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TRANSFORMATION OF THE GENERALIZED WIENER MEASURE UNDER A CLASS OF LINEAR TRANSFORMATIONS

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1. Introduction. Let C_w be the Wiener space consisting of continuous real valued functions $x(t)$ on $[0, 1]$ with $x(0) = 0$. It is the purpose of this paper to investigate the transformation of the generalized Wiener measure on C_w corresponding to the generalized Brownian motion process (i.e. Brownian motion process with nonstationary increments) when the elements of C_w are transformed by a Volterra integral equation of the second kind.

For $0 = t_0 < t_1 < \cdots < t_n \leqq 1$, let $\mathfrak{F}_{t_1 \cdots t_n}$ be the σ-field of subsets of C_w of the type

$$(1.1) \qquad E = \{x \in C_w; [x(t_1), \cdots, x(t_n)] \in B\}, \qquad B \in \mathfrak{B}^n$$

where $\mathfrak{B}^n$ is the σ-field of Borel sets in the n-dimensional Euclidean space R^n. Let $b(t)$ be a strictly increasing continuous function on $[0, 1]$. It is well known that if we define a set function m on $\mathfrak{F}_{t_1 \cdots t_n}$ by

$$(1.2) \qquad m(E) = \frac{1}{\left\{(2\pi)^n \prod_{i=1}^{n} [b(t_i) - b(t_{i-1})]^2\right\}^{1/2}} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{(\xi_i - \xi_{i-1})^2}{b(t_i) - b(t_{i-1})}\right\} d\xi_1 \cdots d\xi_n$$

with $\xi_0 \equiv 0$, then m is well defined on the σ-field $\mathfrak{F}$ generated by the field $\mathfrak{F}_0$ which is the union of all the σ-fields $\mathfrak{F}_{t_1 \cdots t_n}$ and is in fact a probability measure on $(C_w, \mathfrak{F})$. (See for instance K. Itô [4] and P. Lévy [6].) Let $\mathfrak{F}^*$ be the Carathéodory extension of $\mathfrak{F}_0$ relative to m. Then $(C_w, \mathfrak{F}^*, m)$ is a complete probability measure space. We shall refer to $\mathfrak{F}^*$-measurability as Wiener measurability, and to m as the generalized Wiener measure corresponding to b.

The real valued function $X(t, x) = x(t)$, $x \in C_w$, $t \in [0, 1]$ is then a stochastic process with independent increments on the probability space $(C_w, \mathfrak{F}^*, m)$. In fact $X(0, x) = 0$ for every $x \in C_w$, and the increment $X(t'', x) - X(t', x)$ is distributed according to $N(0, b(t'') - b(t'))$, i.e. the probability distribution Φ of the above increment is a normal distribution with density function

$$\Phi'(\eta) = \frac{1}{\{2\pi[b(t'') - b(t')]\}^{1/2}} \exp\left\{-\frac{1}{2}\frac{\eta^2}{b(t'') - b(t')}\right\}, \qquad \eta \in R . \tag{1.3}$$

Furthermore the space of sample functions $X(\cdot, x)$, $x \in C_w$, coincides with the sample space C_w.

Throughout this paper the topology of C_w will be the metric topology defined by the uniform norm $|||x||| = \sup_{[0,1]} |x(t)|$, $x \in C_w$. In this topology C_w is a separable Banach space, an open subset of C_w is always $\mathfrak{F}$-measurable and so is every continuous real valued functional $F[x]$, $x \in C_w$.

Our main results are the following theorems:

THEOREM 1. *Consider the probability space* $(C_w, \mathfrak{F}^*, m)$ *where* $b(t)$ *has a positive and continuous derivative* $b'(t)$ *on* $[0, 1]$. *Let* $F[y]$, $y \in C_w$, *be a bounded and continuous real valued functional on* C_w *which vanishes outside of a bounded subset of* C_w. *Let* $K(t)$ *be a continuous real valued function on* $[0, 1]$ *and define a transformation* T *of* C_w *into* C_w *by*

$$(Tx)(t) = x(t) + \int_0^t b'(s)K(s)x(s)ds , \qquad \text{for } x \in C_w . \tag{1.4}$$

Then

$$\int_{C_w} F[y]m(dy) = \int_{C_w} F[Tx]J[x]m(dx) \tag{1.5}$$

with the "Jacobian" $J[x]$ *given by*

$$J[x] = \exp\left\{-\int_0^1 K(t)X(t, x)dX(t, x)\right\} \exp\left\{-\frac{1}{2}\int_0^1 b'(t)K^2(t)x^2(t)dt\right\} \tag{1.6}$$

where the integral in the first exponential factor is the stochastic integral of the process $K(t)X(t, x)$ *with respect to the process* $X(t, x) = x(t)$.

THEOREM 2. *For the linear operator* T *defined by* (1.4) *which maps* C_w *one-to-one onto* C_w *and is continuous with a continuous inverse* T^{-1} *we have* $T^{-1}\Gamma$, $T\Gamma \in \mathfrak{F}^*$ *for every* $\Gamma \in \mathfrak{F}^*$ *and*

$$m(\Gamma) = \int_{T^{-1}\Gamma} J[x]m(dx) . \tag{1.7}$$

Moreover if $F[y]$, $y \in C_w$, *is a Wiener measurable real valued functional then*

$$\int_{\Gamma} F[y]m(dy) = \int_{T^{-1}\Gamma} F[Tx]J[x]m(dx) \tag{1.8}$$

in the sense that the existence of one side implies that of the other and *the equality of the two. Similarly*

$$m(T\Gamma) = \int_{\Gamma} J[x]m(dx) \tag{1.7'}$$

$$\int_{T\Gamma} F[y]m(dy) = \int F[Tx]J[x]m(dx) . \tag{1.8'}$$

We remark that according to the Volterra integral equation theory (see for instance pp. 145–149, K. Yosida [7])

$$\| T\| \leqq 1 + |||b'|||$$

and

$$\| T^{-1}\| \leqq \exp\{|||b'K|||\} .$$

The transformation of the standard Wiener measure (i.e. when $b(t) = t$ on [0, 1]) under transformations of the elements of C_w by Fredholm integral equations of the second kind has been investigated by R. H. Cameron and W. T. Martin [1]. The results, specialized to transformations by Volterra integral equations of the second kind with kernels depending on one variable only, were applied to evaluate various Wiener integrals by means of Sturm-Liouville differential equations in [2]. Aside from the fact that the measure is the generalized Wiener measure in our case the proofs of our results are considerably different from those of the theorems in [1]. The proofs of Theorem 1 and Theorem 2 are given in §3. In §2 we prove some lemmas for Theorem 1.

2. Lemmas for Theorem 1. Suppose that $b(t)$ has a positive and continuous derivative $b'(t)$ on [0, 1]. For every positive integer n let $t_i = i/n$, $i = 0, 1, 2, \cdots, n$ and let $\tau_i \in (t_{i-1}, t_i)$ be such that $b(t_i) - b(t_{i-1}) = b'(\tau_i)/n$ for $i = 1, 2, \cdots, n$. With τ_i fixed, let $\beta_i = b'(\tau_i)$. Similarly for a real valued continuous function $K(t)$ on [0, 1] let $K_i = K(t_i)$.

Consider the trasformation T_n of C_w defined by

$$(T_n x)(t) = x(t) + \frac{1}{n}\sum_{j=1}^{[nt]} \beta_j K_{j-1} x(t_{j-1}) + \frac{1}{n}\beta_{[nt]+1} K_{[nt]} x(t_{[nt]})(nt - [nt]) \tag{2.1}$$

with the convention that $\beta_{n+1} = \beta_n$. For $t = t_i$ we have $[nt] = i = nt$ so that

$$\begin{cases} (T_n x)(t_i) - x(t_i) = \dfrac{1}{n}\displaystyle\sum_{j=0}^{i-1} \beta_{j+1} K_j x(t_j) , \qquad i = 1, 2, \cdots, n \\ (T_n x)(t_0) - x(t_0) = 0 . \end{cases} \tag{2.2}$$

Thus $(T_n x)(t) - x(t)$ is a polygonal function with n equal steps, $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, n$, whose values at t_i are given by (2.2). The function $(T_n x)(t)$

is a polygonal function with n equal steps if and only if $x(t)$ is.

For later reference we remark at this point that since $x(t)$, $t \in [0, 1]$, $x \in C_w$ is a stochastic process on the probabity space $(C_w, \mathfrak{F}^*, m)$ with m given by (1, 2), we have for every real valued $\mathfrak{B}^n$-measurable function $f[\xi_1, \cdots, \xi_n]$ on R^n

$$\int_{C_w} f\left[x\left(\frac{1}{n}\right), \cdots, x\left(\frac{n}{n}\right)\right] m(dx) \tag{2.3}$$
$$= \frac{1}{\left\{(2\pi)^n \prod_{i=1}^{n} [b(t_i) - b(t_{i-1})]\right\}^{1/2}} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} f[\xi_1, \cdots, \xi_n]$$
$$\times \exp\left\{-\frac{n}{2} \sum_{i=1}^{n} \frac{(\xi_i - \xi_{i-1})^2}{\beta_i}\right\} d\eta_1 \cdots d\eta_n$$

in the sense that the existence of one side implies that of the other and the equality of the two.

LEMMA 1. *Let $H[\eta_1, \cdots, \eta_n]$ be a real valued, bounded and continuous function on R^n and let $G[y]$, $y \in C_w$, be defined by*

$$G[y] = H\left[y\left(\frac{1}{n}\right), \cdots, y\left(\frac{n}{n}\right)\right] \tag{2.4}$$

then

$$\int_{C_w} G[y] m(dy) = \int_{C_w} G[T_n x] \exp\left\{-\sum_{i=1}^{n} K_{i-1} x(t_{i-1}) [x(t_i) - x(t_{i-1})]\right\} \tag{2.5}$$
$$\times \exp\left\{-\frac{1}{2n} \sum_{i=1}^{n} \beta_i K_{i-1}^2 x^2(t_{i-1})\right\} m(dx) .$$

PROOF. According to (2.3),

$$\int_{C_w} G[y] m(dy) = \left\{\frac{n^n}{(2\pi)^n \prod_{i=1}^{n} \beta_i}\right\}^{1/2} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} H[\eta_1, \cdots \eta_n] \tag{2.6}$$
$$\times \exp\left\{-\frac{n}{2} \sum_{i=1}^{n} \frac{(\eta_i - \eta_{i-1})^2}{\beta_i}\right\} d\eta_1 \cdots d\eta_n$$

where the integrals exist from the boundedness of H. Consider the transformation S_n of $\xi = [\xi_1, \cdots, \xi_n] \in R^n$ into $\eta = [\eta_1, \cdots, \eta_n] \in R^n$ defined by

$$\eta = S_n \xi \; ; \qquad \eta_i = \xi_i + \frac{1}{n} \sum_{j=1}^{i-1} \beta_{j+1} K_j \xi_j, \qquad i = 1, 2, \cdots, n . \tag{2.7}$$

The Jacobian of this transformation is equal to 1. Applying (2.7) to the right side of (2.6) we obtain

$$(2.8)\quad \int_{C_w} G[y]m(dy)$$
$$= \left\{\frac{n^n}{(2\pi)^n \prod_{i=1}^{n} \beta_i}\right\}^{1/2} \int_{-\infty}^{\infty} (n) \int_{-\infty}^{\infty} H\Big[\xi_1, \cdots, \xi_n + \frac{1}{n}\sum_{j=1}^{n-1} \beta_{j+1} K_j \xi_j\Big]$$
$$\times \exp\Big\{-\sum_{i=1}^{n} K_{i-1}\xi_{i-1}(\xi_i - \xi_{i-1})$$
$$- \frac{1}{2n}\sum_{i=1}^{n} \beta_i K_{i-1}^2 \xi_{i-1}^2 - \frac{n}{2}\sum_{i=1}^{n} \frac{(\xi_i - \xi_{i-1})^2}{\beta_i}\Big\} d\xi_1 \cdots d\xi_n .$$

On the other hand in the right side of (2.5) we have by (2.4), (2.2)

$$G[T_n x] = H[(T_n x)(t_1), \cdots, (T_n x)(t_n)]$$
$$= H\Big[x(t_1), \cdots, x(t_n) + \frac{1}{n}\sum_{j=1}^{n-1} \beta_{j+1} K_j x(t_j)\Big] .$$

If we apply (2.3) to the right side of (2.5) the result is precisely the right side of (2.8). This proves (2.5).

LEMMA 2. *Let X be a random variable on a probability space $(\Omega, \mathfrak{B}, P)$ which is distributed normally with mean 0 and variance v. Let Y be a random variable on $(\Omega, \mathfrak{B}, P)$ which is measurable with respect to a σ-field $\mathfrak{A} \subset \mathfrak{B}$. If the σ-field $\sigma(X) \subset \mathfrak{B}$ generated by X and the σ-field $\mathfrak{A}$ are independent then*

$$(2.9)\quad E\Big\{\exp\Big\{XY - \frac{1}{2}vY^2\Big\}\Big|\mathfrak{A}\Big\} = 1 .$$

The proof will appear in [3].

LEMMA 3. *Let $X(t, x)$ be the stochastic process on the probability space $(C_w, \mathfrak{F}^*, m)$ and the domain of definition $D = [0, 1]$ defined by $X(t, x) = x(t)$ for $x \in C_w$ and $t \in D$. Let $g(t)$ be a real valued function on D and let $f_n(t, x)$ be an a stochastic process on $(C_w, \mathfrak{F}^*, m)$ and D defined by*

$$(2.10)\quad f_n(t, x) = g\Big(\frac{[nt]}{n}\Big) X\Big(\frac{[nt]}{n}, x\Big) , \text{ for } x \in C_w \text{ and } t \in D .$$

Then the stochastic integral $I(f_n)(t, x)$ of the process $f_n(t, x)$ with respect to the Brownian motion process with nonstationary increments $X(t, x)$ stisfies

$$(2.11)\quad E\Big[\exp\Big\{I(f_n)\Big(\frac{i}{n}, x\Big) - \frac{1}{2}\int_0^{i/n} f_n^2(t, x)db(t)\Big\}\Big] = 1 ,$$
$$\text{for } i = 1, 2, \cdots, n .$$

Proof. Since f_n is a stochastic step function

$$I(f_n)\left(\frac{i}{n}, x\right) = \sum_{j=1}^{i} f_n\left(\frac{j-1}{n}\right)\left\{X\left(\frac{j}{n}, x\right) - X\left(\frac{j-1}{n}, x\right)\right\} .$$

Let

$$Y_j(x) = f_n\left(\frac{j-1}{n}, x\right)\left\{X\left(\frac{j}{n}, x\right) - X\left(\frac{j-1}{n}, x\right)\right\} - \frac{1}{2}\frac{\beta_j}{n} f_n^2\left(\frac{j-1}{n}, x\right) .$$

Observe that

$$\sum_{j=1}^{i} \frac{\beta_j}{n} f_n^2\left(\frac{j-1}{n}, x\right) = \sum_{j=1}^{i} f_n^2\left(\frac{j-1}{n}, x\right)\left\{b(t_j) - b(t_{j-1})\right\} = \int_0^{i/n} f_n^2(t, x)db(t) .$$

Let

$$Z_i(x) = \exp\left\{\sum_{j=1}^{i} Y_j(x)\right\} = \exp\left\{I(f_n)\left(\frac{i}{n}\right) - \frac{1}{2}\int_0^{i/n} f_n^2(t, x)db(t)\right\} .$$

In terms of Z_i, (2.11) becomes $E(Z_i) = 1$ for $i = 1, 2, \cdots, n$.

Let $\mathfrak{A}_i = \sigma\{X(j/n, \cdot), j = 0, 1, 2, \cdots, i\}$ for $i = 0, 1, 2, \cdots, n$. Then $f_n(t, \cdot)$ is $\mathfrak{A}_i$-measurable for $t \in [0, (i+1)/n]$ so that in particular $f_n((i-1)/n, \cdot)$ is $\mathfrak{A}_{i-1}$-measurable for $i = 1, 2, \cdots, n$. The random variable $X(i/n, \cdot) - X((i-1)/n, \cdot)$ is normally distributed with mean 0 and variance $b(t_i) - b(t_{i-1}) = \beta_i/n$. Also the σ-field $\sigma\{X(i/n, \cdot)\}$ and the σ-field $\mathfrak{A}_{i-1}$ are independent. Thus by Lemma 2

$$(2.12) \qquad E[\exp\{Y_i\}|\mathfrak{A}_{i-1}] = 1 \text{ for } i = 1, 2, \cdots, n .$$

We proceed to show that $E(Z_i) = 1$ for $i = 1, 2, \cdots, n$ by induction. First of all, $f_n(0, x) = 0$, $Y_1(x) = 0$, $Z_1(x) = 1$ for $x \in C_w$ so that $E(Z_1) = 1$. Now suppose $E(Z_i) = 1$. Then

$$E(Z_{i+1}) = E[Z_i \exp\{Y_{i+1}\}] = E[E[Z_i \exp\{Y_{i+1}\}|\mathfrak{A}_i]] .$$

Since $Y_1, \cdots, Y_i$ are all $\mathfrak{A}_i$-measurable so is Z_i and consequently

$$E[Z_i \exp\{Y_{i+1}\}|\mathfrak{A}_i] = Z_i E[\exp\{Y_{i+1}\}|\mathfrak{A}_i] = Z_i .$$

Thus $E(Z_{i+1}) = E(Z_i) = 1$. This completes the proof for $E(Z_i) = 1$ by induction.

Let L_n be the linear transformation of C_w into C_w defined by

$$(2.13) \qquad (L_n x)(t) = x(t_{i-1}) + \frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}}(t - t_{i-1})$$

$$\text{for } t \in [t_{i-1}, t_i],\ x \in C_w, \text{ and } i = 1, 2, \cdots, n .$$

Clearly

$$|||L_n x||| = \max_{i=1,\cdots,n}\left|x\left(\frac{i}{n}\right)\right| \leqq |||x||| \text{ and} \tag{2.14}$$

$$\lim_{n\to\infty} |||L_n x - x||| = 0 . \tag{2.15}$$

Also for T and T_n defined by (1.4) and (2.1) respectively, we have

$$\lim_{n\to 0} |||L_n T_n x - Tx||| = 0 . \tag{2.16}$$

This follows from

$$|||L_n T_n x - Tx||| \leqq |||L_n T_n x - L_n Tx||| + |||L_n Tx - Tx||| .$$

where

$$|||L_n T_n x - L_n Tx||| \leqq |||T_n x - Tx|||$$

by (2.14), $\lim_{n\to\infty} |||T_n x - Tx||| = 0$ from the uniform continuity of $b'(t)K(t)$ on $[0, 1]$, and $\lim_{n\to\infty} |||L_n Tx - Tx||| = 0$ by (2.15).

LEMMA 4. *Let $X(t, x)$, $g(t)$ and $f_n(t, x)$ be as defined in Lemma 3. Then the random variables $Z_n(x)$, on $(C_w, \mathfrak{F}^*, m)$ defined by*

$$Z_n(x) = \exp\left\{I(f_n)(1, x) - \int_0^1 f_n^2(t, x)db(t)\right\} \qquad n = 1, 2\cdots . \tag{2.17}$$

are uniformly integrable on C_w. If $g(t)$ is bounded on D then for every $B \geqq 0$ the random variables $Y_n(x)$, $n = 1, 2, \cdots$, defined by

$$Y_n(x) = \chi_{[0,B]}(|||L_n x|||)\exp\{I(f_n)(1, x)\} \tag{2.18}$$

are uniformly integrable on C_w.

PROOF. For $\alpha \geqq 0$ let

$$\Gamma_{\alpha,n} = \{x \in C_w ; Z_n(x) > \alpha\} .$$

To show the uniform integrability of Z_n, $n = 1, 2, \cdots$, we show that for every $\varepsilon > 0$ there exists $A \geqq 0$ such that

$$\int_{\Gamma_{A,n}} Z_n(x)m(dx) < \varepsilon , \quad \text{for} \quad n = 1, 2, \cdots .$$

For each n define a function $I_n(\alpha, x)$ on $[0, \infty) \times C_w$ by

$$I_n(\alpha, x) = \begin{cases} 1 & \text{when} \quad \alpha < Z_n(x) \\ 0 & \text{when} \quad \alpha \geqq Z_n(x) . \end{cases}$$

Then $Z_n(x) = \int_{[0,\infty)} I_n(\alpha, x)d\alpha$ for every $x \in C_w$. Thus for an arbitrary $A \geqq 0$, by Tonelli's Theorem

$$\begin{aligned}\int_{\Gamma_{A,n}} Z_n(x)m(dx) &= \int_{[0,\infty)}\left\{\int_{\Gamma_{A,n}} I_n(\alpha, x)m(dx)\right\}d\alpha \\ &= \int_{[0,\infty)} m(\{x; Z_n(x) > A\} \cap \{x; Z_n(x) > \alpha\})d\alpha \\ &= \int_{[0,A]} m(\{x; Z_n(x) > A\})d\alpha + \int_{(A,\infty)} m(\{x; Z_n(x) > \alpha\})d\alpha \\ &= Am(\Gamma_{A,n}) + \int_{(A,\infty)} m(\Gamma_{\alpha,n})d\alpha\ .\end{aligned}$$

Now for $\alpha > 0$

$$\begin{aligned}m(\Gamma_{\alpha,n}) &\leqq \frac{1}{\alpha^2}\int_{C_w} Z_n^2(x)m(dx) \\ &= \frac{1}{\alpha^2}\int_{C_w} \exp\left\{2\left[I(f_n)(1, x) - \int_0^1 f_n^2(t, x)db(t)\right]\right\}m(dx) = \frac{1}{\alpha^2}\end{aligned}$$

since the last integral is equal to 1 according to Lemma 3 applied to $2f_n$. Then for $A > 2/\varepsilon$

$$\int_{\Gamma_{A,n}} Z_n(x)m(dx) \leqq A\frac{1}{A^2} + \int_{(A,\infty)} \frac{1}{A^2}d\alpha = \frac{2}{A} < \varepsilon \quad \text{for} \quad n = 1, 2, \cdots$$

proving the uniform integrability of Z_n, $n = 1, 2, \cdots$.

Finally consider the case where $g(t)$ is bounded on D. Now

$$\max_{t \in D}\left|X\left(\frac{[nt]}{n}\right)\right| = |||L_n x|||\ .$$

Thus $x \in C_w$, $|||L_n x||| \leqq B$ and $B \geqq 0$ imply $|f_n(t, x)| \leqq |||g|||B$ and

$$\int_0^1 f_n^2(t, x)db(t) \leqq |||g|||^2 B^2[b(1) - b(0)]\ .$$

Then with

$$\gamma = \exp\{|||g|||^2 B^2[b(1) - b(0)]\}$$

we have

$$Y_n(x) = \chi_{[0,B]}(|||L_n x|||)Z_n(x)\exp\left\{\int_0^1 f_n^2(t, x)db(t)\right\} \leqq \gamma Z_n(x)\ .$$

Therefore when $\alpha \geqq \gamma A$

$$\int_{\{x; Y_n(x) > \alpha\}} Y_n(x)m(dx) \leqq \gamma\int_{\{x; Z_n(x) > \alpha/\gamma\}} Z_n(x)m(dx) < \gamma\varepsilon \quad \text{for} \quad n = 1, 2, \cdots$$

proving the uniform integrability of Y_n, $n = 1, 2, \cdots$.

LEMMA 5. *If* $x \in C_w$ *and for some* $M \geqq 0$

$$|||L_n x||| > M \exp\{|||b'K|||\}$$

then

$$|||L_n T_n x||| > M .$$

PROOF. As in the Volterra integral equation theory one can show that T_n defined by (2.1) transforms C_w one-to-one onto C_w, T_n and T_n^{-1} are bounded linear operators and

$$||T_n^{-1}|| \leqq \exp\{|||b'K|||\} .$$

Now for an arbitrary $x \in C_w$ which satisfies $|||x||| > M \exp\{|||b'K|||\}$ for some $M \geqq 0$ we have $|||x||| > M||T_n^{-1}||$. Then $|||T_n x||| > M$ for otherwise we would have $|||T_n x||| \leqq M$ and consequently

$$M||T_n^{-1}|| < |||x||| = |||T_n^{-1}T_n x||| \leqq ||T_n^{-1}||| \ |||T_n x||| \leqq ||T_n^{-1}||M ,$$

a contradiction. Since the above $x \in C_w$ is arbitrary, in particular $|||L_n x||| > M \exp\{|||b'K|||\}$ implies $|||T_n L_n x||| > M$. But by (2.1) and (2.13), $T_n L_n x = L_n T_n x$. Thus $|||L_n T_n x||| > M$.

3. Proof of Theorem 1. From the natural one-to-one correspondence between the polygonal functions on [0, 1] which have n equal steps and vanish at $t = 0$ and the elements of R^n there exists for the real valued functional $F[y]$, $y \in C_w$, a real valued function $H[\eta_1, \cdots, \eta_n]$ on R^n such that $F[L_n y] = H[y(1/n), \cdots, y(n/n)] \equiv G[y]$ for $y \in C_w$. The boundedness and continuity of F on C_w imply the same for H on R^n with respect to the uniform topology of R^n. Now for T_n defined by (2.1) we have

$$G[T_n x] = H\Big[(T_n x)\Big(\frac{1}{n}\Big), \cdots, (T_n x)\Big(\frac{n}{n}\Big)\Big] = F[L_n T_n x], \quad \text{for} \quad x \in C_w$$

so that according to Lemma 1

$$\int_{C_w} F[L_n y] m(dy) = \int_{C_w} F[L_n T_n x] J_n[x] m(dx) \tag{3.1}$$

where

$$J_n[x] = \exp\Big\{-\sum_{i=1}^{n} K_{i-1} x(t_{i-1})[x(t_i) - x(t_{i-1})]\Big\} \times \exp\Big\{-\frac{1}{2n}\sum_{i=1}^{n} \beta_i K_{i-1}{}^2 x^2(t_{i-1})\Big\} . \tag{3.2}$$

We obtain (1.5) by letting $n \to \infty$ in (3.1). This is done as follows.

On the left side of (3.1) since F is bounded on C_w, by applying the Bounded Convergence Theorem and then by (2.15) and the continuity of F we obtain

$$\lim_{n\to\infty}\int_{C_w} F[L_n y]m(dy) = \int_{C_w} F[y]m(dy) . \tag{3.3}$$

On the right side of (3.1) let $M \geqq 0$ be such that $F[x] = 0$ for $|||x||| \geqq M$. By Lemma 5, $|||L_n x||| > B$ with $B = M \exp\{|||b'K|||\}$ implies $|||L_n T_n x||| > M$. Then

$$\int_{C_w} F[L_n T_n x]J_n[x]m(dx) = \int_{C_w} \chi_{[0,B]}(|||L_n x|||)F[L_n T_n x]J_n[x]m(dx) . \tag{3.4}$$

By Lemma 4 the functionals on C_w

$$\chi_{[0,B]}(|||L_n x|||) \exp\left\{-\sum_{i=1}^{n} K_{i-1}x(t_{i-1})[x(t_i) - x(t_{i-1})]\right\},\ n = 1, 2, \cdots$$

are uniformly integrable on C_w. Then since F is bounded on C_w and

$$\exp\left\{-\frac{1}{2n}\sum_{i=1}^{n} \beta_i K_{i-1}^2 x^2(t_{i-1})\right\} \leqq 1 \quad \text{for} \quad x \in C_w,\ n = 1, 2, \cdots ,$$

the functionals on C_w

$$\chi_{[0,B]}(|||L_n x|||)F[L_n T_n x]J_n[x] , \qquad n = 1, 2, \cdots \tag{3.5}$$

are uniformly integrable on C_w.

According to (2.16) and the continuity of F

$$\lim_{n\to\infty} F[L_n T_n x] = F[Tx] \qquad x \in C_w .$$

Also

$$\lim_{n\to\infty} \exp\left\{-\frac{1}{2n}\sum_{i=1}^{n} \beta_i K_{i-1}^2 x^2(t_{i-1})\right\} = \exp\left\{-\frac{1}{2}\int_0^1 b'(t)K^2(t)x^2(t)dt\right\} \qquad x \in C_w .$$

Let $f_n(t, x) = K([nt]/n)X([nt]/n)$, $n = 1, 2, \cdots$, and $f(t, x) = K(t)X(t, x)$, for $x \in C_w$, $t \in [0, 1]$. For each $x \in C_w$, $\lim_{n\to\infty} f_n(t, x) = f(t, x)$ uniformly on $[0, 1]$ so that

$$\lim_{n\to\infty}\int_0^1 [f_n(t, x) - f(t, x)]^2 db(t) = 0$$

and this implies (see for instance pp. 185–186, Itô [5]) that $I(f_n)(1, x)$ converges to $I(f)(1,x)$ in the m measure. Thus the sequence of functionals on C_w given by (3.5) converges in m measure to

$$\chi_{[0,B]}(|||x|||)F[Tx] \exp\{-I(f)(1, x)\} \exp\left\{-\frac{1}{2}\int_0^1 b'(t)K^2(t)x^2(t)dt\right\} .$$

Since the functionals given by (3.5) are integrable and uniformly integrable on C_w the above convergence in measure justifies passing to the limit under the integral in (3.4) and have

$$\begin{aligned}(3.6)\qquad &\lim_{n\to\infty}\int_{C_w} F[L_n T_n x]J_n[x]m(x) \\ &= \int_{C_w}\chi_{[0,B]}(|||x|||)F[Tx]\exp\{-I(f)(1,x)\} \\ &\quad \times \exp\left\{-\frac{1}{2}\int_0^1 b'(t)K^2(t)x^2(t)dt\right\}m(dx)\ .\end{aligned}$$

Now $|||x||| > B$ implies $|||L_n x||| > B$ for sufficiently large n. For such n $|||T_n x||| \geqq \max_{i=1,\ldots,n}|(T_n x)(i/n)| = |||L_n T_n x||| > M$ by Lemma 5. Thus for $|||x||| > B$ we have $|||Tx||| \geqq M$ and consequently $F[Tx] = 0$. Therefore in the integrand on the right side of (3.6) we may drop the factor $\chi_{[0,B]}(|||x|||)$ without disturbing the equality of the two sides. Now (3.1), (3.3) and (3.6) give (1.5).

The proof of Theorem 2 is omitted since it is the same as the proof of Theorem 1 given in [1].

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