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ON SASAKIAN SUBMANIFOLDS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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1. Introduction. In his recent papers [4], [5], and [6], K. Ogiue studied positively curved submanifolds of a complex projective space. The purpose of this paper is to study similar problems for submanifolds of a Sasakian space form.

Let M be a $(2n+1)$-dimensional Sasakian manifold with the structure tensors ϕ, ξ, η, and g. Then we have

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad \phi^2 = -I + \xi \otimes \eta ,$$

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) ,$$

$$d\eta(X, Y) = g(\phi X, Y) ,$$

$$\phi X = \nabla_X \xi, \quad (\nabla_X \phi) Y = \eta(Y)X - g(X, Y)\xi .$$

By a ϕ-holomorphic sectional curvature $H(X)$ of M with respect to a unit vector X orthogonal to ξ, we mean the sectional curvature $K(X, \phi X)$ spanned by the vectors X and ϕX.

A sasakian space form is, by definition, a connected and complete Sasakian manifold of constant ϕ-holomorphic sectional curvature C.

It is known that there are three types of simply connected Sasakian space forms:

1) Elliptic Sasakian space form: $C > -3$; (homeomorphic to a sphere),

2) Parabolic Sasakian space form: $C = -3$; (homeomorphic to a Euclidian space),

3) Hyperbolic Sasakian space form: $C < -3$; (homeomorphic to a real line bundle over a unit disk in C^n).

Simply connected Sasakian space forms are homogeneous contact manifolds which they are regular [1, 8].

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2. Sasakian submanifolds. Let $\tilde{M}$ be a $(2(n+p)+1)$-dimensional Sasakian space form of constant ϕ-holomorphic sectional curvature $\tilde{C}$ with structure tensors $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, and let M be a $(2n+1)$-dimensional differential manifold with an almost contact structure (ϕ, ξ, η, g). We assume that M is immersed in $\tilde{M}$ by f and f satisfies $\tilde{\phi}\cdot f_* = f_*\cdot\phi$, $\tilde{\xi} = f_*\cdot\xi$, $\eta = f^*\tilde{\eta}$ and $g = f^*\tilde{g}$, where f^* denotes the differential of f and f^* the dual map of f_*.

We denote by ∇ (resp. $\tilde{\nabla}$) the covariant differentiation with respect to g (resp. $\tilde{g}$). Then the second fundamental form α of the immersion f is given by

$$\alpha(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y .$$

We can easily see that α satisfies

$$\tilde{\phi}\alpha(X, Y) = \alpha(\phi X, Y) = \alpha(X, \phi Y) , \tag{2.1}$$

$$\alpha(X, \xi) = 0 . \tag{2.2}$$

Let $\nu_1, \cdots, \nu_p, \tilde{\phi}\nu_1, \cdots, \tilde{\phi}\nu_p$ be local fields of orthonormal vectors normal to M. If we set, for $i = 1, 2, \cdots, p$,

$$g(A_i X, Y) = \tilde{g}(\alpha(X, Y), \nu_i) ,$$

$$g(A_{i^*} X, Y) = \tilde{g}(\alpha(X, Y), \tilde{\phi}\nu_i) ,$$

then, $A_1, \cdots, A_p, A_{1^*}, \cdots, A_{p^*}$ are local fields of symmetric linear transformations and they satisfy

$$A_{i^*} = \phi A_i , \tag{2.3}$$

$$\phi A_i = -A_i\phi , \tag{2.4}$$

$$A_i\xi = 0 . \tag{2.5}$$

It is known that (ϕ, ξ, η, g) is a Sasakian structure on M (Tanno [9]). Hereafter, we therefore call M a *Sasakian submanifold* of $\tilde{M}$.

PROPOSITION 2.1 (Tanno [9]). *M is a minimal submanifold of $\tilde{M}$.*

PROOF. It suffices to verify that $\operatorname{tr} A_i = \operatorname{tr} A_{i^*} = 0$. From (2.3) and (2.4), $\operatorname{tr} A_{i^*} = 0$ is evident, and we have $\phi A_i\phi = -\phi^2 A_i = A_i - \xi\otimes\eta A_i$. Hence we have

$$\begin{aligned}\operatorname{tr} A_i &= \operatorname{tr}(\phi A_i\phi + \xi\otimes\eta A_i)\\ &= \operatorname{tr}(A_i\phi^2 + \xi\otimes\eta A_i)\\ &= \operatorname{tr}(-A_i + 2\xi\otimes\eta A_i)\\ &= \operatorname{tr}(-A_i) ,\end{aligned}$$

because $\xi\otimes\eta A_i X = \eta(A_i X)\xi = g(A_i X, \xi)\xi = g(X, A_i\xi) = 0$ by (2.5). q.e.d.

Let R be the curvature tensor field of M. Then, the equation of Gauss is

$$
\begin{aligned}
(2.6)\quad R(X, Y)Z = \sum_{i=1}^{P} \{ & -g(A_iX, Z)A_iY + g(A_iY, Z)A_iX \\
& - g(\phi A_iX, Z)\phi A_iY + g(\phi A_iY, Z)\phi A_iX\} \\
& + \frac{1}{4}(\tilde{C} + 3)\{g(Y, Z)X - g(X, Z)Y\} \\
& + \frac{1}{4}(\tilde{C} - 1)\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\
& - g(Y, Z)\eta(X)\xi - d\eta(X, Z)\phi Y + d\eta(Y, Z)\phi X \\
& - 2d\eta(X, Y)\phi Z\} .
\end{aligned}
$$

Let S and ρ be the Ricci tensor and the scalar curvature of M respectively. Then we have

$$
\begin{aligned}
(2.7)\quad S(X, Y) = & \frac{1}{2}\{n(\tilde{C} + 3) + \tilde{C} - 1\}g(X, Y) - \frac{1}{2}(n + 1)(\tilde{C} - 1)\eta(X)\eta(Y) \\
& - 2\sum_{i=1}^{P} g(A_iX, A_iY)
\end{aligned}
$$

and

$$
(2.8)\qquad \rho = \frac{n}{2}\{(2n + 1)(\tilde{C} + 3) + \tilde{C} - 1\} - 2\,\mathrm{tr}\sum_{i=1}^{P} A_i^2 .
$$

Let $K(X, Y)$ be the sectional curvature of M determined by orthonormal vectors X and Y. Then we have

$$
\begin{aligned}
(2.9)\quad K(X, Y) & = g(R(X, Y)Y, X) \\
& = \sum_{i=1}^{P} \{g(A_iX, X)g(A_iY, Y) - g(A_iX, Y)^2 \\
& \quad + g(\phi A_iX, X)g(\phi A_iY, Y) - g(\phi A_iX, Y)^2\} \\
& \quad + \frac{1}{4}(\tilde{C} + 3) + \frac{1}{4}(\tilde{C} - 1)\{3g(\phi X, Y)^2 - \eta(X)^2 - \eta(Y)^2\} .
\end{aligned}
$$

In particular, the ϕ-holomorphic sectional curvature $H(X)$ of M is given by

$$
(2.10)\qquad H(X) = \tilde{C} - 2\sum_{i=1}^{P} \{g(A_iX, X)^2 + g(\phi A_iX, X)^2\} .
$$

It is easily seen that $K(\xi, X) = \tilde{K}(\tilde{\xi}, X) = 1$.

3. Fiberings of Sasakian submanifolds.

PROPOSITION 3.1. *A Sasakian submanifold of a regular Sasakian manifold is also regular.*

PROOF. Let $\tilde{M}$ be a regular Sasakian manifold and M a Sasakian submanifold of $\tilde{M}$. Let γ be an integral curve of ξ through a point P of M. Then $f(\gamma)$ is an integral curve of $\hat{\xi}=f_*\xi$ through $f(P)\in\tilde{M}$. Assume that γ is not regular at $Q\in\gamma(s)$. Let $U_{f(Q)}$ be an arbitrary open neighborhood of $f(Q)$ in $\tilde{M}$. Then, by assumption, $f^{-1}(U_{f(Q)})$ is piersed at least twice by γ. This implies that $U_{f(Q)}$ cannot be regular neighborhood, which is a contradiction. q.e.d.

By a well known theorem of Boothby-Wang [1], a compact regular Sasakian manifold is a circle bundle over a compact Kaehler manifold. If M/ξ (resp. $\tilde{M}/\tilde{\xi}$) denotes the set of orbits of ξ (resp. $\tilde{\xi}$), then M/ξ (resp. $\tilde{M}/\tilde{\xi}$) is a compact Kaehler manifold.

PROPOSITION 3.2. *Let M be a compact Sasakian submanifold of a compact regular Sasakian manifold $\tilde{M}$. Then M/ξ is a compact Kaehler submanifold of $\tilde{M}/\tilde{\xi}$.*

PROOF. Let f be the immersion of M into $\tilde{M}$, and $\pi\colon M\to M/\xi$ (resp. $\tilde{\pi}\colon\tilde{M}\to\tilde{M}/\tilde{\xi}$) be the natural projection. Then there exists a mapping $F\colon M/\xi\to\tilde{M}/\tilde{\xi}$ such that the following diagram is commutative;

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & \tilde{M} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ M/\xi & \xrightarrow{\ F\ } & \tilde{M}/\tilde{\xi}\ . \end{array}$$

Let (J, G) (resp. $(\tilde{J}, \tilde{G})$) be the Kaehler structure of M/ξ (resp. $\tilde{M}/\tilde{\xi}$). Then we have

$$(3.1)\qquad (JX)^* = \phi X^*,\quad G(X, Y) = g(X^*, Y^*)\quad \text{for}\quad X, Y\in T(M/\xi)\ ,$$

$$(3.2)\qquad (\tilde{J}, \tilde{X})^* = \tilde{\phi}\tilde{X}^*,\quad \tilde{G}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}^*, \tilde{Y}^*)\quad \text{for}\quad \tilde{X}, \tilde{Y}\in T(\tilde{M}/\tilde{\xi})\ ,$$

where $*$ denotes the horizontal lift with respect to the connection η or $\tilde{\eta}$. For any vector X on M/ξ, we have

$$\begin{aligned} F_*(JX) &= F_*(\pi_*\phi X^*) = \tilde{\pi}_*f_*(\phi X^*) = \tilde{\pi}_*\phi f_*X^* = \tilde{J}\tilde{\pi}_*f_*X^* \\ &= \tilde{J}F_*\pi_*X^* = \tilde{J}F_*(X), \end{aligned}$$

which implies that F is a complex immersion.

On the other hand, we have

$$\tilde{G}(F_*X, F_*Y)=\tilde{g}((F_*Y)^*, (F_*, Y)^*)=\tilde{g}(f_*X^*, f_*Y^*)=g(X^*, Y^*)=G(X, Y)\ ,$$

which implies that F is an isometric immersion. q.e.d.

REMARK. Proposition 3.2 was proved in [2] for hypersurfaces.

Let R' be a curvature tensor field of M/ξ. Then we have [3]

$$(3.3)\qquad (R'(X, Y)Z)^* = -\phi^2 R(X^*, Y^*)Z^* - \frac{1}{2}\eta([Y^*, Z^*])\phi X^* + \frac{1}{2}\eta([X^*, Z^*])\phi Y^* + \eta([X^* \cdot Y^*])\phi Z^* .$$

Let $K'(X, Y)$ be the sectional curvature of M/ξ determined by orthonormal vectors X and Y. Then we have

$$(3.4)\qquad K'(X, Y) = K(X^*, Y^*) + 3g(X^*, \phi Y^*)^2 .$$

The holomorphic sectional curvature $H'(X)$ of M/ξ determined by X is given by

$$(3.5)\qquad H'(X) = H(X^*) + 3 .$$

Let S' be the Ricci tensor of M/ξ. Then we have

$$(3.6)\qquad S'(X, Y) = S(X^*, Y^*) - g(R(\xi, X^*)Y^*, \xi) + 3g(X^*, Y^*) .$$

The scalar curvature ρ' of M/ξ is given by

$$(3.7)\qquad \rho' = \rho + 2n .$$

4. **Main results.** Throughout this section, we confine our attention to compact Sasakian submanifolds of a simply connected elliptic Sasakian space form.

Let $\tilde{M}$ be a simply connected elliptic Sasakian space form of constant ϕ-holomorphic sectional curvature $\tilde{C}(\tilde{C} > -3)$ and M be a compact Sasakian submanifold of $\tilde{M}$. Then $\tilde{M}/\xi$ is a complex projective space of constant holomorphic sectional curvature $\tilde{C} + 3$ by (3.5).

THEOREM 4.1. *Let M be a compact Sasakian submanifold of codimension 2 imbedded (resp. immersed) in a simply connected elliptic Sasakian space form of constant ϕ-holomorphic sectional curvature $\tilde{C}$. If* $\dim M \geqq 5$ *(resp.* $\dim M \geqq 9$*) and if the sectional curvature K of M satisfies $K(X, Y) + 3g(\phi X, Y)^2 > 0$ for each pair of orthonormal vectors X and Y, then M is totally geodesic.*

PROOF. By Proposition 3.2, M/ξ is a compact Kaehler hypersurface imbedded (resp. immersed) in a complex projective space. Our assumption, together with (3.4), implies that every sectional curvature of M/ξ is positive. Hence, by Theorem 3.3 in [4], (resp. Theorem in [5]) M/ξ is a totally geodesic submanifold of codimension 2 of the complex projective space so that $H' = \tilde{C} + 3$. This, together with (3.5), implies $H = \tilde{C}$. Therefore, by (2, 10), we have $A_1 = 0$, that is, M is totally geodesic. q.e.d.

THEOREM 4.2. *Let M be a* $(2n + 1)$*-dimensional compact Sasakian* *submanifold immersed in a simply connected elliptic Sasakian space form of constant ϕ-holomorpic sectional curvature $\tilde{C}$ of dimension* $2(n+p)+1$. *If every ϕ-holomorphic sectional curvature of M is greater than $\tilde{C} - ((n+2)/2(n+2p))(\tilde{C}+3)$, then, M is totally geodesic.*

PROOF. M/ξ is an n-dimensional compact Kaehler submanifold immersed in a complex projective space of constant holomorphic sectional curvature $\tilde{C}+3$ of dimension $n+p$. By assumption and (3.5), we have

$$H' > (\tilde{C}+3)\Big(1 - \frac{n+2}{2(n+2p)}\Big) .$$

By virtue of Theorem in [6], M/ξ is totally geodesic. By the argument similar to Theorem 4.1, M is totally geodesic. q.e.d.

The same argument as Theorem 4.2, combined with Theorem in [10] implies the following.

THEOREM 4.3. *Let M be a* 5-*dimensional compact Sasakian submanifold immersed in an* 11-*dimensional simply connected elliptic Sasakian space form of constant ϕ-holomorphic sectional curvature $\tilde{C}$. If every ϕ-holomorphic sectional curvature of M is greater than* $(2/3)\tilde{C} - 1$, *then M is totally geodesic.*

THEOREM 4.4. *Let M be a* $(2n+1)$-*dimensional compact Sasakian submanifold immersed in a simply connected elliptic Sasakian space form of constant ϕ-holomorphic sectional curvature $\tilde{C}$. If every Ricci curvature of M is greater than* $(n/2)(\tilde{C}+3) - 2$, *then M is totally geodesic.*

PROOF. For a unit vector X in M/ξ, we have from (3.6) that

$$S'(X, X) = S(X^*, X^*) + 2 .$$

This, together with our assumption, implies $S'(X, X) > (n/2)(\tilde{C}+3)$. Hence, by virtue of Theorem 1 in [6], M/ξ is a totally geodesic submanifold of the complex projective space. By the argument similar to Theorem 4.1, M is totally geodesic. q.e.d.

THEOREM 4.5. *Let M be a* $(2n+1)$-*dimensional compact Sasakian submanifold of codimension* 2 *immbedded in a simply connected elliptic Sasakian space form of constant ϕ-holomorphic sectional curvature $\tilde{C}$. If the scalar curvature of M is greater than* $(\tilde{C}+3)n^2 - 2n$ *almost everywhere on M, then, M is totally geodesic.*

PROOF. From (3.7), we have $\rho' > (\tilde{C}+3)n^2$, which together with Corollary 2.2 in [4], implies that M is totally geodesic. q.e.d.

BIBRIOGRAPHY

[1] W. M. BOOTHBY AND H. C. WANG, On contact manifolds, Ann. of Math. 68 (1958) 721-734.

[2] K. KENMOTSU, Invariant submanifolds in a Sasakian manifold, Tôhoku Math. J., 21 (1969) 495-500.

[3] K. OGIUE, On fiberings of almost contact manifolds, Kôdai Math. Sem. Rep., 17 (1965) 53-62.

[4] ———, Differential geometry of algebraic manifolds, Differential Geometry in honor of Professor Kentaro Yano (1972), 355-372.

[5] ———, Positively curved complex hypersurface immersed in a complex projective space, to appear.

[6] ———, Positively curved complex submanifolds immersed in a complex projective space I, II, to appear.

[7] S. SASAKI, Almost contact manifolds, Lecture Note, Tôhoku Univ.

[8] S. TANNO, The automorphism groups of almost contact Riemannian manifolds, Tôhoku Math. J., 21 (1969) 21-38.

[9] ———, Isometric immersions of Sasakian manifolds in sphere, Kôdai Math. Sem. Rep., 21 (1969) 448-458.

[10] ———, 2-dimensional complex submanifolds immersed in complex projective spaces, to appear.

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