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ON THE MEAN CURVATURE FOR HOLOMORPHIC 2p-PLANE IN KÄHLERIAN SPACES

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Introduction. Let M^n be an n dimensional Riemannian space, and denote by $\rho(X, Y)$ the sectional curvature of a 2-plane spanned by vectors X and Y. For a q-plane π at a point P, we take an orthonormal base $\{e_\lambda\}$ of the tangent space $T_p(M)$ such that $e_1, \cdots, e_q$ span π. Such a base is called an adapted base for π. The mean curvature $\rho(\pi)$ for π is defined by

$$\rho(\pi) = \frac{1}{q(n-q)} \sum_{a=q+1}^{n} \sum_{i=1}^{q} \rho(e_i, e_a)$$

and independent of the choice of adapted bases for π.

In a recent paper [3], we have proved the following

THEOREM. *In an $n\,(>2)$ dimensional Riemannian space M^n, if the mean curvature for q-plane is independent of the q-plane at each point, then*

(i) *M^n is an Einstein space, for $q = 1$ or $n - 1$,*

(ii) *M^n is of constant curvature, for $n - 1 > q > 1$ and $2q \neq n$,*

(iii) *M^n is conformally flat, for $n - 1 > q > 1$ and $2q = n$.*

The converse is also true.

The purpose of this paper is to prove an analogous theorem in Kählerian spaces taking holomorphic $2p$-plane in place of q-plane in the above theorem.

1. Preliminaries.

In [3], the following has been proved.

LEMMA A. *Let $A = (a_{ij})$ be an $m \times m$ symmetric matrix whose diagonal elements are all zero. If $1 < p < m - 1$ and A satisfies*

$$\sum_{h,k=1}^{p} a_{i_h i_k} = 0$$

for any $i_1 < \cdots < i_p$ taken from $\{1, \cdots, m\}$, then A is the zero matrix.

We shall generalize this lemma as follows:

LEMMA 1.1. *Let $B = (b_{ij})$ be an $m \times m$ symmetric matrix. If $1 <$*

$p < m-1$ and B satisfies

$$\sum_{h,k=1}^{p} b_{i_h i_k} = 0$$

for any $i_1 < \cdots < i_p$ taken from $\{1, \cdots, m\}$, then B satisfies

$$b_{ij} = -\frac{1}{2(p-1)}(b_{ii} + b_{jj}) , \qquad i \neq j .$$

PROOF. The matrix $A = (a_{ij})$ defined by

$$a_{ii} = 0 ,$$

$$a_{ij} = b_{ij} + \frac{1}{2(p-1)}(b_{ii} + b_{jj}) , \qquad i \neq j ,$$

satisfies all the conditions in Lemma A, which proves our lemma.

Let M^n be an n dimensional Riemannian space with positive definite metric $g_{\lambda\mu}$. We shall denote by $R_{\lambda\mu\nu}{}^{\kappa}$, $R_{\mu\nu} = R_{\lambda\mu\nu}{}^{\lambda}$ and R the Riemannian curvature tensor, the Ricci tensor and the scalar curvature respectively.[1] Putting $R_{\lambda\mu\nu\omega} = g_{\omega\kappa} R_{\lambda\mu\nu}{}^{\kappa}$, we shall denote by $\hat{R}$ the tensor $(R_{\lambda\mu\nu\omega})$.

Let X and Y be orthonormal vectors at a point P of M^n, and the sectional curvature for the 2-plane spanned by them is given by

$$\rho(X, Y) = - R_{\lambda\mu\nu\omega} \xi^{\lambda} \eta^{\mu} \xi^{\nu} \eta^{\omega} = - \hat{R}(X, Y, X, Y) ,$$

where ξ^{λ} (resp. η^{μ}) denotes the components of X (resp. Y) with respect to the natural base.

Consider a q-plane π in the tangent space $T_p(M)$ and an adapted base $\{e_{\lambda}\}$ for π. Let ξ^{λ}_i, $i = 1, \cdots, q$, be components of e_{λ} with respect to a natural base. We consider the simple q-vector determined by $e_1, \cdots, e_q$ at P and denote it by π again. Its components with respect to the natural base are determined within sign and are given by

$$\pi^{\lambda_1 \cdots \lambda_q} = \begin{vmatrix} \xi^{\lambda_1}_1 & \cdots & \xi^{\lambda_1}_q \\ \vdots & & \vdots \\ \xi^{\lambda_q}_1 & \cdots & \xi^{\lambda_q}_q \end{vmatrix} .$$

The ambiguity of signs does not matter in the following discussions.

With respect to the adapted base $\{e_{\lambda}\}$ in consideration, the components $\pi^{\lambda_1 \cdots \lambda_q} = \pi_{\lambda_1 \cdots \lambda_q}$ turn to

[1] Tensors are written in terms of their components with respect to a natural base and the Greek indices run from 1 to n, if not otherwise stated. The summation convention will be assumed for these indices when they appear in components of tensor with respect to natural bases. When we consider a tensor with respect to orthonormal bases, its components are written with lower indices only and Σ is not omitted.

$$\pi_{\lambda_1\cdots\lambda_q} = \begin{cases} \operatorname{sign}(\lambda_1, \cdots, \lambda_q) , & \text{if } (\lambda_1, \cdots, \lambda_q) \text{ is a permutation of } \{1, \cdots, q\} , \\ 0 , & \text{other cases} , \end{cases}$$

because of $\xi_i^\lambda = \delta_{i\lambda}$. Therefore we have

$$(1.1) \qquad \sum_{\rho_3, \cdots, \rho_q = 1}^{n} \pi_{\lambda\mu\rho_3\cdots\rho_q}\pi_{\nu\omega\rho_3\cdots\rho_q} = \begin{cases} (q-2)!\,(\delta_{\lambda\nu}\delta_{\mu\omega} - \delta_{\lambda\omega}\delta_{\mu\nu}) , & \text{if } \lambda, \mu, \nu, \omega = 1, \cdots, q , \\ 0 , & \text{other cases} , \end{cases}$$

with respect to the adapted base.

Let $\hat{L} = (L_{\lambda\mu\nu\omega})$ be the tensor given by

$$L_{\lambda\mu\nu\omega} = 2(q-1)R_{\lambda\mu\nu\omega} + R_{\lambda\nu}g_{\mu\omega} - R_{\lambda\omega}g_{\mu\nu} + g_{\lambda\nu}R_{\mu\omega} - g_{\lambda\omega}R_{\mu\nu}$$

for $1 < q < n$, and consider a quadratic form $L_q(u)$ of skew symmetric tensor $u_{\lambda_1\cdots\lambda_q}$:

$$L_q(u) = L_{\lambda\mu\nu\omega}u^{\lambda\mu\rho_3\cdots\rho_q}u^{\nu\omega}{}_{\rho_3\cdots\rho_q} .$$

This form appears, for example, in the following theorem.[2)]

If $L_q(u)$ is positive definite in a compact Riemannian space, there exists no harmonic q-form other than the zero form.

A geometrical meaning of $\hat{L}$ is given in terms of the mean curvature for q-plane as

$$\rho(\pi) = \frac{1}{4(n-q)q!}L_q(\pi) ,$$

where π in $L_q(\pi)$ means the simple q-vector, [3].

2. K-curvature-like tensor. Hereafter we shall consider a Kählerian space K^{2m} of complex dimension m (>1). K^{2m} is a $2m$ ($=n$) dimensional Riemannian space admitting a parallel tensor field $J = (\varphi_\lambda^\mu)$ such that

$$\varphi_\lambda^\alpha\varphi_\alpha^\mu = -\delta_\lambda^\mu , \qquad \varphi_{\lambda\mu}(=\varphi_\lambda^\alpha g_{\alpha\mu}) = -\varphi_{\mu\lambda} .$$

A tensor $\hat{U} = (U_{\lambda\mu\nu\omega})$ of type (0, 4) will be called K-curvature-like, if it satisfies

$$(2.1) \qquad U_{\lambda\mu\nu\omega} = -U_{\mu\lambda\nu\omega} = -U_{\lambda\mu\omega\nu} ,$$

$$(2.2) \qquad U_{\lambda\mu\nu\omega} + U_{\mu\nu\lambda\omega} + U_{\nu\lambda\mu\omega} = 0 ,$$

$$(2.3) \qquad U_{\lambda\mu\nu\alpha}\varphi_\omega^\alpha = -U_{\lambda\mu\alpha\omega}\varphi_\nu^\alpha .$$

As is well known,

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2) Yano and Bochner [4], p. 64 and Mogi [1].

$$U_{\lambda\mu\nu\omega} = U_{\nu\omega\lambda\mu}\ , \qquad \varphi_\lambda^\alpha U_{\alpha\mu\nu\omega} = -\varphi_\mu^\alpha U_{\lambda\alpha\nu\omega}$$

hold good, and (2.3) means that $\hat{U}$ is hybrid with respect to the last two indices.

The Riemannian curvature tensor $\hat{R} = (R_{\lambda\mu\nu\omega})$ of K^{2m} is an example of K-curvature-like tensor. Other examples are given by the following $\hat{Q}$ and $\hat{T}$:

$$\begin{aligned} Q_{\lambda\mu\nu\omega} &= g_{\lambda\omega}R_{\mu\nu} - g_{\mu\omega}R_{\lambda\nu} + R_{\lambda\omega}g_{\mu\nu} - R_{\mu\omega}g_{\lambda\nu} \\ &\quad + \varphi_{\lambda\omega}S_{\mu\nu} - \varphi_{\mu\omega}S_{\lambda\nu} + S_{\lambda\omega}\varphi_{\mu\nu} - S_{\mu\omega}\varphi_{\lambda\nu} - 2\varphi_{\lambda\mu}S_{\nu\omega} - 2S_{\lambda\mu}\varphi_{\nu\omega}\ , \\ T_{\lambda\mu\nu\omega} &= g_{\lambda\omega}g_{\mu\nu} - g_{\mu\omega}g_{\lambda\nu} + \varphi_{\lambda\omega}\varphi_{\mu\nu} - \varphi_{\mu\omega}\varphi_{\lambda\nu} - 2\varphi_{\lambda\mu}\varphi_{\nu\omega}\ , \end{aligned}$$

where $S_{\lambda\mu}$ is a skew symmetric tensor defined by

(2.4) $$S_{\lambda\mu} = \varphi_\lambda^\alpha R_{\alpha\mu}\ ,$$

and satisfies $\varphi_\lambda^\alpha S_{\alpha\mu} = -R_{\lambda\mu}$.

$\hat{Q}$ and $\hat{T}$ satisfy

(2.5) $$g^{\lambda\omega}Q_{\lambda\mu\nu\omega} = 2(m+2)R_{\mu\nu} + Rg_{\mu\nu}\ ,$$

(2.6) $$g^{\lambda\omega}T_{\lambda\mu\nu\omega} = 2(m+1)g_{\mu\nu}\ .$$

Let X be a unit vector at a point P. A 2-plane spanned by X and JX is called a holomorphic 2-plane, and the sectional curvature $\rho(X, JX)$ of such a plane is called a holomorphic sectional curvature.

If the holomorphic sectional curvature of a Kählerian space K^{2m} has a value independent of the holomorphic 2-plane at each point, the space is called a space of constant holomorphic curvature. Such a space satisfies

(2.7) $$\hat{R} = \alpha\hat{T}$$

for a scalar function α, and the converse is also true, [4], [5].

As (2.7) is proved algebraically by means of (2.1)~(2.3), we have the following

LEMMA 2.1. *If $\hat{U}$ is a K-curvature-like tensor and $\hat{U}(X, JX, X, JX)$ is independent of the unit vector X at a point P, then*

$$\hat{U} = \alpha\hat{T}$$

holds good at P, where α is a scalar.

Let $K_{\lambda\mu\nu}{}^\kappa$ be the Bochner curvature tensor [2], then $\hat{K} = (K_{\lambda\mu\nu\omega})$ is a K-curvature-like tensor given by

$$\hat{K} = \hat{R} - \frac{1}{2(m+2)}\hat{Q} + \frac{R}{4(m+1)(m+2)}\hat{T}\ .$$

Now we shall introduce a K-curvature-like tensor $\hat{M}$ by

$$\hat{M} = 4(p+1)\hat{R} - \hat{Q}\ , \qquad (p\text{: an integer})\ ,$$

which will play a leading role in this paper.

LEMMA 2.2. *In a Kählerian space K^{2m} the equation*

$$\hat{M} = \alpha \hat{T}$$

is valid for a scalar function α, if and only if

(i) *for $2p \neq m$, K^{2m} is a space of constant holomorphic curvature,*

(ii) *for $2p = m$, the Bochner curvature tensor vanishes identically.*

For both cases, the value of α is given by

$$\alpha = -\frac{m-p}{m(m+1)}R\ ,$$

and hence α is constant for the case (i).

PROOF. Let us assume that $\hat{M} = \alpha\hat{T}$, then

$$4(p+1)R_{\lambda\mu\nu\omega} - Q_{\lambda\mu\nu\omega} = \alpha T_{\lambda\mu\nu\omega} \tag{2.8}$$

holds. Transvecting $g^{\lambda\omega}$ with (2.8) and taking account of (2.5) and (2.6) we have

$$2(2p-m)R_{\mu\nu} = \{R + 2(m+1)\alpha\}g_{\mu\nu}\ . \tag{2.9}$$

Thus, if $2p \neq m$, we have

$$R_{\mu\nu} = \frac{R + 2(m+1)\alpha}{2(2p-m)}g_{\mu\nu} \tag{2.10}$$

which means that K^{2m} is an Einstein space. Hence it holds that

$$R_{\mu\nu} = \frac{R}{2m}g_{\mu\nu}\ . \tag{2.11}$$

Substituting (2.11) into (2.8), we know K^{2m} to be of constant holomorphic curvature. The value of α is obtained from (2.10) and (2.11). If $2p = m$, we have from (2.9)

$$\alpha = -\frac{R}{2(m+1)}$$

and eliminating α in (2.8) by the last equation we get the case (ii). The converse is almost trivial. q.e.d.

3. ***J*-base.** Let P be an arbitrary point of a Kählerian space K^{2m} and consider an orthonormal base $\{e_\lambda\}$ of the tangent space $T_p(K^{2m})$ such that

$$e_{i^*} = Je_i\ , \qquad i = 1, \cdots, m\ ; \qquad i^* = i + m\ .$$

Such a base will be called a J-base.

In this section, components of all tensors are given with respect to J-bases and so appear with lower indices only.

Taking account of $g_{\lambda\mu} = \delta_{\lambda\mu}$ and $e_\lambda = (\delta_{\lambda\mu})$, we have

$$\begin{cases} \varphi_{ii^*} = -\varphi_{i^*i} = 1\ , \\ \varphi_{i\lambda} = 0 \ \text{ for } \lambda \neq i^*\ . \end{cases} \tag{3.1}$$

The Ricci tensor $R_{\lambda\mu}$ and $S_{\lambda\mu}$ in (2.4) satisfy

$$R_{ij} = R_{i^*j^*}\ , \qquad R_{ij^*} = -R_{i^*j}\ , \tag{3.2}$$

$$S_{ij} = S_{i^*j^*} = R_{i^*j}\ , \quad S_{ij^*} = -S_{i^*j} = R_{ij}\ . \tag{3.3}$$

From the hybrid property of a K-curvature-like tensor $\hat{U}$, we know that its components $U_{\lambda\mu\nu\omega} = \hat{U}(e_\lambda, e_\mu, e_\nu, e_\omega)$ satisfy

$$U_{ij\nu\omega} = U_{i^*j^*\nu\omega}\ , \qquad U_{ij^*\nu\omega} = -U_{i^*j\nu\omega}\ . \tag{3.4}$$

Taking account of (2.1), (2.2) and (3.4) we can get

$$U_{ijij} + U_{ij^*ij^*} = U_{ii^*jj^*}\ . \tag{3.5}$$

The sectional curvatures $\rho(e_\lambda, e_\mu) = -\hat{R}(e_\lambda, e_\mu, e_\lambda, e_\mu)$ satisfy by virtue of (3.4)

$$\rho(e_i, e_j) = \rho(e_{i^*}, e_{j^*})\ , \qquad \rho(e_i, e_{j^*}) = \rho(e_{i^*}, e_j)\ . \tag{3.6}$$

LEMMA 3.1. *Let $\hat{U}$ be a K-curvature-like tensor at a point P in a Kählerian space K_{2m}, $m \geqq 3$. If $\hat{U}$ satisfies*

$$U_{ii^*jj^*} = a(U_{ii^*ii^*} + U_{jj^*jj^*}) + b\ , \qquad i \neq j\ , \tag{3.7}$$

$(i, j = 1, \cdots, m)$ for any J-base, then

$$\hat{U} = \alpha\hat{T}$$

holds good for a scalar α, where a and b are scalars independent of J-bases and $a \neq 0, 1/4$.

PROOF. Let $\{e_\lambda\}$ be a J-base at a point P and i, j, k be different from one another. If we put

$$e_{j'} = \frac{1}{\sqrt{2}}(e_j + e_k)\ , \qquad e_{k'} = \frac{1}{\sqrt{2}}(e_j - e_k)\ ,$$

the following equations hold good:

$$(ii^*j'j'^*) = a\{(ii^*) + (j'j'^*)\} + b\ , \tag{3.8}$$

where we have put $(\lambda\mu\nu\omega) = \hat{U}(e_\lambda, e_\mu, e_\nu, e_\omega)$ and $(ii^*) = (ii^*ii^*)$ for simplicity.

On the other hand, we have

$$\begin{aligned} 2(ii^*j'j'^*) &= \hat{U}(e_i, e_{i^*}, e_j + e_k, e_{j^*} + e_{k^*}) \\ &= (ii^*jj^*) + 2(ii^*jk^*) + (ii^*kk^*) \\ &= 2a(ii^*) + a\{(jj^*) + (kk^*)\} + 2b + 2(ii^*jk^*)\ , \\ 4(j'j'^*) &= \hat{U}(e_j + e_k, e_{j^*} + e_{k^*}, e_j + e_k, e_{j^*} + e_{k^*}) \\ &= (jj^*) + (kk^*) + 2(jj^*kk^*) \\ &\qquad + 4\{(jk^*jk^*) + (jj^*jk^*) + (kk^*kj^*)\} \\ &= (2a + 1)\{(jj^*) + (kk^*)\} + 2b \\ &\qquad + 4\{(jk^*jk^*) + (jj^*jk^*) + (kk^*kj^*)\}\ . \end{aligned}$$

Substituting these equations into (3.8) we get

$$\begin{aligned} 4(ii^*jk^*) &= a(2a - 1)\{(jj^*) + (kk^*)\} + 2ab \\ &\qquad + 4a\{(jk^*jk^*) + (jj^*jk^*) + (kk^*kj^*)\}\ . \end{aligned} \tag{3.9}$$

If we replace e_k in (3.9) by $-e_k$, it follows that

$$\begin{aligned} -4(ii^*jk^*) &= a(2a - 1)\{(jj^*) + (kk^*)\} + 2ab \\ &\qquad + 4a\{(jk^*jk^*) - (jj^*jk^*) - (kk^*kj^*)\}\ . \end{aligned}$$

Adding the last equation to (3.9) side by side, we obtain

$$(2a - 1)\{(jj^*) + (kk^*)\} + 2b + 4(jk^*jk^*) = 0\ . \tag{3.10}$$

Now let us notice that (3.7) is valid for $i, j = 1, \cdots, m, 1^*, \cdots, m^*$ $(i \neq j)$. Then we may replace k, k^* in (3.10) by $k^*, -k$ respectively. Thus we have

$$(2a - 1)\{(jj^*) + (kk^*)\} + 2b + 4(jkjk) = 0\ .$$

Adding this equation to (3.10) and taking account of (3.5) and (3.7) we get

$$(4a - 1)\{(jj^*) + (kk^*)\} + 4b = 0\ .$$

Hence

$$(jj^*) = -\frac{4b}{4a - 1} - (kk^*)$$

which implies for $m \geqq 3$ that

$$(jj^*) = -\frac{2b}{4a - 1}\ , \qquad j = 1, \cdots, m\ .$$

Thus the proof is completed on taking account of Lemma 2.1. q.e.d.

4. Geometrical interpretation of $\hat{M}$. Consider a $2p$-plane π at a point P of a Kählerian space K^{2m}. If we can find p vectors $X_1, \cdots, X_p$ such that $X_1, \cdots, X_p, JX_p, \cdots, JX_1$ span π, then π is called holomorphic. In this section, π will always mean a holomorphic $2p$-plane. A J-base $\{e_\lambda\}$ of $T_p(K^{2m})$ will be said an adapted J-base for π, if π is spanned by $e_1, \cdots, e_p, e_{1^*}, \cdots, e_{p^*}$.

Now we shall calculate

$$M_p(\pi) = M_{\lambda\mu\nu\omega}\pi^{\lambda\mu\rho_3\cdots\rho_{2p}}\pi^{\nu\omega}{}_{\rho_3\cdots\rho_{2p}}$$

for the simple $2p$-vector π.

The right hand side of $M_p(\pi)$ being a tensor equation, it is sufficient for the calculation to do with respect to an adapted J-base $\{e_\lambda\}$.

As (1.1) is still valid relative to $\{e_\lambda\}$, we have

$$\begin{aligned} 4(p+1)R_{\lambda\mu\nu\omega}\pi^{\lambda\mu\rho_3\cdots\rho_{2p}}\pi^{\nu\omega}{}_{\rho_3\cdots\rho_{2p}} &= 8(p+1)(2p-2)!\sum R_{\lambda\mu\lambda\mu}{}^{3)} \\ &= -16(p+1)(2p-2)!\sum_{i,j=1}^{p}\{\rho(e_i, e_j)+\rho(e_i, e_{j^*})\}\ , \\ Q_{\lambda\mu\nu\omega}\pi^{\lambda\mu\rho_3\cdots\rho_{2p}}\pi^{\nu\omega}{}_{\rho_3\cdots\rho_{2p}} &= -16(p+1)(2p-2)!\sum_{i=1}^{p}R_{ii}\ , \end{aligned}$$

where we have used (3.1), (3.2), (3.3) and (3.6). Thus we get

$$M_p(\pi) = 16(p+1)(2p-2)!\sum_{a=p+1}^{m}\sum_{i=1}^{p}\{\rho(e_i, e_a)+\rho(e_i, e_{a^*})\}\ .$$

On the other hand, the mean curvature for π is

$$\rho(\pi) = \frac{1}{2p(m-p)}\sum_{a=p+1}^{m}\sum_{i=1}^{p}\{\rho(e_i, e_a)+\rho(e_i, e_{a^*})\}$$

because of its definition and (3.6). Hence we obtain

$$M_p(\pi) = 32(m-p)(p+1)p(2p-2)!\,\rho(\pi) \tag{4.1}$$

which gives $\hat{M}$ a geometrical meaning.

5. A generalization of F. Schur's theorem. We shall determine, in this section, Kählerian spaces in which the mean curvature $\rho(\pi)$ for $2p$-plane is independent of the holomorphic $2p$-plane π. This will be done by making use of (4.1).

Let us assume that $\rho(\pi)$ takes a value—say $p\beta/8(m-p)(p+1)$—which depends only on the point. Then from (4.1) it follows that

$$M_{\lambda\mu\nu\omega}\pi^{\lambda\mu\rho_3\cdots\rho_{2p}}\pi^{\nu\omega}{}_{\rho_3\cdots\rho_{2p}} = 4p^2(2p-2)!\,\beta\ . \tag{5.1}$$

Let $\{e_\lambda\}$ be a J-base at a point P and π the holomorphic $2p$-plane spanned by $e_1, \cdots, e_p, e_{1^*}, \cdots, e_{p^*}$. With respect to $\{e_\lambda\}$, (5.1) becomes

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3) Here, Σ means the sum for $\lambda, \mu = 1, \cdots, p, 1^*, \cdots, p^*$.

$$\sum M_{\lambda\mu\nu\omega}(\delta_{\lambda\nu}\delta_{\mu\omega} - \delta_{\lambda\omega}\delta_{\mu\nu}) = 4p^2\beta \ ,$$

where $\sum$ is taken over $1, \cdots, p, 1^*, \cdots, p^*$. Then we have

$$\sum_{i,j=1}^{p} (M_{ijij} + M_{ij^*ij^*}) = p^2\beta \tag{5.2}$$

by virtue of (3.4). By the assumption the analogous equations to (5.2) are valid for any $2p$-plane spanned by $e_{i_1}, \cdots, e_{i_p}, e_{i_1^*}, \cdots, e_{i_p^*}$. Now let

$$b_{ij} = M_{ijij} + M_{ij^*ij^*} - \beta = M_{ii^*jj^*} - \beta \ , \qquad i \neq j \ ,$$
$$b_{ii} = M_{ii^*ii^*} - \beta \ ,$$

then the $m \times m$ symmetric matrix $B = (b_{ij})$ satisfies all the conditions in Lemma 1.1. Hence we have

$$M_{ii^*jj^*} = -\frac{1}{2(p-1)}(M_{ii^*ii^*} + M_{jj^*jj^*}) + \frac{p\beta}{p-1} \ .$$

Thus it follows by Lemma 3.1 that

$$\hat{M} = \alpha\hat{T}$$

for a scalar α, if $m \geqq 3$. Consequently, by Lemma 2.2 we get the following theorem. Its converse part is obtained by straight-forward calculations making use of (4.1).

THEOREM. *In a Kählerian space K^{2m} ($m \geqq 4$), if the mean curvature for $2p$-plane is independent of the holomorphic $2p$-plane at each point, then*

(i) *K^{2m} is a space of constant holomorphic curvature, for $1 < p < m-1$ and $2p \neq m$,*

(ii) *the Bochner curvature tensor of K^{2m} vanishes identically, for $1 < p < m-1$ and $2p = m$.*

The converse is also true.

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