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ON MAXIMAL FAMILIES OF COMPACT COMPLEX SUBMANIFOLDS OF COMPLEX MANIFOLDS

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Introduction. Let W be a complex manifold. Let V be a compact complex submanifold of W. Let $\tilde{F}$ be the sheaf over V of germs of holomorphic sections of the normal bundle F of V in W. Let $H^1(V, \tilde{F})$ be the first cohomology group of $\tilde{F}$. In 1962, Kodaira proved the following theorem:

THEOREM (K. Kodaira [4]). *If $H^1(V, \tilde{F}) = 0$, then there exists a maximal family $\{V_s\}_{s\in S}$ of compact complex submanifolds of W such that $V_o = V$ for a point $o \in S$ where the parameter space S is a complex manifold.*

The main purpose of this paper is to drop the assumption $H^1(V, \tilde{F}) = 0$. We get:

THEOREM 1. *There exists a maximal family $\{V_s\}_{s\in S}$ of compact complex submanifolds of W such that $V_o = V$ for a point $o \in S$ where the parameter space S is an analytic space.*

The idea of the proof is due to Kuranishi's proof of his theorem on the existence of the local moduli spaces of complex structures [6]. (See also [7].)

Fixing W, we can easily patch these maximal families together and get the following theorem.

THEOREM 2. *Let W be a complex manifold. Then the set of all compact complex submanifolds of W forms a (not necessarily connected) analytic space $S(W)$ in a natural way.*

Our space $S(W)$ is naturally identified with an open subspace of the Douady space [1].

For each point $s \in S(W)$, we denote V_s the corresponding compact complex submanifold of W. Using our concrete construction of maximal families, we get:

THEOREM 3. *Let W be a complex manifold. Let $S(W)$ be the analytic space in Theorem 2. Then,*

$$\{(s, t) \in S(W) \times S(W) \mid V_s \subset V_t\}$$

is a closed subvariety of $S(W) \times S(W)$.

COROLLARY. *Let* V *be a compact complex submanifold of a complex manifold* W. *Then*

$$\{s \in S(W) \mid V_s \supset V\}$$

is a closed subvariety of $S(W)$.

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1. Preliminaries. Let W be a $(r + d)$-dimensional (connected) complex manifold. Let V be a d-dimensional (connected) compact complex submanifold of W. We may assume that V is covered by a finite number of open subsets $\{W_i\}_{i \in I}$ of W, each of which has a local coordinate system:

$$(w_i, z_i) = (w_i^1, \cdots, w_i^r, z_i^1, \cdots, z_i^d)$$

such that V is defined in W_i by the equation $w_i = 0$. We put $U_i = W_i \cap V$. Let

$$w_i = f_{ik}(w_k, z_k) ,$$

$$z_i = g_{ik}(w_k, z_k)$$

be the coordinate transformations in $W_i \cap W_k$, where f_{ik} and g_{ik} are vector-valued holomorphic functions of $(w_k, z_k) \in W_i \cap W_k$. We define matrix-valued holomorphic functions $F_{ik}(z_k)$ by

$$F_{ik}(z_k) = (\partial f_{ik}/\partial w_k)_{(0,z_k)} \qquad \text{for } z_k \in U_i \cap U_k .$$

Then we get the following identities:

$$F_{ij}(z_j)F_{jk}(z_k) = F_{ik}(z_k) \quad \text{for} \quad z_k \in U_i \cap U_j \cap U_k \quad \text{and} \quad z_j = g_{ik}(0, z_k) .$$

Thus the system $\{F_{ik}\}$ defines a holomorphic vector bundle F on V. We call this bundle *the normal bundle of* V *in* W. We denote $\tilde{F}$ the sheaf of germs of holomorphic sections of F.

Now we consider another compact complex submanifold V' of W covered by $\{W_i\}_{i \in I}$. We assume that V' is defined in W_i by the equation:

$$w_i = \phi_i(z_i)$$

where ϕ_i is a vector-valued holomorphic function of $z_i \in U_i$. These ϕ_i

must satisfy the following compatibility conditions:

$$f_{ik}(\phi_k(z_k), z_k) = \phi_i(g_{ik}(\phi_k(z_k),, z_k)) \quad \text{for} \quad (\phi_k(z_k), z_k) \in W_i \cap W_k .$$

We want to consider families of such V'.

DEFINITION 1.1. Let X and S be analytic spaces* and let $\pi: X \to S$ be a proper surjective holomorphic map. The triple (X, π, S) is called *a family of compact complex manifolds* if and only if there are an open covering $\{X_\alpha\}$ of X, open subsets Ω_α of C^n, and holomorphic isomorphisms

$$\eta_\alpha\colon X_\alpha \to \Omega_\alpha \times S_\alpha$$

where $S_\alpha = \pi(X_\alpha)$ is open in S, such that the diagram

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\eta_\alpha} & \Omega_\alpha \times S_\alpha \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & S_\alpha & \end{array}$$

commutes for each α. S is called *the parameter space of the family* (X, π, S).

DEFINITION 1.2. Let W be a complex manifold. A family (X, π, S) of compact complex manifolds is called *a family of compact complex submanifolds of* W if and only if X is an analytic subvariety of $W \times S$ and π is the the restriction to X of the projection map: $W \times S \to S$.

For each point $s \in S$ of a family (X, π, S) of compact complex submanifolds of W, the fiber $\pi^{-1}(s)$ can be written as $\pi^{-1}(s) = V_s \times s$ where V_s is a compact complex submanifold of W. We identify $\pi^{-1}(s)$ with V_s and write the family as $\{V_s\}_{s \in S}$ to simplify the notations.

DEFINITION 1.3. A family $\{V_s\}_{s \in S}$ of compact complex submanifolds of a complex manifold W is said to be *maximal at* $s_0 \in S$ if and only if for any family $\{V_t\}_{t \in T}$ of compact complex submanifolds of W with a point $t_0 \in T$ such that $V_{t_0} = V_{s_0}$, there exist a neighbourhood U of t_0 in T and a holomorphic map f of U into S such that $f(t_0) = s_0$ and such that

$$V_{f(t)} = V_t \qquad \text{for all } t \in U .$$

A family of compact complex submanifolds of W is called *a maximal family* if and only if it is maximal at every point of the parameter space.

* By an analytic space we mean a reduced, connected, Hausdorff complex analytic space.

Now, let W, V, W_i and (w_i, z_i) be as above. In order to prove Theorem 1, we may take the parameter spaces of families as small as we want. Thus, by the implicit mapping theorem, we may restrict our attention to families $\{V_s\}_{s \in S}$ of compact complex submanifolds of W such that $V_0 = V$ for a point $o \in S$ and such that, for each point $s \in S$, V_s is defined in W_i by the equation:

$$w_i = \phi_i(z_i, s)$$

where ϕ_i is a vector-valued holomorphic function of $(z_i, s) \in U_i \times S$.

2. **Some lemmas.** Let W, V, $\{W_i\}_{i \in I}$, (w_i, z_i) and $\tilde{F}$ be as above. We may assume that, for each $i \in I$, $\bar{W}_i$ (the closure of W_i in W) is compact, the local coordinate system (w_i, z_i) is extended to an open set $\tilde{W}_i \supset \bar{W}_i$ and that $W_i = \{(w_i, z_i) \in \tilde{W}_i \mid |w_i| < 1 \text{ and } |z_i| < 1\}$ where

$$|w_i| = \sup\{|w_i^\lambda| : \lambda = 1, \cdots, r\}$$

and

$$|z_i| = \sup\{|z_i^\alpha| : \alpha = 1, \cdots, d\} .$$

We also assume that V is defined in $\tilde{W}_i$ by the equation $w_i = 0$. We put $\tilde{U}_i = V \cap \tilde{W}_i$. Then

$$U_i = W_i \cap V = \{(0, z_i) \in \tilde{U}_i \mid |z_i| < 1\} .$$

We may assume that, for each positive integer n and for each n-tuple $(i_1, \cdots, i_n)$ of indices, $U_{i_1} \cap \cdots \cap U_{i_n}$ and $\tilde{U}_{i_1} \cap \cdots \cap \tilde{U}_{i_n}$ are connected and Stein, unless they are empty.

Let $C^p = C^p(V, \tilde{F}, \{U_i\})$ be the (not necessarily skew symmetric) p-th cochain group of $\tilde{F}$ on the nerve of the covering $\{U_i\}$. We introduce a norm $\|\ \|$ in C^p. For each $\xi = \{\xi_{i_0 \cdots i_p}\} \in C^p$, we define $\|\xi\|$ by

$$\|\xi\| = \sup\{|\xi^\lambda_{i_0 \cdots i_p}(z)| : \lambda = 1, \cdots, r, z \in U_{i_0} \cap \cdots \cap U_{i_p}, (i_0, \cdots, i_p) \in I^{p+1}\}$$

where $\xi^\lambda_{i_0 \cdots i_p}$ is the representation of the component $\xi_{i_0 \cdots i_p}$ of ξ with respect to the coordinate (w_{i_0}, z_{i_0}). We put

$$C^p(\|\ \|) = \{\xi \in C^p \mid \|\xi\| < +\infty\} .$$

It is easy to see that $C^p(\|\ \|)$ is a Banach space and the coboundary map δ maps $C^p(\|\ \|)$ continuously into $C^{p+1}(\|\ \|)$. We put

$$\begin{aligned} Z^p &= \{\xi \in C^p \mid \delta\xi = 0\} , \\ B^p &= \delta C^{p-1} , \\ H^p &= Z^p / B^p , \\ Z^p(\|\ \|) &= \{\xi \in C^p(\|\ \|) \mid \delta\xi = 0\} , \end{aligned}$$

$$B^p(\|\ \|) = B^p \cap C^p(\|\ \|)\ ,$$
$$H^p(\|\ \|) = Z^p(\|\ \|)/B^p(\|\ \|)\ .$$

LEMMA 2.1. *H^p and $H^p(\|\ \|)$ are canonically isomorphic to $H^p(V, \tilde{F})$ (the p-th cohomology group of $\tilde{F}$).*

PROOF. H^p is canonically isomorphic to $H^p(V, \tilde{F})$ by Leray's theorem. Since $Z^p(\|\ \|)$ is a subgroup of Z^p and

$$B^p(\|\ \|) = B^p \cap C^p(\|\ \|) = B^p \cap Z^p(\|\ \|)\ ,$$

we have the canonical injection:

$$H^p(\|\ \|) \longrightarrow H^p\ .$$

Let $\tilde{C}^p$ be the (not necessarily skew symmetric) p-th cochain group of $\tilde{F}$ on the nerve of the covering $\{\tilde{U}_i\}_{i \in I}$. We put

$$\tilde{Z}^p = \{\xi \in \tilde{C}^p \mid \delta\xi = 0\}\ ,$$
$$\tilde{B}^p = \delta\tilde{C}^{p-1}\ ,$$
$$\tilde{H}^p = \tilde{Z}^p/\tilde{B}^p\ .$$

Then $\tilde{H}^p$ is canonically isomorphic to H^p. Since the restriction maps $\left\{\mathrm{res}\,\frac{\tilde{U}_i}{U_i}\right\}$ map $\tilde{C}^p$ into $C^p(\|\ \|)$, $\tilde{Z}^p$ into $Z^p(\|\ \|)$ and $\tilde{B}^p$ into $B^p(\|\ \|)$, we have a homomorphism:

$$\tilde{H}^p \longrightarrow H^p(\|\ \|)\ .$$

It is clear that the diagram

$$\begin{array}{ccc} \tilde{H}^p & \longrightarrow & H^p(\|\ \|) \\ & {\scriptstyle\cong}\searrow \quad \swarrow & \\ & H^p & \end{array}$$

is commutative. Hence $H^p(\|\ \|) \cong H^p$. *q.e.d.*

Let e be a small positive number such that the open sets

$$W_i^e = \{(w_i, z_i) \in W_i \mid |w_i| < 1\ ,\ |z_i| < 1 - e\},\ i \in I\ ,$$

again cover V. We put $U_i^e = W_i^e \cap V = \{(0, z_i) \in W_i \mid |z_i| < 1 - e\}$.

Besides C^p, we must consider additive groups C_e^p. An element $\xi = \{\xi_{i_0 \cdots i_p}\} \in C_e^p$ is a function which associates to each $(p+1)$-ple $(i_0, \cdots, i_p)$ of indices in I a holomorphic section $\xi_{i_0 \cdots i_p}$ on $U_{i_0}^e \cap \cdots \cap U_{i_{p-1}}^e \cap U_{i_p}$. In particular, $C_e^0 = C^0$. We define the coboundary map

$$\delta_e\colon C_e^p \longrightarrow C_e^{p+1}$$

by

$$(\delta_e \xi)_{i_0 \cdots i_{p+1}}(z) = \sum_\nu (-1)^\nu \xi_{i_0 \cdots i_{\nu-1} i_{\nu+1} \cdots i_{p+1}}(z)$$

for

$$z \in U^e_{i_0} \cap \cdots \cap U^e_{i_p} \cap U_{i_{p+1}} .$$

We introduce a norm $|\ |_e$ in C^p_e. For each $\xi = \{\xi_{i_0 \cdots i_p}\} \in C^p_e$, we define $|\xi|_e$ by

$$|\xi|_e = \sup \{|\xi^\lambda_{i_0 \cdots i_p}(z)|: \\ \lambda = 1, \cdots, r,\ z \in U^e_{i_0} \cap \cdots \cap U^e_{i_{p-1}} \cap U_{i_p},\ (i_0, \cdots, i_p) \in I^{p+1}\}$$

where $\xi^\lambda_{i_0 \cdots i_p}$ is the representation of the component $\xi_{i_0 \cdots i_p}$ of ξ with respect to the coordinate (w_{i_0}, z_{i_0}). In particular, we define

$$|\xi|_e = \|\xi\| \qquad \text{for } \xi \in C^0_e = C^0 .$$

We put

$$C^p_e(|\ |_e) = \{\xi \in C^p_e \mid |\xi|_e < +\infty\} .$$

It is easy to see that $C^p_e(|\ |_e)$ is a Banach space and the coboundary map δ_e maps $C^p_e(|\ |_e)$ continuously into $C^{p+1}_e(|\ |_e)$. We put

$$\begin{aligned} Z^p_e &= \{\xi \in C^p_e \mid \delta_e \xi = 0\} , \\ B^p_e &= \delta_e C^{p-1}_e , \\ H^p_e &= Z^p_e / B^p_e , \\ Z^p_e(|\ |_e) &= \{\xi \in C^p_e(|\ |_e) \mid \delta_e \xi = 0\} , \\ B^p_e(|\ |_e) &= B^p_e \cap C^p_e(|\ |_e) , \\ H^p_e(|\ |_e) &= Z^p_e(|\ |_e) / B^p_e(|\ |_e) . \end{aligned}$$

LEMMA 2.2. *There is a canonical identification*: $Z^1_e(|\ |_e) = Z^1(\|\ \|)$ *and the norms* $|\ |_e$ *and* $\|\ \|$ *are equivalent in them.*

PROOF. Each element $\xi = \{\xi_{ik}\} \in Z^1(\|\ \|)$ corresponds to the element $\xi' = \{\xi'_{ik}\} \in Z^1_e(|\ |_e)$ with $\xi'_{ik} = \xi_{ik} \mid U^e_i \cap U_k$. It is clear that $|\xi'|_e \leqq \|\xi\|$. Conversely, let $\xi' = \{\xi'_{ik}\} \in Z^1_e(|\ |_e)$. We take a point $z \in U_i \cap U_k$. Since $\{U^e_i\}$ is a covering of V, there is an index j such that $z \in U^e_j$. We define an element $\xi_{ik}(z)$ of the fiber F_z of F by

$$\xi_{ik}(x) = \xi'_{jk}(z) - \xi'_{ji}(z) . \tag{1}$$

We show that $\xi_{ik}(z)$ does not depend on the choice of the index j. Let us take another index l such that $z \in U^e_l$. Since $\xi' \in Z^1_e(|\ |_e)$,

$$\begin{aligned} \{\xi'_{jk}(z) - \xi'_{ji}(z)\} - \{\xi'_{lk}(z) - \xi'_{li}(z)\} &= \{\xi'_{jk}(z) - \xi'_{lk}(z)\} - \{\xi'_{ji}(z) - \xi'_{li}(z)\} \\ &= \xi'_{jl}(z) - \xi'_{jl}(z) \\ &= 0 . \end{aligned}$$

Thus $\xi_{ik}(z)$ is a well defined holomorphic section of F on $U_i \cap U_k$. We set $\xi = \{\xi_{ik}\}$. We express (1) in the coordinate in U_i:

$$\xi_{ik}(z_i) = F_{ij}(z_j)\xi'_{jk}(z_j) - F_{ij}(z_j)\xi'_{ji}(z_j)$$

where $z = (0, z_i)$ and $z_j = g_{ji}(0, z_i)$. Then we get

$$\begin{aligned} |\xi_{ik}(z_i)| &\leqq r\,||F||\,(|\xi'_{jk}(z_j)| + |\xi'_{ji}(z_j)|) \\ &\leqq 2r\,||F||\,|\xi'|_e \end{aligned}$$

where $||F|| = \sup\{|F^{\lambda}_{ij\nu}(z_j)|;\ \lambda, \nu = 1, \cdots, r, i, j \in I, z_j \in U_i \cap U_j\}$. Hence $||\xi|| \leqq 2r\,||F||\,|\xi'|_e$.

We show that ξ is a cocycle. Let us take a point $z \in U_i \cap U_j \cap U_k$. There is an index l such that $z \in U_l^e$. Thus

$$\begin{aligned} \xi_{jk}(z) - \xi_{ik}(z) + \xi_{ij}(z) &= \{\xi'_{lk}(z) - \xi'_{lj}(z)\} - \{\xi'_{lk}(z) - \xi'_{li}(z)\} + \{\xi'_{lj}(z) - \xi'_{li}(z)\} \\ &= 0\,. \end{aligned}$$

Now, $\xi' \in Z_e^1(|\ |_e)$ corresponds to $\xi \in Z^1(||\ ||)$. *q.e.d.*

The following lemma is a slight modification of Kuranishi's Proposition 2.5′, [5].

LEMMA 2.3. *There is a continuous linear map*

$$E\colon B_e^2(|\ |_e) \to C_e^1(|\ |_e)$$

such that $\delta_e E =$ *the identity map on* $B_e^2(|\ |_e)$.

PROOF. First of all, we define additive groups $C_e^p(q, q')$, $p, q, q' = 0, 1, 2, \cdots$. An element $\xi \in C_e^p(q, q')$ is a function which associates to each $(p+1)$-ple $(i_0, \cdots, i_p)$ of indices a C^∞-differential (q, q')-form $\xi_{i_0 \cdots i_p}$ on $U_{i_0}^e \cap \cdots \cap U_{i_{p-1}}^e \cap U_{i_p}$ with coefficients in $\tilde{F}$. We define a norm $|\xi|_e$ by

$$\begin{aligned} |\xi|_e = \sup\{&|\xi^{\lambda}_{i_0 \cdots i_p, j_1 \cdots j_q k_1 \cdots k_{q'}}(z)| : \lambda = 1, \cdots, r, \\ &z \in U_{i_0}^e \cap \cdots \cap U_{i_{p-1}}^e \cap U_{i_p},\ (i_0, \cdots, i_p) \in I^{p+1}, \\ &1 \leqq j_1 < \cdots < j_q \leqq d, 1 \leqq k_1 < \cdots < k_{q'} \leqq d\} \end{aligned}$$

where $\xi^{\lambda}_{i_0 \cdots i_p, j_1 \cdots j_q k_1 \cdots k_{q'}}$ is the coordinate expression in U_{i_0} of the component

$$\xi_{i_0 \cdots i_p} = \sum \xi_{i_0 \cdots i_p, j_1 \cdots j_q k_1 \cdots k_{q'}} dz_{i_0}^{j_1} \wedge \cdots \wedge dz_{i_0}^{j_q} \wedge d\bar{z}_{i_0}^{k_1} \wedge \cdots \wedge d\bar{z}_{i_0}^{k_{q'}}\,.$$

We also define a map

$$\delta_e\colon C_e^p(q, q') \to C_e^{p+1}(q, q')$$

by

$$(\delta_e \xi)_{i_0 \cdots i_{p+1}} = \sum (-1)^{\nu} \xi_{i_0 \cdots i_{\nu-1} i_{\nu+1} \cdots i_{p+1}}\,.$$

It is clear that C_e^p is a subgroup of $C_e^p(0, 0)$ and

$$\delta_e\colon C_e^p \to C_e^{p+1}$$

is the restriction map of

$$\delta_e\colon C_e^p(0,0) \to C_e^{p+1}(0,0)$$

defined above.

Let $\{q_i\}_{i\in I}$ be a partition of unity subordinate to the covering $\{U_i^e\}_{i\in I}$. Given $\xi \in B_e^2(|\ |_e)$ we define an element $\eta = \{\eta_{jl}\} \in C_e^1(0,0)$ by $\eta_{jl} = \sum_{i\in I} q_i \xi_{ijl}$. Then

$$\begin{aligned}|\eta|_e &\leqq \sum_{\nu,i} \sup \{|q_i(z_i)F_{ji\nu}^{\lambda}(z_i)\xi_{ijl}^{\nu}(z_i)|:\ z_i \in U_i^e \cap U_j^e \cap U_l\} \\ &\leqq c_1 |\xi|_e\end{aligned}$$

where c_1 is a constant. We claim that $\delta_e\eta = \xi$.

$$\begin{aligned}(\delta_e\eta)_{jkl} = \eta_{kl} - \eta_{jl} + \eta_{jk} &= \sum_i q_i(\xi_{ikl} - \xi_{ijl} + \xi_{ijk}) \\ &= \sum_i q_i\xi_{jkl} = \xi_{jkl}\,.\end{aligned}$$

Let $\bar\partial\eta = \{\bar\partial\eta_{jk}\}$. Then $\bar\partial\eta$ is an element of $C_e^1(0,1)$ and satisfies $\delta_e\bar\partial\eta = 0$, for $\bar\partial(\eta_{jk}) - \bar\partial(\eta_{ik}) + \bar\partial(\eta_{ij}) = \bar\partial(\xi_{ijk}) = 0$. Let $\lambda_j = \sum_i q_i\bar\partial\eta_{ij}$. Then

$$\lambda_j - \lambda_k = \sum_i q_i(\bar\partial(\eta_{ij}) - \bar\partial(\eta_{jk})) = \sum_i q_i\bar\partial\eta_{kj} = \bar\partial\eta_{kj}\,.$$

Since $\lambda_j = \sum_i q_i\bar\partial\eta_{ij} = \sum_{i,k} q_i\bar\partial(q_k\xi_{kij}) = \sum_{i,k} q_i\xi_{kij}\bar\partial q_k$, we can find a constant c_2 such that

$$(1)\qquad |\bar\partial_{\beta_1}\cdots\bar\partial_{\beta_p}\lambda_j| \leqq c_2|\xi|_e$$

where $\bar\partial_{\beta_1} = \partial/\partial\bar z^{\beta_1}$ etc. and $p = 0, 1, \cdots, d$. We now denote by κ_j the Newlander-Nirenberg operator on U_j ([8] or p. 186 [9]), and use its properties; for a (vector valued) $(0,1)$-form λ_j on U_j,

$$(\mathrm{N}_1)\qquad |\kappa_j\lambda_j| \leqq c_3 \sup|\bar\partial_{\beta_1}\cdots\bar\partial_{\beta_p}\lambda_j| \qquad \text{with } c_3 \text{ a constant}\,,$$

$$(\mathrm{N}_2)\qquad \lambda_j = (\bar\partial\kappa_j + \kappa_j\bar\partial)\lambda_j\,.$$

From (1) and (N_1) above, it follows that

$$|\kappa_j\lambda_j| \leqq c_4|\xi|_e \qquad \text{with } c_4 \text{ a constant}\,.$$

Now we get $\bar\partial\lambda_j = \bar\partial\lambda_k$ on $U_j \cap U_k$. Hence $\bar\partial\lambda_j$ defines a global $C^\infty-(0,2)$ form ω with coefficients in $\tilde F$.

Let $0 < \alpha < 1$ be a constant and $|\omega|_{d+\alpha}$ be the Kodaira-Nirenberg-Spencer norm [3]. Then by estimating $|\omega|_{d+\alpha}$ on U_j^e we have

$$|w|_{d+\alpha} \leqq c_5|\xi|_e \qquad \text{with } c_5 \text{ a constant}\,.$$

We introduce a Hermitian metric on V and let $\bar\partial^*$ and G be the adjoint operator of $\bar\partial$ and the Green operator respectively. Let ξ'_{ijl} be the restriction of ξ_{ijl} on $U_i^e \cap U_j^e \cap U_l^e$. Since $\xi \in B_e^2(|\ |_e)$, $\xi' = \{\xi'_{ijl}\}$ is a coboundary of $\tilde{F}$ on the nerve of the covering $\{U_i^e\}$. It is clear that ω corresponds to ξ' by Dolbeault's isomorphism. Since ξ' is a coboundary,

$$\omega = \bar{\partial}\bar{\partial}^* G\omega .$$

We put $\pi = \bar{\partial}^* G\omega$. Then there is a constant c_6 such that

$$|\pi|_{d+\alpha} \leqq c_6 |\xi|_e .$$

Let us denote π_i the restriction of π on U_i. Then we have

$$|\bar{\partial}_{\beta_1} \cdots \bar{\partial}_{\beta_p} \pi_i| \leqq c_7 |\xi|_e$$

where c_7 is a constant, by (N_1) above, we get

$$|\kappa_j \pi_j| \leqq c_8 |\xi|_e$$

where c_8 is a constant.

We put $\lambda_i' = \lambda_i - \pi_i$. Then we have

$$\bar{\partial}\lambda_i' = \omega - \omega = 0 ,$$
$$\bar{\partial}\eta_{ij} = \lambda_j - \lambda_i = \lambda_j' - \lambda_i' .$$

Hence we have

$$\begin{aligned}\bar{\partial}(\eta_{ij} - \kappa_j\lambda_j' + \kappa_i\lambda_i') &= \lambda_j' - \lambda_i' - \bar{\partial}\kappa_j\lambda_j' + \bar{\partial}\kappa_i\lambda_i' \\ &= (\lambda_j' - \bar{\partial}\kappa_j\lambda_j') - (\lambda_i' - \bar{\partial}\kappa_i\lambda_i') \\ &= \kappa_j\bar{\partial}\lambda_j' - \kappa_i\bar{\partial}\lambda_i' \\ &= 0\end{aligned}$$

by (N_2) above.

Now, we define $\beta = \{\beta_{ij}\}$ by $\beta_{ij} = \eta_{ij} - \kappa_j\lambda_j' + \kappa_i\lambda_i'$. Then it is an element of C_e^1 and there is a constant c such that

$$|\beta|_e \leqq c |\xi|_e .$$

We define E: $\xi \rightarrow \beta$. We claim $\delta_e\beta = \xi$.

$$\begin{aligned}(\delta_e\beta)_{ijk} &= \eta_{jk} - \kappa_k\lambda_k' + \kappa_j\lambda_j' - \eta_{ik} + \kappa_k\lambda_k' - \kappa_i\lambda_i' + \eta_{ij} - \kappa_j\lambda_j' + \kappa_i\lambda_i' \\ &= \xi_{ijk} .\end{aligned}$$

q.e.d.

Using the map E in Lemma 2.3, we define a map

$$\Lambda: C_e^1(|\ |_e) \rightarrow Z_e^1(|\ |_e)$$

by $\Lambda = 1 - E\delta_e$. Then Λ is a projection map.

Since the proof of the following lemma is similar to (and simpler than) that of Lemma 2.3, we omit it.

LEMMA 2.4. *There is a continuous linear map*

$$E_0: B_e^1(|\ |_e) \rightarrow C_e^0(|\ |_e) = C^0(\|\ \|)$$

such that $\delta_e E_0 =$ *the identity on* $B_e^1(|\ |_e)$. *Finally, we prove the following lemma.*

LEMMA 2.5.

(1) *There is a canonical identification*: $B^1(\|\ \|) = B_e^1(|\ |_e)$,

(2) $H_e^1(|\ |_e)$ *is canonically isomorphic to* $H^1(V, \tilde{F})$,

(3) $\delta_e C_e^0(|\ |_e) = B_e^1(|\ |_e)$,

(4) $B_e^1(|\ |_e)$ *is closed in* $Z_e^1(|\ |_e)$.

PROOF. First of all, we show (1). Each element $\xi = \{\xi_{ik}\} \in B^1(\|\ \|)$ corresponds to the element $\xi' = \{\xi'_{ik}\} \in B_e^1(|\ |_e)$ with $\xi'_{ik} = \xi_{ik} | U_i^e \cap U_k$. It is clear that $|\xi'|_e \leqq \|\xi\|$. Conversely, we take an element $\xi \in B_e^1(|\ |_e)$. By Lemma 2.4, $E_0\xi$ is an element of $C^0(\|\ \|)$ and each component of $\xi = \delta_e E_0 \xi$ is the restriction of the corresponding component of $\delta(E_0\xi) \in C^1(\|\ \|)$. Since $U_i \cap U_k$ is connected for each pair (i, k), the extension $\delta(E_0\xi)$ is uniquely determined by ξ. We associate $\delta(E_0\xi)$ to ξ. Thus we get (1). (2) follows from (1), Lemma 2.1 and Lemma 2.2. $\delta_e(E_0\xi) = \xi$ shows (3).

To prove (4), let $\{\xi^{(n)}\}$ be a sequence in $B_e^1(|\ |_e)$ which converges to $\xi \in Z_e^1(|\ |_e)$. We put $\eta^{(n)} = E_0\xi^{(n)} \in C_e^0(|\ |_e)$, $n = 1, 2, \cdots$. Then

$$|\eta^{(n)}|_e \leqq c\, |\xi^{(n)}|_e \leqq M$$

where c and M are constants. Thus, for all point $z_i \in U_i$,

$$|\eta_i^{(n)}(z_i)| \leqq M\,, \qquad n = 1, 2, \cdots$$

where $\eta^{(n)} = \{\eta_i^{(n)}\}$.

By Montel's theorem, there is a subsequence

$$n_1, n_2, \cdots \longrightarrow \infty$$

such that $\eta_i^{(n_\nu)}(z_i)$ converges absolutely and uniformly on each compact subset of U_i for each $i \in I$.

We put $\eta_i(z_i) = \lim_\nu \eta_i^{(n_\nu)}(z_i)$. Then η_i is holomorphic on U_i. We put $\eta = \{\eta_i\}$ and regard η as an element of C^0. For each fixed $z_i \in U_i$, we have

$$|\eta_i(z_i)| = |\lim_\nu \eta_i^{(n_\nu)}(z_i)| \leqq \limsup_\nu |\eta_i^{(n_\nu)}(z_i)| \leqq M\,.$$

Thus $|\eta|_e \leqq M$ so that $\eta \in C_e^0(|\ |_e)$. Now, for each fixed $z_i \in U_i^e \cap U_k$, we have

$$F_{ik}(z_k)\eta_k^{(n_\nu)}(z_k) - \eta_i^{(n_\nu)}(z_i) = \xi_{ik}^{(n_\nu)}(z_i)$$

where

$$z_k = g_{ki}(0, z_i)\,.$$

Letting $\nu \longrightarrow \infty$, we have

$$F_{ik}(z_k)\eta_k(z_k) - \eta_i(z_i) = \xi_{ik}(z_i) .$$

Hence $\delta_e\eta = \xi$. *q.e.d.*

It is well known that $H^1(V, \tilde{F})$ is of finite dimensional. Hence, by (2) and (4) of Lemma 2.5, there is a subspace $H_e^1(|\ |_e)$* of $Z_e^1(|\ |_e)$ isomorphic to $H^1(V, \tilde{F})$ such that $Z_e^1(|\ |_e)$ splits into the direct sum of $B_e^1(|\ |_e)$ and $H_e^1(|\ |_e)$:

$$Z_e^1(|\ |_e) = B_e^1(|\ |_e) \oplus H_e^1(|\ |_e) .$$

Let

$$B\colon Z_e^1(|\ |_e) \to B_e^1(|\ |_e)$$

and

$$H\colon Z_e^1(|\ |_e) \to H_e^1(|\ |_e)$$

be the projection maps corresponding to the splitting.

3. Proof of Theorem 1. Let W, V, $\{\tilde{W}_i\}_{i\in I}$, $\{W_i\}_{i\in I}$, $\{W_i^e\}_{i\in I}$, (w_i, z_i) and $\tilde{F}$ be as above. We assume that a compact complex submanifold V' covered by $\{W_i\}_{i\in I}$ is defined in W_i by the equation:

$$w_i = \phi_i(z_i) .$$

Then, for such V', we associate an element

$$\phi = \{\phi_i\} \in C^0(||\ ||) .$$

ϕ must satisfy the compatibility conditions:

$$f_{ik}(\phi_k(z_k), z_k) = \phi_i(g_{ik}(\phi_k(z_k), z_k)) \text{ for } (\phi_k(z_k), z_k) \in W_i \cap W_k .$$

Conversely, an element $\phi = \{\phi_i\} \in C^0(||\ ||)$ which satisfies $||\phi|| < 1$ and the above compatibiliy conditions defines a complex submanifold V_ϕ of W by the equation:

$$w_i = \phi_i(z_i) .$$

We show that there is a small number $\varepsilon > 0$ such that V_ϕ is compact if $||\phi|| < \varepsilon$. For this purpose, we need the following lemma. The proof will be given at the end of this section.

LEMMA 3.1. *There is a small positive number ε such that if $||\phi|| < \varepsilon$ and if ϕ defines a submanifold V_ϕ, then V_ϕ is covered by $\{W_k^e\}_{k\in I}$.*

Now we show that V_ϕ is compact if $||\phi|| < \varepsilon$ where ε satisfies Lemma 3.1. Let $\{P^\nu\}_{\nu=1,2,\dots}$ be an arbitrary sequence of points of V_ϕ. By Lemma 3.1, $\{P^\nu\}_{\nu=1,2,\dots} \subset \bigcup W_i^e$. We want to choose a subsequence of $\{P^\nu\}_{\nu=1,2,\dots}$

* We use the same notation for the convenience.

converging to point of V_ϕ. Since the number $\#(I)$ of indices is finite, we may assume that P^ν belongs to a fixed W_i^e for all ν. We write $P^\nu = (w_i^\nu, z_i^\nu)$ in the local coordinate (w_i, z_i). Then

$$w_i^\nu = \phi_i(z_i^\nu) .$$

For each P^ν, we associate a point Q^ν in V defined by

$$Q^\nu = (0, z_i^\nu) \in U_i^e .$$

Since V is compact, we may assume that $\{Q^\nu\}_{\nu=1,2,\dots}$ itself converges to a point

$$Q = (0, z_i) \in U_i .$$

Now, we put

$$P = (\phi_i(z_i), z_i) \in W_i .$$

Then $P \in V_\phi$ and

$$\phi_i(z_i) = \phi_i(\lim_\nu z_i^\nu) = \lim_\nu \phi_i(z_i^\nu) = \lim_\nu w_i^\nu .$$

Hence $\{P^\nu\}_{\nu=1,2,\dots}$ converges to P. This shows that V_ϕ is compact.

Now, we need the following two lemmas. The proofs will be given at the end of this section.

LEMMA 3.2. *There is a small positive number ε such that if $|w_k| < \varepsilon$, then $(w_k, z_k) \in W_i \cap W_k$ for all $z_k \in U_i^e \cap U_k$.*

LEMMA 3.3. *Let e' be a small positive number greater than e such that the open sets*

$$W_i^{e'} = \{(w_i, z_i) \in W_i \mid |w_i| < 1,\ |z_i| < 1 - e'\} \qquad i \in I ,$$

again cover V. Then there is a small positive number ε such that if $|w_k| < \varepsilon$ and if $(w_k, z_k) \in W_i^{e'} \cap W_\kappa^{e'}$, then $z_k \in U_i^e \cap U_k$.

Now, let $B(\varepsilon)$ be the open ε-ball of $C^0(\|\ \|) = C_e^0(|\ |_e)$ with the center 0, where ε satisfies Lemmas 3.1, 3.2 and 3.3. We define a map

$$K\colon\ B(\varepsilon) \to C_e^1(|\ |_e)$$

by

$$(K\phi)_{ik}(z_i) = f_{ik}(\phi_k(z_k), z_k) - \phi_i(g_{ik}(\phi_k(z_k), z_k)) \qquad \text{for } z_i \in U_i^e \cap U_k ,$$

where $z_k = g_{ki}(0, z_i)$. Since $(\phi_k(z_k), z_k) \in W_i \cap W_k$ by Lemma 3.2, K maps $B(\varepsilon)$ into C_e^1. It is clear that $|K\phi|_e < 1 + \varepsilon$ so that K maps $B(\varepsilon)$ into $C_e^1(|\ |_e)$.

We assume that $\phi \in B(\varepsilon)$ satisfies $K\phi = 0$. If $z_k \in U_k$ satisfies

$$(\phi_k(z_k), z_k) \in W_i^{e'} \cap W_k^{e'} ,$$

then $z_k \in U_i^e \cap U_k$ by Lemma 3.3 so that

$$f_{ik}(\phi_k(z_k), z_k) = \phi_i(g_{ik}(\phi_k(z_k), z_k)) .$$

Thus the equations: $w_i = \phi_i(z_i)$ define a compact complex submanifold V_ϕ.

Conversely, we assume that $\phi \in B(\varepsilon)$ defines a compact complex submanifold V_ϕ defined by the equations: $w_i = \phi_i(z_i)$, then

$$f_{ik}(\phi_k(z_k), z_k) - \phi_i(g_{ik}(\phi_k(z_k), z_k)) = 0$$
$$\text{for } (\phi_k(z_k), z_k) \in W_i \cap W_k .$$

Hence $K\phi = 0$ by Lemma 3.2.

Thus the problem is reduced to analyze the set

$$\{\phi \in B(\varepsilon) \mid K\phi = 0\} .$$

LEMMA 3.4. *There is a small positive number $\varepsilon' < \varepsilon$ such that*

$$K: B(\varepsilon') \to C_e^1(|\ |_e)$$

is an analytic map and $K'(0) = \delta_e$.

PROOF. We want to show that there is a small positive number $\varepsilon' < \varepsilon$ such that for any affine line L in $C^0(\|\ \|)$, K is analytic map of $L \cap B(\varepsilon')$ into $C_e^1(|\ |_e)$. This implies that $K: B(\varepsilon') \to C_e^1(|\ |_e)$ is analytic. (See e.g., Proposition 2, [1]).

We take a point $\phi^0 \in L \cap B(\varepsilon')$. Then L can be written as

$$L(s) = \phi^0 + s\phi^1$$

where $s \in \boldsymbol{C}$ and $\phi^1 \in C^0(\|\ \|)$. We may assume that $\phi^1 \in B(\varepsilon')$ and $L(s) \in B(\varepsilon')$ for all $s \in \varDelta$ where $\varDelta$ is the unit disc in $\boldsymbol{C}$. Now, putting $z_k = g_{ki}(0, z_i)$, we have

$$\begin{aligned}(KL(s))_{ik}(z_i) &= f_{ik}(\phi_k^0(z_k) + s\phi_k^1(z_k), z_k)\\ &\quad -\phi_i^0(g_{ik}(\phi_k^0(z_k) + s\phi_k^1(z_k), z_k))\\ &\quad -s\phi_i^1(g_{ik}(\phi_k^0(z_k) + s\phi_k^1(z_k), z_k)) .\end{aligned}$$

We put

$$\begin{aligned}A(s)_{ik}(z_i) &= f_{ik}(\phi_k^0(z_k) + s\phi_k^1(z_k), z_k) , \quad A(s) = \{A(s)_{ik}\} ,\\ B(s)_{ik}(z_i) &= \phi_i^0(g_{ik}(\phi_k^0(z_k) + s\phi_k^1(z_k), z_k)) , \quad B(s) = \{B(s)_{ik}\} ,\\ C(s)_{ik}(z_i) &= s\phi_i^1(g_{ik}(\phi_k^0(z_k) + s\phi_k^1(z_k), z_k)) , \quad C(s) = \{C(s)_{ik}\} .\end{aligned}$$

We show that $B(s)$ is an analytic map of $\varDelta$ into $C_e^1(|\ |_e)$. Similar arguments show that $A(s)$ and $C(s)$ are analytic. We put

$$y = y(s) = g_{ik}(\phi_k^0(z_k) + s\phi_k^1(z_k), z_k) - g_{ik}(\phi_k^0(z_k), z_k) .$$

We also put $w_k = \phi_k^0(z_k)$, and $x = s\phi_k^1(z_k)$. Then

$$y = y(s) = g_{ik}(w_k + x, z_k) - g_{ik}(w_k, z_k) .$$

If $|w_k + x| < \varepsilon$, then $|g_{ik}(w_k + x, z_k)| < 1$ for all $z_k \in U_i^e \cap U_k$ by Lemma 3.2. Let ε' be a positive number smaller than ε. Then by Cauchy's estimate,

$$y \ll \sum x_1^{\nu_1} \cdots x_r^{\nu_r}/(\varepsilon - \varepsilon')^{\nu_1+\cdots+\nu_r} = D(x) \qquad \text{for } |w_k| < \varepsilon'$$

where $\sum$ is extended over all non-negative integers $\nu_1, \cdots, \nu_r$ with $\nu_1 + \cdots + \nu_r \geqq 1$ and $\ll$ means that the absolute values of the coefficients of y in the formal power series in $x_1, \cdots, x_r$ are less than the absolute values of the corresponding coefficients of $D(x)$. Hence

$$y = y(s) \ll \sum (\varepsilon' s)^{\nu_1+\cdots+\nu_r}/(\varepsilon - \varepsilon')^{\nu_1+\cdots+\nu_r} = E(s) .$$

$E(s)$ converges for $s \in \Delta$ and is equal to

$$\left(\frac{1}{1 - \varepsilon' s/(\varepsilon - \varepsilon')}\right)^r - 1$$

provided $\varepsilon' < \varepsilon/2$.

Taking ε' sufficiently small, we may assume that

$$|E(s)| < e/2 \qquad \text{for all } s \in \Delta .$$

Thus,

$$(1) \qquad |y(s)| < e/2 \qquad \text{for all } s \in \Delta .$$

Next, if $|w_k| < \varepsilon$ then by Cauchy's estimate,

$$g_{ik}(w_k, z_k) - g_{ik}(0, z_k) \ll \sum (w_k^1)^{\nu_1} \cdots (w_k^r)^{\nu_r}/\varepsilon^{\nu_1+\cdots+\nu_r} .$$

Thus, if $|w_k| < \varepsilon'$, then

$$|g_{ik}(w_k, z_k) - g_{ik}(0, z_k)| < \left(\frac{1}{1 - \varepsilon'/\varepsilon}\right)^r - 1 .$$

Taking ε' sufficiently small, we may assume that

$$\left(\frac{1}{1 - \varepsilon'/\varepsilon}\right)^r - 1 < e/2 .$$

thus

$$(2) \qquad |g_{ik}(w_k, z_k)| < 1 - e/2 \qquad \text{for } |w_k| < \varepsilon' \text{ and } z_k \in U_i^e \cap U_k .$$

Now,

$$(B(s) - B(0))_{ik}(z_i) = \phi_i^0(z_i' + y) - \phi_i^0(z_i')$$

where $z_i' = g_{ik}(\phi_k^0(z_k), z_k)$.

(2) shows that $|z_i'| < 1 - e/2$ when $\|\phi^0\| < \varepsilon'$. Thus, again by Cauchy's estimate, we have

$$\phi_i^0(z_i' + y) - \phi_i^0(z_i') \ll \sum \varepsilon' y_1^{\nu_1} \cdots y_d^{\nu_d}/(e/2)^{\nu_1+\cdots+\nu_d} = F(y) .$$

Thus

$$(B(s) - B(0))_{ik}(z_i) \ll F(E(s)) .$$

Hence

$$B(s) - B(0) \ll F(E(s)) .$$

$F(y)$ converges for $|y| < e/2$ so that $F(E(s))$ converges for all $s \in \Delta$ by (1). This shows that $B(s)$ is analytic.

Similar arguments show that $A(s)$ and $C(s)$ are analytic provided ε' is sufficiently small. Since ϕ^0 is an arbitrary point of $L \cap B(\varepsilon')$, K is analytic on $L \cap B(\varepsilon')$.

Finally, we show that $K'(0) = \delta_e$. Since $K(0) = 0$, $K\phi - K0 = K\phi$. Now

$$\begin{aligned}(K\phi)_{ik}(z_i) &= f_{ik}(\phi_k(z_k), z_k) - \phi_i(g_{ik}(\phi_k(z_k), z_k)) \\ &= [f_{ik}(\phi_k(z_k), z_k) - f_{ik}(0, z_k)] - \phi_i(z_i) \\ &\quad - [\phi_i(g_{ik}(\phi_k(z_k), z_k)) - \phi_i(g_{ik}(0, z_k))] \\ &= F_{ik}(z_k)\phi_k(z_k) + o(\phi) - \phi_i(z_i) \\ &\quad - (\partial\phi_i/\partial z_i)_{z_i}(\partial g_{ik}/\partial w_k)_{(0,z_k)}\phi_k(z_k) + o(\phi)\end{aligned}$$

where $z_k = g_{ki}(0, z_i)$ and $o(\phi)$ is some function of ϕ (and of z_k) such that

$$|o(\phi)|/\|\phi\| \to 0 \qquad \text{as } \|\phi\| \to 0 .$$

There is a constant M_1 such that

$$|(\partial g_{ik}/\partial w_k)_{(0,z_k)}| \leqq M_1 \qquad \text{for } z_k \in U_i^e \cap U_k .$$

On the other hand, there is a constant M_2 such that

$$|(\partial\phi_i/\partial z_i)_{z_i}| \leqq M_2\|\phi\| \qquad \text{for } z_i \in U_i^e .$$

Thus,

$$(\partial\phi_i/\partial z_i)_{z_i}(\partial g_{ik}/\partial w_k)_{(0,z_k)}\phi_k(z_k) = o(\phi) .$$

Hence

$$(K\phi)_{ik}(z_i) = (\delta_e\phi)_{ik}(z_i) + o(\phi)$$

so that

$$K\phi = \delta_e\phi + o(\phi) \qquad q.e.d.$$

Let ε' be a small positive number such that Lemma 3.4 holds. We define an analytic map $L\colon B(\varepsilon') \to C^0(\| \ \|)$ by

$$L\phi = \phi + E_0 B\Lambda K\phi - E_0\delta_e\phi$$

where E_0, B and Λ are continuous linear maps defined in § 2. Then we get

$$L(0) = 0\ ,$$
$$L'(0) = 1 + E_0 B\Lambda\delta_e - E_0\delta_e = 1 + E_0\delta_e - E_0\delta_e = 1\ .$$

By the inverse mapping theorem, L is an analytic isomorphism of an open neighbourhood Ω of 0 in $B(\varepsilon')$ onto an open neighbourhood Ω' of 0 in $C^0(\| \ \|)$. We put

$$M = \{\phi \in \Omega \mid K\phi = 0\}\ .$$

LEMMA 3.5. $L(M) \subset H^0(V, \tilde{F})$.

PROOF. If $\phi \in M$, then $\delta_e L\phi = \delta_e(\phi - E_0\delta_e\phi) = \delta_e\phi - \delta_e\phi = 0$ *q.e.d.*
We put $\Phi = L^{-1}\colon \Omega' \to \Omega$ and put

$$S = \{s \in H^0(V, \tilde{F}) \cap \Omega' \mid K(\Phi(s)) = 0\}\ .$$

Then it is clear that $S = L(M)$ and $\Phi(S) = M$. Let $s \in H^0(V, \tilde{F}) \cap \Omega'$. We put $\phi = \Phi s$. Then

$$0 = \delta_e s = \delta_e L\phi = \delta_e\phi + \delta_e E_0 B\Lambda K\phi - \delta_e E_0\delta_e\phi = B\Lambda K\phi\ .$$

Let H be the projection map defined at the end of § 2. Then

$$\begin{aligned} K\Phi(s) &= B\Lambda K\phi + H\Lambda K\phi + E\delta_e K\phi \\ &= H\Lambda K\phi + E\delta_e K\phi \\ &= H\Lambda K\Phi(s) + E\delta_e K\Phi(s)\ . \end{aligned}$$

In other words, $K\Phi(s)$ has no coboundary part.

LEMMA 3.6. *Taking Ω' sufficiently small, we have*

$$S = \{s \in H^0(V, \tilde{F}) \cap \Omega' \mid H\Lambda K\Phi(s) = 0\}\ .$$

PROOF. Let e' be a small positive number greater than e such that the open sets

$$W_i^{e'} = \{(w_i, z_i) \in W_i \mid |w_i| < 1,\ |z_i| < 1 - e'\}\ ,\quad i \in I\ ,\quad \text{cover } V\ .$$

We put $U_i^{e'} = W_i^{e'} \cap V$. We introduce a norm $|\ |_{e'}$ in C_e^2 as follows: for each $\xi = \{\xi_{ijk}\} \in C_e^2$, we define $|\xi|_{e'}$ by

$$|\xi|_{e'} = \sup\{|\xi_{ijk}^\lambda(z)|\colon \lambda = 1, \cdots, r,\ z \in U_i^{e'} \cap U_j^e \cap U_k,\ (i, j, k) \in I^3\}$$

where ξ_{ijk}^λ is the representation of the component ξ_{ijk} of ξ with respect to the coordinate (w_i, z_i). Then it is clear that

(1) $$|\xi|_{e'} \leqq |\xi|_e \qquad \text{for } \xi \in C_e^2(|\ |_e) .$$

We show that there is a constant $c_0 > 0$ such that

(2) $$|\xi|_e \leqq c_0 |\xi|_{e'} \qquad \text{for } \xi \in Z_e^2(|\ |_e) .$$

Let $z_i \in U_i^e \cap U_j^e \cap U_k$. Since $\{U_i^{e'}\}$ covers V, there is an index l such that $z_i \in U_l^{e'}$. Since $\xi \in Z_e^2(|\ |_e)$, we have

$$\xi_{ijk}(z_i) = F_{il}(z_l)\xi_{ljk}(z_l) - F_{il}(z_l)\xi_{lik}(z_l) + F_{il}(z_l)\xi_{lij}(z_l) ,$$

where $z_l = g_{li}(0, z_i)$. Thus

$$|\xi|_e \leqq 3r \,||F||\, |\xi|_{e'}$$

where $||F|| = \sup\{|F_{ik\nu}^{\lambda}(z_k)|: \lambda, \nu = 1, \cdots, r, i, k \in I, z_k \in U_i \cap U_k\}$.

Now, let $z_i \in U_i^e \cap U_j^e \cap U_k$. Taking $||\phi||$ sufficiently small, we may assume that $(\phi_k(z_k), z_k) \in W_i^{e/2} \cap W_j^{e/2} \cap W_k$ where $z_k = g_{ki}(0, z_i)$ and

$$W_i^{e/2} = \{(w_i, z_i) \in W_i \mid |w_i| < 1,\ |z_i| < 1 - e/2\} .$$

This follows from Lemma 3.2 by replacing W_i to $W_i^{e/2}$ (and W_j to $W_j^{e/2}$). We put $\zeta_j = g_{jk}(\phi_k(z_k), z_k)$. Then

$$\phi_j(\zeta_j) = f_{jk}(\phi_k(z_k), z_k) - (K\phi)_{jk}(z_j) \quad \text{where } z_j = g_{ji}(0, z_i) .$$

Again, taking $||\phi||$ sufficiently small, we may assume that

$$\zeta_j \in U_i^{e/3} \cap U_j^{e/2} \quad \text{and} \quad (\phi_j(\zeta_j), \zeta_j) \in W_i^{e/4} \cap W_j^{e/2}$$

where

$$U_i^{e/3} = \{(0, z_i) \in U_i \mid |z_i| < 1 - e/3\} ,$$
$$W_i^{e/5} = \{(w_i, z_i) \in W_i \mid |w_i| < 1,\ |z_i| < 1 - e/4\} .$$

We put

$$\begin{aligned} R^1(k\phi, \phi) &= \{R^1(k\phi, \phi)_{ijk}\} , \\ R^1(K\phi, \phi)_{ijk}(z_i) &= f_{ij}(\phi_j(\zeta_j), \zeta_j) - f_{ik}(\phi_k(z_k), z_k) + F_{ij}(z_j)(K\phi)_{jk}(z_k) \\ &= f_{ij}(\phi_j(\zeta_j), \zeta_j) - f_{ij}(f_{jk}(\phi_k(z_k), z_k), \zeta_j) + F_{ij}(z_j)(K\phi)_{jk}(z_j) . \end{aligned}$$

$R^1(K\phi, \phi)$ is an element of $C_e^2(|\ |_e)$. Then it is easy to see that there is a constant c_1 such that

(3) $$|R^1(K\phi, \phi)|_e \leqq c_1 |K\phi|_e \,||\phi|| ,$$

provided $|K\phi|_e$ and $||\phi||$ are sufficiently small.

This follows from the mean value theorem applied on the real and the imaginary parts of the functions $f_{ij}^{\lambda}(w_j, z_j)$, $\lambda = 1, \cdots, r$.

In a similar way, if we put

$$R^2(K\phi, \phi) = \{R^2(K\phi, \phi)_{ijk}\} \in C_e^2(|\ |_e) ,$$
$$R^2(K\phi, \phi)_{ijk}(z_i) = \phi_i(g_{ik}(\phi_k(z_k), z_k)) - \phi_i(g_{ij}(\phi_j(\zeta_j), \zeta_j))$$

then, we can show that there is a constant c_2 such that

$$(4) \qquad | R^2(K\phi, \phi) |_e \leqq c_2 | K\phi |_e \| \phi \| .$$

Now, we put

$$R^3(K\phi, \phi) = \{R^3(K\phi, \phi)_{ijk}\} \in C_e^2(|\ |_e) ,$$
$$R^3(K\phi, \phi)_{ijk}(z_i) = f_{ij}(\phi_j(\zeta_j), \zeta_j) - \phi_i(g_{ij}(\phi_j(\zeta_j), \zeta_j)) - (K\phi)_{ij}(z_i) .$$

We assume that z_i belongs to $U_i^{e'} \cap U_j^e \cap U_k$. Then, taking $\| \phi \|$ sufficiently small, we may assume that $\zeta_j = g_{jk}(\phi_k(z_k), z_k) \in U_i^e \cap U_j^{e/2}$. Thus

$$R^3(K\phi, \phi)_{ijk}(z_i) = (K\phi)_{ij}(\zeta_i) - (K\phi)_{ij}(z_i)$$
$$\text{where } \zeta_i = g_{ij}(0, \zeta_j) .$$

Applying the mean value theorem on the real and the imaginary parts of the functions $(K\phi)_{ij}^\lambda(z_i)$, $\lambda = 1, \cdots, r$, we have

$$| R^3(K\phi, \phi)_{ijk}(z_i) | \leqq c_3 | K\phi |_e \| \phi \|$$

with c_3 constant. Hence

$$(5) \qquad | R^3(K\phi, \phi) |_{e'} \leqq c_3 | K\phi |_e \| \phi \| .$$

Now, we assume that z_i belongs to $U_i^e \cap U_j^e \cap U_k$. Then

$$\begin{aligned}(K\phi)_{ij}(z_i) &= f_{ik}(\phi_k(z_k), z_k) - \phi_i(g_{ik}(\phi_k(z_k), z_k)) \\ &= f_{ij}(\phi_j(\zeta_j), \zeta_j) - \phi_i(g_{ij}(\phi_j(\zeta_j), \zeta_j)) \\ &\quad + F_{ij}(z_j)(K\phi)_{jk}(z_j) - (R^1(K\phi, \phi) + R^2(K\phi, \phi))_{ijk}(z_i) \\ &= (K\phi)_{ij}(z_i) + F_{ij}(z_j)(K\phi)_{jk}(z_j) \\ &\quad - (R^1(K\phi, \phi) + R^2(K\phi, \phi) - R^3(K\phi, \phi))_{ijk}(z_i) .\end{aligned}$$

Hence we have

$$\delta_e K\phi = R^1(K\phi, \phi) + R^2(K\phi, \phi) - R^3(K\phi, \phi) .$$

By (1), (3), (4) and (5), we have

$$| \delta_e K\phi |_{e'} \leqq c_4 | K\phi |_e \| \phi \|$$

with c_4 a constant. Thus, by (2),

$$| \delta_e K\phi |_e \leqq c_0 c_4 | K\phi |_e \| \phi \| .$$

Thus there is a constant c such that

$$| E\delta_e K\phi |_e \leqq c | K\phi |_e \| \phi \| .$$

Now, let $s \in H^0(V, \tilde{F}) \cap \Omega'$. Taking Ω' sufficiently small, we may assume that

$$\| \Phi(s) \| < 1/c \qquad \text{for } s \in H^0(V, \tilde{F}) \cap \Omega' .$$

Now, $K\Phi(s) = H\Lambda K\Phi(s) + E\delta_e K\Phi(s)$. We assume that $H\Lambda K\Phi(s) = 0$. Then

$$| K\Phi(s) |_e = | E\delta_e K\Phi(s) |_e \leqq c \, | K\Phi(s) |_e \, \| \Phi(s) \| .$$

If $K\Phi(s) \neq 0$, then $1 \leqq c \| \Phi(s) \|$, i.e., $1/c \leqq \| \Phi(s) \|$, a contradiction. Hence $K\Phi(s) = 0$. *q.e.d.*

Now, for each $s \in S$, $\Phi(s) = \{\Phi_i(z_i, s)\}$ defines a compact complex submanifold V_s and $\Phi_i(z_i, s)$ is a vector valued holomorphic function of

$$(z_i, s) \in U_i \times S .$$

This is easily seen, because for each fixed $z_i \in U_i$,

$$\phi \in C^0(\| \ \|) \longrightarrow \phi_i(z_i) \in C^r$$

is a continuous linear map, so that

$$s \in S \longrightarrow \Phi(s) \longrightarrow \Phi_i(z_i, s)$$

is an analytic map. Thus $\{V_s\}_{s \in S}$ forms a family (X, π, S) of compact complex submanifolds of W.

We show that (X, π, S) is a maximal family. Let $s_0 \in S$. Let (Y, μ, T) be a family of compact complex submanifolds of W with a point $p \in T$ such that $\mu^{-1}(p) = V_{s_0}$. Let $w'_i = w_i - \Phi_i(z_i, s_0)$. Then (shrinking T if necessary) we may assume that there are vector valued holomorphic functions $\theta'_i(z_i, t)$ on $U_i \times T$ such that $\theta'_i(z_i, p) = 0$ and that the equation $w'_i = \theta'_i(z_i, t)$ defines the submanifold $\mu^{-1}(t)$. We put

$$\theta_i(z_i, t) = \theta'_i(z_i, t) + \Phi_i(z_i, s_0)$$

and

$$\theta(t) = \{\theta_i(z_i, t)\} \in C^0(\| \ \|) .$$

Then it is easy to see that

$$\theta\colon t \in T \longrightarrow \theta(t) \in C^0(\| \ \|)$$

is an analytic map. We may assume that $\theta(T) \subset \Omega$. We have $K(\theta(t)) = 0$. Let $f(t) = L(\theta(t))$. Then f is a holomorphic map of T into S with $f(p) = s_0$. We have $\Phi(f(t)) = \theta(t)$. Hence we get $\mu^{-1}(t) = \pi^{-1}(f(t))$.

This completes the proof of Theorem 1.

REMARK 3.1. It is clear that the map f, with the property: $\mu^{-1}(t) = \pi^{-1}(f(t))$ for $t \in T$, is uniquely determined.

REMARK 3.2. If $H^1(V, \tilde{F}) = 0$, then $S = H^0(V, \tilde{F}) \cap \Omega'$ by Lemma 3.6. This is Kodaira's case (see the Introduction).

PROOF OF LEMMA 3.1. Let $\pi_k: \tilde{W}_k \to \tilde{U}_k$ be the projection map defined by $\pi_k(w_k, z_k) = z_k$. For each positive integer n, we set

$$W_k(n) = \{(w_k, z_k) \in \tilde{W}_k \mid |w_k| < 1/n, |z_k| < 1\}$$

and

$$A_k(n) = \overline{W_k(n)} \qquad \text{(the closure in } W) .$$

Since $A_k(n)$ is compact,

$$A(n) = \bigcup_{k \in I} A_k(n)$$

is also compact. It is clear that $A(n)$ contains V. We show that

$$\bigcap_{n=1}^{\infty} A(n) = V .$$

Let $b \in \bigcap_{n=1}^{\infty} A(n)$. Then there are an index $k \in I$ and a subsequence

$$n_1 < n_2 < \cdots \text{ such that } b \in A_k(n_\nu) \qquad \text{for } \nu = 1, 2, \cdots .$$

Then

$$|w_k(b)| \leqq 1/n_\nu , \qquad \nu = 1, 2, \cdots .$$

Hence $w_k(b) = 0$ so that $b \in V$. If we find an integer n such that

$$A(n) \subset \bigcup_k W_k^e$$

then $\varepsilon = 1/n$ satisfies Lemma 3.1. Thus the proof of Lemma 3.1 reduces to the following lemma. The proof is straightforward.

LEMMA 3.7. *Let A be a compact subset of a Hausdorff space X. Let $A(n)$ be compact subsets of X such that*

(1) $$A(1) \supset A(2) \supset \cdots \supset A ,$$

(2) $$\bigcap_{n=1}^{\infty} A(n) = A .$$

Let U be an open neighbourhood of A. Then there exists an integer n such that $U \supset A(n)$.

PROOF OF LEMMA 3.2. Let $W_k(n)$ be as in the proof of Lemma 3.1. We put

$$A(n) = \overline{W_k(n)} \cap \pi_k^{-1}(\bar{U}_i^e \cap \bar{U}_k) .$$

Here, the closure is taken in W.

First of all, we show that $A(n)$ is a closed subset of W. Let $\{b_\nu\}_{\nu=1,2,\cdots}$ be a sequence of points of $A(n)$ converging to a point b of W. b belongs to $\overline{W_k(n)}$. Since $\pi_k(b_\nu) \in \bar{U}_i^e \cap \bar{U}_k$ converges to $\pi_k(b)$, b also belongs to $\pi_k^{-1}(\bar{U}_i^e \cap \bar{U}_k)$. This shows that $A(n)$ is closed. Since $A(n) \subset \overline{W_k(n)}$ and the later is compact, $A(n)$ is also compact. We put $A = \bar{U}_i^e \cap \bar{U}_k$. In order to apply lemma 3.7 to $X = W$, $U = W_i$, it is enough to see that $\bigcap_{n=1}^\infty A(n) = A$. Let $b \in \bigcap_{n=1}^\infty A(n)$. Then

$$|w_k(b)| \leqq 1/n, \quad n = 1, 2, \cdots$$

and

$$|z_k(b)| \leqq 1 .$$

Thus $w_k(b) = 0$ so that $b \in \bar{U}_k$. Since $b \in \pi_k^{-1}(\bar{U}_i^e \cap \bar{U}_k)$, $\pi_k(b) = b \in \bar{U}_i^e \cap \bar{U}_k$. Hence $b \in A$.

Thus there is an integer n such that $A(n) \subset W_i$. Hence

$$W_k(n) \cap \pi_k^{-1}(U_i^e \cap U_k) \subset A(n) \subset W_i .$$

On the other hand, $W_k(n) \cap \pi_k^{-1}(U_i^e \cap U_k)$ is contained in W_k. Hence

$$W_k(n) \cap \pi_k^{-1}(U_i^e \cap U_k) \subset W_i \cap W_k .$$

Now, $\varepsilon = 1/n$ satisfies the requirement. *q.e.d.*

PROOF OF LEMMA 3.3. Let $W_k(n)$ be as above. In order to prove the lemma, it is enough to find an integer n such that

$$W_i^{e'} \cap W_k^{e'} \cap W_k(n) \subset \pi_k^{-1}(U_i^e \cap U_k) .$$

We put

$$A(n) = \overline{W_i^{e'}} \cap \overline{W_k^{e'}} \cap \overline{W_k(n)}$$

and

$$A = \overline{U_i^{e'}} \cap \overline{U_k^{e'}} .$$

$A(n)$ and A are compact.

We claim that $\bigcap_{n=1}^\infty A(n) = A$. Let $b \in \bigcap_{n=1}^\infty A(n)$. Then $|w_k(b)| \leqq 1/n$, $n = 1, 2, \cdots$. Hence $w_k(b) = 0$. $b \in \overline{W_i^{e'}} \cap \overline{W_k^{e'}}$ implies that $|z_k(b)| \leqq 1 - e'$ and $|z_i(b)| \leqq 1 - e'$. On the other hand,

$$w_i(b) = f_{ik}(w_k(b), z_k(b)) = f_{ik}(0, z_k(b)) = 0 .$$

Hence $b \in \overline{U_i^{e'}} \cap \overline{U_k^{e'}} = A$. Now, we apply Lemma 3.7 to the case $X = W$, $U = \pi_k^{-1}(U_i^e \cap U_k)$. Thus there is an integer n such that

$$A(n) \subset \pi_k^{-1}(U_i^e \cap U_k) .$$

Hence

$$W_i^{e'} \cap W_k^{e'} \cap W_k(n) \subset A(n) \subset \pi_k^{-1}(U_i^e \cap U_k) . \qquad q.e.d.$$

4. **Proof of Theorem 2.** Let W be a complex manifold. Let $S(W)$ be the set of all compact complex submanifolds of W. Let V be an element of $S(W)$. Let (X, π, S) be the maximal family with the center V constructed in § 3. Two different points $s \neq t$ in S have different fibers $\pi^{-1}(s) \neq \pi^{-1}(t)$. Thus, there is a unique injective map

$$S \to S(W)$$

defined by $s \to \pi^{-1}(s)$. We want to take this map as a local chart around $V \in S(W)$. Using the maximality of (X, π, S) and Remark 3.1, these local charts patch up to give a (locally finite dimensional, not necessarily connected) analytic space structure on $S(W)$.

We prove that the underlying topological space of $S(W)$ is a Hausdorff space. For this purpose we need the following two lemmas.

LEMMA 4.1. *Let W be a metric space with metric d. Let $C(W)$ be the set of all compact subsets of W. For any two elements A and B in $C(W)$, we associate a number $d'(A, B)$ defined by*
$d'(A, B) = \sup\{d(x, B) \mid x \in A\} + \sup\{d(A, y) \mid y \in B\}$. *Then d' is a metric on $C(W)$.*

PROOF. It is easy to check that d' satisfies the three axioms for metric. *q.e.d.*

LEMMA 4.2. *Let (X, π, S) be a family of compact complex submanifolds of W. With an Hermitian metric on W, we regard W as a metric space (W, d). Let $o \in S$. Then $d'(\pi^{-1}(s), \pi^{-1}(o))$ is a continuous function of $s \in S$, where d' is the metric in $C(W)$ introduced in Lemma 4.1.*

PROOF. It suffices to prove that

$$d'(\pi^{-1}(s),, \pi^{-1}(o)) \to 0 \quad \text{as} \quad s \to o .$$

It is known [7] that there is an open neighbourhood S' of o in S and a continuous retraction

$$R: \pi^{-1}(S') \to \pi^{-1}(o)$$

such that $R \mid \pi^{-1}(s)$ is a C^∞-diffeomorphism of $\pi^{-1}(s)$ onto $\pi^{-1}(o)$ for each $s \in S'$. We fix on point $s \in S'$. We take a point $P \in \pi^{-1}(s)$. Then

$$d(P, \pi^{-1}(o)) \leqq d(P, R(P)) .$$

Hence

$$\sup\{d(P, \pi^{-1}(o)) \mid P \in \pi^{-1}(s)\} \leqq \sup\{d(P, R(P)) \mid P \in \pi^{-1}(s)\} .$$

The right hand side is finite, for R is continuous and $\pi^{-1}(s)$ is compact. In a similar way, we get

$$\sup\{d(Q, \pi^{-1}(s)) \mid Q \in \pi^{-1}(o)\} \leqq \sup\{d(Q, (R \mid \pi^{-1}(s))^{-1}(Q)) \mid Q \in \pi^{-1}(o)\}\,.$$

Hence

$$d'(\pi^{-1}(s), \pi^{-1}(o)) \leqq 2 \sup\{d(P, R(P)) \mid P \in \pi^{-1}(s)\}\,.$$

We show that

$$\sup\{d(P, R(P)) \mid P \in \pi^{-1}(s)\} \to 0 \quad \text{as} \quad s \to o\,.$$

We assume the converse. Then there are a positive number $\varepsilon > 0$, a sequence $\{s_n\}_{n=1,2,\dots}$ of points of S' converging to o and a sequence $\{P_n\}_{n=1,2,\dots}$ of points of $\pi^{-1}(S')$ with $P_n \in \pi^{-1}(s_n)$, $n = 1, 2, \cdots$ such that

$$d(P_n, R(P_n)) > \varepsilon\,, \quad n = 1, 2, \cdots\,.$$

Since each fiber $\pi^{-1}(s)$ is compact, there is a subsequence $n_1 < n_2 < \cdots$ such that $\{P_{n_\nu}\}_{\nu=1,2,\dots}$ converges to a point $P \in \pi^{-1}(o)$. Then

$$\varepsilon \leqq d(P, R(P)) = d(P, P) = 0, \text{ a contradiction.} \qquad q.e.d.$$

Now, it is easy to prove that $S(W)$ is a Hausdorff space. Using an Hermitian metric on W, we regard W as a metric space (W, d). Let d' be the metric on the set $C(W)$ of all compact subsets of W defined in Lemma 4.1. Lemma 4.2 asserts that the identity map

$$I\colon S(W) \to (S(W), d')$$

is a continuos map where $(S(W), d')$ is a metric space with the metric d'. Since $(S(W), d')$ is a Hausdorff space, $S(W)$ is also a Hausdorff space. This completes the proof of Theorem 2.

Henceforth, for each point $t \in S(W)$, we denote V_t the corresponding compact complex submanifold in W. Let

$$X(W) = \{(P, t) \in W \times S(W) \mid P \in V_t\}\,.$$

We first show that $X(W)$ is closed in $W \times S(W)$. Let (P_n, t_n), $n = 1, 2, \cdots$ be a sequence in $X(W)$ which converges to a point $(P, t) \in W \times S(W)$. We claim that $P \in V_t$.

$$\begin{aligned} d(P, V_t) &\leqq d(P, P_n) + d(P_n, V_t) \\ &\leqq d(P, P_n) + \sup\{d(Q, V_t) \mid Q \in V_{t_n}\} \\ &\leqq d(P, P_n) + d'(V_{t_n}, V_t) \\ &\to 0 \quad \text{as} \quad n \to \infty \quad \text{by Lemma 4.2.} \end{aligned}$$

Hence $d(P, V_t) = 0$ so that $P \in V_t$.

Let $(P, t) \in X(W)$. There is a maximal family (X, π, S) where S is an open neighbourhood of t in $S(W)$ such that

$$X = \{(Q, s) \in W \times S \mid Q \in V_s\} .$$

X is a subvariety of $W \times S$ containing (P, t). Since

$$X = X(W) \cap (W \times S) ,$$

X is an open subset of $X(W)$ containing (P, t). Hence $X(W)$ is a subvariety of $W \times S(W)$. Let $\tilde{\pi}$: $W \times S(W) \rightarrow S(W)$ be the projection map. Then (X, π, S) is the restriction of $(X(W), \tilde{\pi}, S(W))$ to the open subset S of $S(W)$. Thus we conclude that $(X(W), \tilde{\pi}, S(W))$ is a maximal family of compact complex submanifolds of W.

It is clear that the family $(X(W), \tilde{\pi}, S(W))$ has the following universal property:

For any family (X, π, S) of compact complex submanifolds of W, there is a unique holomorphic map

$$f\colon S \rightarrow S(W)$$

such that $\pi^{-1}(s) = \tilde{\pi}^{-1}(f(s))$ for all $s \in S$.

In the rest of this section, we show that $S(W)$ is identified with an open subset of the Douady space $D = D(W)$. Douady [1] constructed a flat family (Z, η, D) of all (not necessarily reduced) compact complex subvarieties of W. We consider the reduced analytic space D_{red} associated to D. Let

$$S'(W) = \{t \in D_{\mathrm{red}} \mid \eta^{-1}(t) \text{ is non-singular}\} .$$

By Theorem 3.1 and Corollary 3.3 in VI, [2], we see that $S'(W)$ is an open subset of D_{red} and that the triple $(\eta_{\mathrm{red}}^{-1} (S'(W)), \eta_{\mathrm{red}}, S'(W))$ forms a family of compact complex submanifods of W in our sense where η_{red}: $Z_{\mathrm{red}} \rightarrow D_{\mathrm{red}}$ is the holomorphic map associated to η.

We note that $\eta_{\mathrm{red}}^{-1}(t) = \eta^{-1}(t)_{\mathrm{red}} = \eta^{-1}(t)$ for $t \in S'(W)$. Let $(X(W), \tilde{\pi}, S(W))$ be the family which we constructed above. There is a unique holomorphic map

$$f\colon S'(W) \rightarrow S(W)$$

such that $\eta_{\mathrm{red}}^{-1}(t) = \tilde{\pi}^{-1}(f(t))$ for all $t \in S'(W)$. On the other hand, by the universal property of the family (Z, η, D), there is a unique morphism

$$g\colon S(W) \rightarrow D$$

such that $\tilde{\pi}^{-1}(s) = \eta^{-1}(g(s))$ for all $s \in S(W)$.

Since $S(W)$ is reduced, there is a unique holomorphic map

$$g_0\colon S(W) \to D_{\text{red}}$$

such that the diagram

$$\begin{array}{ccc} S(W) & \xrightarrow{g_0} & D_{\text{red}} \\ & \searrow^{g} & \downarrow \\ & & D \end{array}$$

is commutative, where $D_{\text{red}} \to D$ is the canonical morphism. Now it is clear that $g_0(S(W)) = S'(W)$ and that

$$g_0 f = I_{S'(W)} ,$$
$$f g_0 = I_{S(W)}$$

where $I_{S'(W)}$ and $I_{S(W)}$ are the identity maps of $A'(W)$ and $S(W)$ respectively. Hence f is a holomorphic isomorphism. Thus we conclude that $S(W)$ is identified with an open subspace of D_{red}.

5. **Proof of Theorem 3.** Let $S(W)$ be the analytic space constructed in § 4. We put

$$A = \{(s, t) \in S(W) \times S(W) \mid V_s \subset V_t\} .$$

We first show that A is closed in $S(W) \times S(W)$. Let (s_n, t_n), $n = 1, 2, \cdots$ be a sequence of points in A converging to a point $(s, t) \in S(W) \times S(W)$. We assume that $V_s \not\subset V_t$. Then there is a point $P \in V_s - V_t$. We put $\varepsilon = d(P, V_t) > 0$ where d is an hermitian metric on W. Let d' be the metric on the set of all compact subsets in W defined in Lemma 4.1. By Lemma 4.2, we can choose n so large that

$$(1) \qquad d'(V_s, V_{s_n}) < \varepsilon/2$$

$$(2) \qquad d'(V_t, V_{t_n}) < \varepsilon/2 .$$

By (1), there is a point $P_n \in V_{s_n}$ such that $d(P, P_n) < \varepsilon/2$. We note that $P_n \in V_{s_n} \subset V_{t_n}$ by the assumption. Hence

$$\begin{aligned} d'(V_t, V_{t_n}) &= \sup\{d(Q, V_{t_n}) \mid Q \in V_t\} + \sup\{d(Q', V_t) \mid Q' \in V_{t_n}\} \\ &\geqq \sup\{d(Q', V_t) \mid Q' \in V_{t_n}\} \\ &\geqq d(P_n, V_t) \geqq d(P, V_t) - d(P_n, P) \\ &= \varepsilon - d(P_n, P) > \varepsilon/2 . \end{aligned}$$

This contradicts to (2). Thus $V_s \subset V_t$. Hence A is closed in $S(W) \times S(W)$.

Now, let X and V be two compact complex submanifolds in W such that $V \subset X$. Let $\{\tilde{W}_i\}_{i \in I}$ and $\{W_i\}_{i \in I}$ be finite coverings of V by open subsets $\tilde{W}_i$ and W_i in W such that $\tilde{W}_i \supset \bar{W}_i$ (the closure in W) for all $i \in I$. Moreover, we assume that there is on each $\tilde{W}_i$ a local coordinate system

$$(u_i, w_i, z_i) = (u_i^1, \cdots, u_i^q, w_i^1, \cdots, w_i^r, z_i^1, \cdots, z_i^d)$$

such that

$$W_i = \{(u_i, w_i, z_i) \in \tilde{W}_i \mid |u_i| < 1, |w_i| < 1, |z_i| < 1\}$$

and such that X and V are defined in $\tilde{W}_i$ by the equations

$$X: u_i = 0 ,$$
$$V: u_i = w_i = 0 .$$

We assume that $\{W_i \cap V\}$ and $\{\tilde{W}_i \cap V\}$ satisfy the similar conditions to those in § 2.

Let $\{\tilde{W}_\gamma\}_{\gamma \in \Gamma}$ and $\{W_\gamma\}_{\gamma \in \Gamma}$ be finite collections of open subsets of W having the following properties:

(a) $\tilde{W}_\gamma \supset \bar{W}_\gamma$ (the closure in W) for all $\gamma \in \Gamma$,

(b) each $\tilde{W}_\gamma$ does not intersect with V,

(c) $\{W_i\}_{i \in I} \cup \{W_\gamma\}_{\gamma \in \Gamma}$ is a covering of X,

(d) there is on each $\tilde{W}_\gamma$ a local coordinate system

$$(u_\gamma, v_\gamma) = (u_\gamma^1, \cdots, u_\gamma^q, v_\gamma^1, \cdots, v_\gamma^{r+d})$$

such that

$$W_\gamma = \{(u_\gamma, v_\gamma) \in \tilde{W}_\gamma \mid |u_\gamma| < 1, |v_\gamma| < 1\}$$

and such that X is defined in $\tilde{W}_\gamma$ by the equation: $u_\gamma = 0$.

(e) $\{W_i \cap X\} \cup \{W_\gamma \cap X\}$ and $\{\tilde{W}_i \cap X\} \cup \{\tilde{W}_\gamma \cap X\}$ satisfy the similar conditions to those in § 2.

Let F be the normal bundle of X in W and let G be the restriction of F on V. Let H and N be the normal bundles of V in W and in X respectively. Then we have an exact sequence

$$(3) \qquad 0 \to N \to H \to G \to 0 .$$

Let $C^0(V, H) = C^0(V, H, \{W_i \cap V\}_{i \in I})$, etc., be the 0-th cochain groups of the sheaf $\tilde{H}$, etc., over V of germs of holomorphic sections of H, etc., on the nerve of the covering $\{W_i \cap V\}_{i \in I}$. Then, by (3), we have a canonical isomorphism

$$(4) \qquad C^0(V, H) \cong C^0(V, N) \oplus C^0(V, G) .$$

Let

$$\alpha: C^0(V, H) \rightarrow C^0(V, N) ,$$
$$\beta: C^0(V, H) \rightarrow C^0(V, G)$$

be the projection maps with respect to (4). Let $\| \ \|$ be norms in $C^0(V, H)$, etc., defined as in § 2. Let $C^0(V, H, \| \ \|)$, etc., be the Banach spaces of elements in $C^0(V, H)$, etc., with finite norms. Then it is clear that α and β are continuous linear maps of $C^0(V, H, \| \ \|)$ into $C^0(V, N, \| \ \|)$ and $C^0(V, G, \| \ \|)$ respectively.

We put $Y = X \cap (\bigcup_{i \in I} W_i)$. Then Y is an open subset of X containing V. Let $C^0(X, F)$ (resp. $C^0(Y, F \mid Y)$) be the 0-th cochain group of the sheaf $\tilde{F}$ over X (resp. $\widetilde{F \mid Y}$ over Y) of germs of holomorphic sections of F (resp. $F \mid Y$) on the nerve of the covering $\{W_i \cap X\}_{i \in I} \cup \{W_\gamma \cap X\}_{\gamma \in \Gamma}$ (resp. $\{W_i X\}_{i \in I}$). Then we have a canonical linear map

$$l: C^0(X, F) \rightarrow C^0(Y, F \mid Y) .$$

We introduce a norm $\| \ \|$ in $C^0(X, F)$ (resp. $C^0(Y, F \mid Y)$) and define a Banach space $C^0(X, F, \| \ \|)$ (resp. $C^0(Y, F \mid Y, \| \ \|)$) as in § 2. Then l maps $C^0(X, F, \| \ \|)$ continuously into $C^0(Y, F \mid Y, \| \ \|)$.

Now, let X' and V' be compact complex submanifolds of W near from X and V respectively, defined in W_i by the equations:

$$X': u_i = \lambda_i(w_i, z_i) ,$$
$$V': u_i = \psi_i(z_i), \quad w_i = \phi_i(z_i) .$$

Then $V' \subset X'$ if and only if

$$\psi_i(z_i) = \lambda_i(\phi_i(z_i), z_i) \quad \text{for} \quad z_i \in W_i \cap V .$$

We assume that X' is defined in W_γ, by the equation:

$$u_\gamma = \lambda_\gamma(v_\gamma) .$$

We may consider

$$\lambda = \{\lambda_i\}_{i \in I} \cup \{\lambda_\gamma\}_{\gamma \in \Gamma} \in C^0(X, F, \| \ \|) .$$

Then $l\ (\lambda) = \{\lambda_i\}_{i \in I} \in C^0(Y, F \mid Y, \| \ \|)$. We may also consider

$$\eta = \{(\psi_i, \phi_i)\} \in C^0(V, H, \| \ \|) ,$$
$$\psi = \{\psi_i\} = \beta\eta \in C^0(V, G, \| \ \|) ,$$
$$\phi = \{\phi_i\} = \alpha\eta \in C^0(V, N, \| \ \|) .$$

Now, we define a map

$$Q: B(1) \times C^0(X, F, \| \ \|) \rightarrow C^0(V, G, \| \ \|)$$

by

$$Q(\eta, \lambda)_i(z_i) = \beta(\eta)_i(z_i) - l(\lambda)_i(\alpha(\eta)_i(z_i), z_i) ,$$

where $B(1)$ is the 1-ball of $C^0(V, H, \| \|)$ with the center 0. Then it is easy to see that Q is analytic in an open neighbourhood $\Omega_1 \times \Omega_2$ of (0, 0). On the other hand, (taking Ω_1 and Ω_2 sufficiently small), η and λ satisfy the equations:

$$K_1\eta = 0 \quad \text{and} \quad K_2\lambda = 0$$

where

$$K_1\colon \Omega_1 \to C_e^1(V, H, |\ |_e) ,$$
$$K_2\colon \Omega_2 \to C_e^1(X, F, |\ |_e)$$

are analytic maps defined as in § 3. ($C_e^1(V, H, |\ |_e)$ and $C_e^1(X, F, |\ |_e)$ are defined as in § 2.) By the arguments in § 3, we know that the sets

$$\{\eta \in \Omega_1 \mid K_1\eta = 0\} \quad \text{and} \quad \{\lambda \in \Omega_2 \mid K_2\lambda = 0\}$$

are (finite dimensional) analytic spaces which are taken as local charts of $S(W)$ in neighbourhoods of V and X respectively. Now,

$$A(V, X) = \{(\eta, \lambda) \in \Omega_1 \times \Omega_2 \mid K_1\eta = 0,\ K_2\lambda = 0,\ Q(\eta, \lambda) = 0\}$$

is a finite dimensional subvariety of $\Omega_1 \times \Omega_2$.

It is clear that, using $A(V, X)$ as a local chart of the set A in a neighbourhood of the point (V, X), we can give an analytic space structure in A. It is also clear that the analytic space A thus defined is a closed subvariety of $S(W) \times S(W)$. This proves Theorem 3.

Let o be a point of $S(W)$. Then

$$\{t \in S(W) \mid V_t \supset V_0\} = A \cap (o \times S(W)) .$$

This proves the corollary of Theorem 3.

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