space* if we have

$$C_{kji}{}^h = 0 \tag{1.6}$$

and

$$\nabla_k L_{ji} - \nabla_j L_{ki} = 0 . \tag{1.7}$$

It is well-known that (1.6) holds automatically for $n = 3$ and (1.7) can be derived from (1.6) for $n > 3$.

If there exist, on a conformally flat space, two functions α and β such that α is positive and

$$L_{ji} = -\frac{\alpha^2}{2} g_{ji} + \beta(\nabla_j \alpha)(\nabla_i \alpha) , \tag{1.8}$$

then the space M_n is called a *special conformally flat space*. In particular, if β is a function of α, then the special conformally flat space M_n is called a subprojective space. (See, Schouten [6], p. 329.)

Let M_n be a hypersurface of a euclidean $(n+1)$-space E_{n+1} defined by

$$X = X(u^1, u^2, \cdots, u^n) ,$$

where X denotes the position vector of E_{n+1} representing a point of M_n. We put

$$X_i = \partial_i X$$

and denote by N the unit normal vector field along M_n. Then the metric tensor of M_n is given by

$$g_{ji} = X_j \cdot X_i , \tag{1.9}$$

the dot denoting the inner product of vectors in E_{n+1} and the second fundamental tensor $h_j{}^i$ is given by

$$\partial_j N = -h_j{}^i X_i . \tag{1.10}$$

The Gauss equation and the Codazzi equation are then respectively given by

$$K_{kji}{}^{h} = h_k{}^h h_{ji} - h_j{}^h h_{ki} \tag{1.11}$$

and

$$\nabla_k h_{ji} - \nabla_j h_{ki} = 0 \ , \tag{1.12}$$

h_{ji} being covariant components of the second fundamental tensor.

2. Special conformally flat spaces. The main purpose of this section is to prove the following:

THEOREM 1. *Every conformally flat hypersurface in a euclidean space is special. Conversely, every simply connected special conformally flat space can be isometrically immersed in a euclidean space as a hypersurface.*

PROOF. Suppose that M_n is a hypersurface of a euclidean $(n+1)$-space E_{n+1}. Then it has been proved by S. Nishikawa and Y. Maeda [3] that M_n is conformally flat if and only if at each point of M the second fundamental form is one of the following types:

(i) $h = \alpha g$,

(ii) h has two distinct eigenvalues of multiplicity $n-1$ and 1 respectively.

If (i) occurs, then M_n is obviously special. In a neighborhood in which (ii) occurs, there exists a non-zero vector field v_i such that the second fundamental tensor is given in the following form:

$$h_{ji} = \alpha g_{ji} + \beta v_j v_i \ , \tag{2.1}$$

where α and β are functions. From (2.1) we can prove that the Ricci tensor and the scalar curvature are given respectively by

$$K_{ji} = [(n-1)\alpha^2 + \alpha\beta v_t v^t] g_{ji} + (n-2)\alpha\beta v_j v_i \tag{2.2}$$

and

$$K = n(n-1)\alpha^2 + 2(n-1)\alpha\beta v_t v^t \ . \tag{2.3}$$

Therefore, we have

$$L_{ji} = -\frac{\alpha^2}{2} g_{ji} - \alpha\beta v_j v_i \ . \tag{2.4}$$

Combining (2.1) and (2.4), we obtain

$$\alpha h_{ji} = -L_{ji} + \frac{\alpha^2}{2} g_{ji} \ . \tag{2.5}$$

By taking covariant derivative of (2.5) and applying (1.7) and (1.12), we obtain

$$\alpha_k h_{ji} - \alpha_j h_{ki} = \alpha(\alpha_k g_{ji} - \alpha_j g_{ki}) , \tag{2.6}$$

where $\alpha_k = \nabla_k \alpha$.

Substituting (2.1) into (2.6), we obtain

$$\alpha_k v_j - \alpha_j v_k = 0 , \tag{2.7}$$

which yields

$$v_j = f\alpha_j , \tag{2.8}$$

where f is a function on M_n. Substituting (2.8) into (2.4) we see that the conformally flat space M_n is special. This proves the first part of the theorem.

Conversely, suppose that M_n is a simply connected special conformally flat space with

$$L_{ji} = -\frac{\alpha^2}{2} g_{ji} + \beta(\nabla_j \alpha)(\nabla_i \alpha) , \tag{2.9}$$

α and β being defined globally.

We define a covariant tensor h_{ji} of order 2 by

$$h_{ji} = \frac{\alpha}{2} g_{ji} - \frac{1}{\alpha} L_{ji} . \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$h_{ji} = \alpha g_{ji} - \frac{\beta}{\alpha}(\nabla_j \alpha)(\nabla_i \alpha) . \tag{2.11}$$

Taking covariant derivative of (2.10) and applying (1.7) and (2.9), we easily obtain

$$\nabla_k h_{ji} - \nabla_j h_{ki} = 0 . \tag{2.12}$$

This shows that h_{ji} satisfy the Codazzi equations. On the other hand, by using formulas (1.5), (1.6), (2.9), (2.11) and a straightforward computation, we can prove that

$$K_{kji}{}^h = h_k{}^h h_{ji} - h_j{}^h h_{ki} . \tag{2.13}$$

Thus h_{ji} satisfy also the Gauss equations. Therefore, by the fundamental theorem of differential geometry, we see that the space M_n can be isometrically immersed in a euclidean space as a hypersurface. This completes the proof of the theorem.

As a consequence of Theorem 1, we obtain

COROLLARY 1. *Every simply connected subprojective space can be isometrically immersed in a euclidean space as a hypersurface.*

3. Canal hypersurfaces. The envelope of one-parameter family of hyperspheres of a euclidean space E_{n+1} is called a *canal hypersurface.*

In the following, we suppose M_n to be a canal hypersurface given as the envelope of the hyperspheres

$$(X - x(s))\cdot(X - x(s)) = r(s)^2\,, \qquad r(s) > 0\,, \tag{3.1}$$

where $x(s)$ and $r(s)$ are respectively centers and radii of the hyperspheres. Then the canal hypersurface M_n is given by (3.1) and

$$(X - x(s))\cdot x'(s) = -rr'\,, \tag{3.2}$$

where $x' = dx/ds$ and $r' = dr/ds$. Without loss of generality, we may assume that the canal hypersurface M_n is also given by a vector-valued function

$$X = X(u^1, u^2, \cdots, u^{n-1}, s) \tag{3.3}$$

satisfying (3.1) and (3.2), where $\{u^1, u^2, \cdots, u^{n-1}, u^n = s\}$ is a local coordinate system on M_n. Then, taking partial derivative of (3.1), we obtain

$$X_b\cdot(X - x(s)) = 0\,, \qquad a, b, c, d, \cdots = 1, 2, \cdots, n-1\,, \tag{3.4}$$

and

$$X_n\cdot(X - x(s)) = 0\,, \tag{3.5}$$

by virtue of (3.2), where $X_b = \partial X/\partial u^b$ and $X_n = \partial X/\partial s$. From (3.4) and (3.5), we see that the unit normal N to the canal hypersurface is parallel to $X - x(s)$. Thus we may write

$$X = x(s) - r(s)N(u^1, u^2, \cdots, u^{n-1}, s)\,. \tag{3.6}$$

By taking partial derivative of (3.2) with respect to u^b, we have

$$X_b\cdot x'(s) = 0\,. \tag{3.7}$$

Since N is a unit normal vector field, we have Weingarten equations (1.10), from which

$$\partial_b N = -h_b{}^a X_a - h_b{}^n X_n \tag{3.8}$$

and

$$\partial_n N = -h_n{}^a X_a - h_n{}^n X_n\,. \tag{3.9}$$

From (3.6) and (3.8), we find

$$X_b = rh_b{}^a X_a + rh_b{}^n X_n\,, \tag{3.10}$$

from which

$$h_b{}^a = \frac{1}{r}\delta_b^a\,, \qquad h_b{}^n = 0\,. \tag{3.11}$$

Also, from (3.6) and (3.9), we have

$$X_n = x' - r'N + rh_n{}^a X_a + rh_n{}^n X_n\,. \tag{3.12}$$

Thus (3.7) and (3.12) imply

$$\begin{aligned} g_{bn} &= rh_n{}^a g_{ba} + rh_n{}^n g_{bn} \\ &= rh_n{}^i g_{ib} = rh_{bn}\,. \end{aligned} \tag{3.13}$$

Thus, from (3.11) and (3.13), we obtain

$$h_{ci} = \frac{1}{r} g_{ci}\,. \tag{3.14}$$

Thus, if we put

$$\bar\alpha = \frac{1}{r}\,, \qquad \bar\beta = h_{nn} - \frac{1}{r} g_{nn}\,, \tag{3.15}$$

then we obtain

$$h_{ji} = \bar\alpha g_{ji} + \bar\beta(\nabla_j s)(\nabla_i s)\,, \tag{3.16}$$

where $\bar\alpha$ is a function of s.

Thus, from Gauss equations (1.11) and (3.16), we find by a direct computation that

$$\begin{aligned} K_{kji}{}^h = {} & \bar\alpha^2(\delta_k^h g_{ji} - \delta_j^h g_{ki}) + \bar\alpha\bar\beta\{\delta_k^h(\nabla_j s) - \delta_j^h(\nabla_k s)\}(\nabla_i s) \\ & + \bar\alpha\bar\beta\{(\nabla_k s) g_{ji} - (\nabla_j s) g_{ki}\}(\nabla^h s)\,, \end{aligned} \tag{3.17}$$

where $\nabla^h s = (\nabla_i s) g^{ih}$,

$$\begin{aligned} K_{ji} = {} & \{(n-1)\bar\alpha^2 + \bar\alpha\bar\beta(\nabla_l s)(\nabla^l s)\} g_{ji} \\ & + (n-2)\bar\alpha\bar\beta(\nabla_j s)(\nabla_i s)\,, \end{aligned} \tag{3.18}$$

$$K = n(n-1)\bar\alpha^2 + 2(n-1)\bar\alpha\bar\beta(\nabla_l s)(\nabla^l s) \tag{3.19}$$

and

$$L_{ji} = -\frac{1}{2}\bar\alpha^2 g_{ji} - \bar\alpha\bar\beta(\nabla_j s)(\nabla_i s)\,. \tag{3.20}$$

Thus, by substituting (3.17) and (3.20) into (1.5), we can easily find that the conformal curvature tensor $C_{kji}{}^h$ vanishes. On the other hand, from (3.16) and (3.20), we have

(3.21) $$L_{ji} = \frac{\bar{\alpha}^2}{2} g_{ji} - \bar{\alpha} h_{ji} .$$

Hence, by the Codazzi equations, we easily find that

$$\nabla_k L_{ji} - \nabla_j L_{ki} = 0 .$$

Consequently, we have proved the following:

THEOREM 2. *Every canal hypersurface of a euclidean space, singularities excluded, is a special conformally flat space.*

REMARK. If the canal hypersurface is subprojective for the induced conformally flat structure, i.e., if the function

$$\bar{\beta} = h_{nn} - \frac{1}{r} g_{nn}$$

is a function of s, then the canal hypersurface is a surface of revolution, i.e., the locus of centers $x(s)$ is a straight line. The proof is as follows:

From (3.12), we have

$$g_{nn} = X_n \cdot x' + r h_{nn} ,$$

that is,

(3.22) $$X_n \cdot x' = -r\bar{\beta}$$

and

(3.23) $$N \cdot x' = r' .$$

From (3.7), (3.22), and (3.23), we find

(3.24) $$x' = -r\bar{\beta} g^{ni} X_i + r' N .$$

If we choose s as the arc length of the locus of centers $x(s)$, then by taking partial derivative of (3.2) with respect to s, we obtain

$$(X - x) \cdot x'' = -r'^2 - r r'' + 1 + r\bar{\beta}$$

by virtue of (3.22). Therefore, $(X - x) \cdot x'$ and $(X - x) \cdot x''$ are both functions of s only. Hence, for a fixed s, we have $(X - x) \cdot x' =$ constant and $(X - x) \cdot x'' =$ constant. Since, for a fixed s, $x(s) =$ constant and X defines an $(n-1)$-sphere in E_{n+1}, by the constancy of $(X - x) \cdot x'$ and $(X - x) \cdot x''$, we see that x' and x'' are parallel. This implies that $x'' = 0$. Hence the canal hypersurface is a surface of revolution.

REFERENCES

[1] B. KAGAN, Sur les espaces sous-projectifs, C. R., 191 (1930), 548-550.
[2] ———, Über eine Erweiterung des Begriffes vom projektiven Räume und dem zugehörigen Absolut, Abh. des Sem. für Vektor und Tensor analysis, Moskau, 1 (1933), 12-101.
[3] S. NISHIKAWA AND Y. MAEDA, Conformally flat hypersurfaces in a conformally flat Riemannian manifold, to appear.
[4] P. RACHEVSKY, Sur les espaces sous-projectifs, C.R., 191 (1930), 547-548.
[5] ———, Caractères tensoriels de l'espace sous-projectif, Abh. des Sem. für Vektor und Tensor analysis, Moskau, 1 (1933), 126-140.
[6] J. A. SCHOUTEN, Ricci-Calculus, Springer-Verlag, Berlin, 1954.
[7] H. SHAPIRO, Sur les espaces sous-projectifs, C.R., 191 (1930), 551-552.
[8] ———, Über die Metrik der subprojektiven Räume, Abh. des Sem. für Vektor und Tensor analysis, Moskau, 1 (1933), 102-124.

MICHIGAN STATE UNIVERSITY